SOFIC GROUPS ARE NOT LOCALLY EMBEDDABLE INTO FINITE MOUFANG LOOPS

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Received April 13, 2020. Published online March 9, 2021.
Communicated by Radomír Halaš

Abstract. We shall show that there exist sofic groups which are not locally embeddable into finite Moufang loops. These groups serve as counterexamples to a problem and two conjectures formulated in the paper by M. Vodička, P. Zlatoš (2019).

Keywords: group; diassociative IP loop; Moufang loop; finite embeddability property; local embeddability

MSC 2020: 20E25, 20N05, 05B15, 05C25

1. Introduction

A class $\mathbf{K}$ of groupoids has the *finite embeddability property* (FEP for short) if for every algebra $(G,\cdot)\in\mathbf{K}$ and each nonempty finite subset $X\subseteq G$ there is a finite algebra $(H,*)\in\mathbf{K}$ extending $(X,\cdot)$, i.e. $X\subseteq H$ and $x\cdot y=x*y$ for all $x,y\in X$ such that $x\cdot y\in X$. A groupoid $(G,\cdot)$ is *locally embeddable* into a class of groupoids $\mathbf{M}$ if for every nonempty finite set $X\subseteq G$ there is $(H,*)\in\mathbf{M}$ such that $X\subseteq H$ and $x\cdot y=x*y$ for all $x,y\in X$ such that $x\cdot y\in X$.

The notion of FEP was firstly introduced by Henkin in [6] for general algebraic systems and the more general notion of locall embeddability was introduced by Mal'tsev in [10], [11]. Clearly, class $\mathbf{K}$ has FEP if and only if every grupoid in $\mathbf{K}$ is locally embeddable into the class of finite groupoids in $\mathbf{K}$.

There are strong connections between the finite (local) embeddability properties and sofic groups (originally introduced by Gromov in his work on coarse geometry),

The first author was supported by the NSP of the Slovak Republic, SAIA. The second author was supported by VEGA grant No. 1/0333/17 of Slovak Republic.

DOI: 10.21136/MB.2021.0065-20

which are the widest known class of groups satisfying the surjunctivity condition for group automata over them, see [2]. The notion of a sofic group is given later in Definition 2.6.

These notions have been intensively studied; we shall mention here just some papers which motivated our results. Gordon and Glebsky in [5] characterized sofic groups as the groups which are "approximately locally embeddable" into finite quasigroups. Ziman in [17] has shown that the class of all loops with antiautomorphic inverses, i.e. loops with two-sided inverses satisfying the identity $(xy)^{-1} = y^{-1}x^{-1}$, has the FEP.

The present paper is a direct reaction to the article by Vodička and Zlatoš (see [15]) in which they study local embeddability of groups into finite IP loops (the precise definition is given in Definition 2.1). The main result of this paper is that the class of all IP loops has FEP. Known results (including the mentioned ones) lead to some natural questions and the authors in the discussion in the last part of their paper formulate some problems and conjectures.

We will give answers to one of their problem, namely Problem 2 and to both conjectures formulated as follows.

Problem 2. *Does the class of all Moufang loops have the FEP?*

Conjecture 1. *Every group is locally embeddable into finite Moufang loops.*

Conjecture 2. *A group is sofic if and only if it is locally embeddable into finite Moufang loops.*

We are going to give a negative answer to Problem 2 and Conjecture 1. By showing that there are sofic groups not locally embeddable into Moufang loops we will refute one implication in Conjecture 2; the remaining implication remains open. Thus, it seems to make sense to examine more closely the class of all groups locally embeddable into finite Moufang loops in the future. Clearly, it is an extension of the class of all groups locally embeddable into finite groups (LEF groups) studied by Gordon and Vershik (see [14]), however, it is not clear whether this extension is proper. And it is still reasonable to expect that all such groups are sofic.

2. Results

Definition 2.1. An algebra $(L, \cdot, 1, ^{-1})$ with a binary operation of multiplication "$\cdot$", a distinguished element 1 denoting the unit, and a unary operation $^{-1}$ of taking inverses is called an *IP loop* if it satisfies the identities:

$$1x = x = x1 \quad \text{and} \quad x^{-1}(xy) = y = (yx)x^{-1}.$$

Then the identitites $x^{-1}x = 1 = xx^{-1}$ and $(x^{-1})^{-1} = x$ easily follow.

The IP loop L is *diassociative* (*DIP loop* for short) if for any terms $s(x, y)$, $t(x, y)$, $u(x, y)$ in just two variables x, y using "$\cdot$", 1, $^{-1}$ the identity

$$s(tu) = (st)u$$

is satisfied in L.

Equivalently, the IP loop L is a DIP loop if and only if any IP subloop $\langle x, y\rangle$ generated by two elements $x, y \in L$ has the operation associative, it means that $\langle x, y\rangle$ is a group.

Definition 2.2 ([13])**.** A loop is called a *Moufang loop* if it satisfies one (and therefore all) of the following four equivalent identities:

$$x(y(xz)) = ((xy)x)z, \quad (xy)(zx) = (x(yz))x,$$
$$x(y(zy)) = ((xy)z)y, \quad (xy)(zx) = x((yz)x).$$

The definition of Moufang loops, the equivalence of their defining identities as well as the proof of Moufang's theorem can be found e.g. in [13]. The diassociativity of Moufang loops is a consequence of a Moufang's theorem. A short proof of Moufang's theorem can be found in [4].

We shall use some graph theoretical notions in the next proof (see [9]). Let G be a group, $S \subseteq G$ be a finite symmetrical generating set of G not containing the identity element 1 of G. Let $\Gamma = \Gamma(G, S)$ be a Caley graph of the group G with edges colored by elements of the generating set S. We shall use the distance $\operatorname{dist}(g, h)$ between two elements $g, h \in G$, which is defined as the length of a shortest path in the graph Γ which connects vertices g, h (in fact, it is the length of a shortest word consisting of elements of S which can express the group element gh^{-1} in G, the identity element 1 is by definition expressed as a word of length 0). The function dist is a metric on G. For $a \in G$ we put $|a| = \operatorname{dist}(1, a)$.

Denote $\mathrm{B}_\Gamma(r) = \{a \in G\colon\ |a| \leqslant r\}$, for any non-negative integer r, the ball of radius r. In particular, $\mathrm{B}_\Gamma(0) = \{1\}$, $\mathrm{B}_\Gamma(1) = \{1\} \cup S$. Since set S is finite, $\mathrm{B}_\Gamma(r)$

is finite for any r. Clearly, $a \in \mathrm{B}_\Gamma(|a|)$ for any $a \in G$. As dist is a metric on G, for $m, n \geqslant 0$ and $g \in \mathrm{B}_\Gamma(m)$, $h \in \mathrm{B}_\Gamma(n)$ we have $g \circ h \in \mathrm{B}_\Gamma(m+n)$.

The following theorem is our main tool.

Theorem 2.3. *Let $(G, \circ)$ be a group with two generators. Then the following conditions are equivalent:*

(i) *G is locally embeddable into finite groups;*

(ii) *G is locally embeddable into finite Moufang loops;*

(iii) *G is locally embeddable into finite diassociative IP loops.*

Proof. (i) $\Rightarrow$ (ii): This is true because every group is a Moufang loop.

(ii) $\Rightarrow$ (iii): This is also true because it is known that every Moufang loop is a diassociative IP loop.

(iii) $\Rightarrow$ (i): Let $(G, \circ)$ be a group with two generators x, y. Assume that G is locally embeddable into finite DIP loops. Let $X = \{a_1, \dots, a_n\} \subseteq G$ be a finite set. We will prove that we can find a finite DIP loop $(H', *')$ which is two-generated (that means that it is a group), $X \subseteq H'$ and $a_i *' a_j = a_i \circ a_j$ for any $a_i, a_j \in X$ such that $a_i \circ a_j \in X$.

The new DIP loop $(H', *')$ on two generators will be constructed in the following way. Let 1 be the identity element of G, $S = \{x, y, x^{-1}, y^{-1}\}$. We shall use the Caley graph $\Gamma = \Gamma(G, S)$, the distance dist and the norm $|a|$ of an element $a \in G$ introduced above. Denote

$$d = \max\{|a_1|, \dots, |a_n|\}.$$

Then $X \subseteq \mathrm{B}_\Gamma(d)$. Let $X' = \mathrm{B}_\Gamma(d)$. The partial operation restricted from G to the set X' contains all information which is necessary to generate any element $b \in X'$ as a group element using generators x, y. The element $b \in X'$ has $|b| = \mathrm{dist}(1, b) = k \leqslant d$ and that means that we can write b as a word $g_1 \circ \ldots \circ g_k$, where $g_i \in S$ for $i = 1, \dots, k$. This is a reduced word (that means that for two consecutive elements we always have $g_i \circ g_{i+1} \neq 1$), otherwise it would not be a shortest way to write the element b in the group G. The elements of the path connecting 1 and b in the Caley graph Γ are

$$b_0 = 1, \quad b_1 = g_1, \quad b_2 = g_1 \circ g_2, \dots, \quad b_{k-1} = g_1 \circ g_2 \circ \ldots \circ g_{k-1}, \quad b_k = b = g_1 \circ g_2 \circ \ldots \circ g_k$$

and we know that for $0 \leqslant l \leqslant k-1$, $b_l, g_{l+1} \in X'$ and also $b_l \circ g_{l+1} = b_{l+1} \in X'$.

The set $X' \subseteq G$ is finite and, by assumption, there exits a finite DIP loop $(H, *)$ such that $X' \subseteq H$ and $a * b = a \circ b$ for any $a, b \in X'$ such that $a \circ b \in X'$.

According to our construction of X', we can calculate all the elements of X', in particular $a_1, \dots, a_n$, inside of H, starting from 1 and subsequently multiplied by elements from the set $\{x, y, x^{-1}, y^{-1}\} \subseteq X'$ and these calculations coincide with those in G.

Let $H' = \langle x, y \rangle$ be the IP subloop of the loop $(H, *)$, generated by $x, y \in H$. Then $X' \subseteq H'$. As $(H, *)$ is a finite DIP loop, $(H', *)$ is a finite group. Since $X \subseteq X' \subseteq H'$, it is a finite group containg X. This shows that the group G is locally embeddable into finite groups. $\square$

Now we introduce so called Baumslag-Solitar groups which will serve as a counterexample to Problem 2 and both conjectures of Vodička and Zlatoš.

Definition 2.4 ([1]). The *Baumslag-Solitar* groups are the two-generated groups $\mathrm{BS}(m, n) = \langle a, b\colon\ a^{-1}b^m a = b^n \rangle$ for $|m|, |n| > 1$, $|m| \neq |n|$.

By the results which can be found in [12], Baumslag-Solitar groups are not residually finite and therefore they are not locally embeddable into finite groups. They are two-generated and by Theorem 2.3 we have:

Corollary 2.5.

(a) *The Baumslag-Solitar groups* $\mathrm{BS}(m, n)$ *for* $|m|, |n| > 1$, $|m| \neq |n|$ *are not locally embeddable into Moufang loops.*

(b) *The class of Moufang loops does not have the FEP.*

This disproves Problem 2 and, at the same time, it means that the answer to Conjecture 1 is no.

For Conjecture 2 we need to recall some more definitions and results.

Definition 2.6 ([3]). A group G is called *sofic* if for every finite subset F of G and every $\varepsilon > 0$ there exist an integer $n \geqslant 1$ and a map $\varphi\colon\ G \to S_n$ such that

(a) for every $g \in F \setminus \{e\}$, $\mathrm{dist}(\varphi(g), \mathrm{id}) > 1 - \varepsilon$, where e is the identity element of G,

(b) for all $g_1, g_2 \in F$, $\mathrm{dist}(\varphi(g_1^{-1}g_2), \varphi(g_1)^{-1}\varphi(g_2)) < \varepsilon$,

where $\mathrm{dist}(\sigma, \tau)$ denotes the normalized Hamming distance between permutations $\sigma, \tau \in S_n$, i.e. the number of points not fixed by $\sigma^{-1}\tau$, divided by n.

Definition 2.7 ([3], [16]). A group G is *amenable* if and only if for every finite set $K \subseteq G$ and every $\varepsilon > 0$ there is a (K, ε)-invariant set, it means a finite set $F \subseteq G$ such that $|KF \setminus F| < \varepsilon |F|$. A *monotile* for a group G is a finite set $T \subseteq G$ such that G is a disjoint union of right translates of T. We will say that a group G is *monotileably amenable* (*MTA* for short) if for every finite set $K \subseteq G$ and every $\varepsilon > 0$ there is a monotile T for G that is (K, ε)-invariant.

The notion of MTA groups was introduced in [16]. It is known that all amenable groups are sofic (see e.g. [2], Proposition 7.5.6) which means that all MTA groups are sofic as well. Weiss in [16] proved that every residually finite amenable group and every solvable (hence every Abelian, too) group is an MTA group.

Moreover, Weiss using his methods and a result from [1] showed that specifically the Baumslag-Solitar group $\mathrm{BS}(2,3)$ is an MTA group, see [16]. Hence, $\mathrm{BS}(2,3)$ is a two-generated sofic group, which is not locally embeddable into finite groups. By Theorem 2.3 we have:

Corollary 2.8. *The MTA (hence sofic) group* $\mathrm{BS}(2,3)$ *is not locally embeddable into finite Moufang loops.*

This fact disproves Conjecture 2. In fact, this means more, namely that there are MTA groups (and amenable groups as well) which are not locally embeddable into finite Moufang loops.

To describe more counterexamples we need to use some additional tools.

Definition 2.9 ([7], [8]). Let G be a group with presentation $G = \langle S \mid R \rangle$, let $\theta\colon H \to G$ be an injective homomorphism from a subgroup H of G and t be a new symbol not contained in S. We denote

$$G*_\theta = \langle S, t \mid R, t^{-1}ht = \theta(h)\ \forall\, h \in H \rangle$$

the group called the *HNN extension* of G relative to θ. The original group G is called the base group for the construction, the subgroups H and $\mathrm{im}(\theta)$ are the associated subgroups. The new generator t is called the stable letter.

The next proposition can be found in [3] as Corollary 3.6.

Proposition 2.10. *If* $K = G*_\theta$ *is an HNN extension of a sofic group* G *relative to an injective homomorphism* $\theta\colon H \to G$*, where* H *is a monotileably amenable subgroup of* G*, then* K *is sofic.*

We are going to use this proposition to show that the Baumslag-Solitar groups $\mathrm{BS}(m,n)$ are sofic.

Let $G = \langle a \rangle \cong (\mathbb{Z}, +)$, $H = \langle a^m \rangle$ and $\theta\colon H \to G$ be defined by

$$\theta(a^m) = a^n, \text{ it means that for } k \in \mathbb{Z},\ \theta((a^m)^k) = (a^n)^k.$$

Then θ is clearly an injective homomorphism and by the definition of HNN extension we see that

$$\mathrm{BS}(m,n) \cong G *_\theta .$$

As we have already mentioned, the Abelian group H is an MTA group, $(\mathbb{Z}, +)$ is sofic and by Proposition 2.10, $G*_\theta$ is sofic. Thus, the Baumslag-Solitar groups $\mathrm{BS}(m,n)$ with $|m|, |n| > 1$, $|m| \neq |n|$, are sofic groups which are not locally embeddable into finite groups and by Theorem 2.3 we have:

Corollary 2.11. *The Baumslag-Solitar groups* $\mathrm{BS}(m,n)$ *with* $|m|,|n| > 1$, $|m| \neq |n|$ *are sofic groups which are not locally embeddable into finite Moufang loops.*

3. Final remarks

Our results together with the original problems and conjectures by Vodička and Zlatoš suggest some challenging questions following up our discussion from the last section of the introduction:

(1) Is every group locally embeddable into the class of all finite Moufang loops already sofic?
(2) Does the class of all groups locally embeddable into the class of all finite Moufang loops coincide with the class of all LEF groups or is it a proper extension of it?
(3) Does the class of all monoassociative IP loops have the FEP?
(4) Is every group locally embeddable into the class of all finite monoassociative IP loops?

Of course the first alternative from (2) implies (1), however (1) could be true even if this is not the case. Similarly, (the positive answer to) (3) implies (the positive answer to) (4), however (4) could be true, even if (3) fails. Moreover, if (4) is true then the class of all monoassociative IP loops would become a hot candidate for a minimal variety into the finite members of which all the groups can be locally embedded.

A c k n o w l e d g e m e n t. We thank the anonymous referees for their detailed reviews and suggestions which helped to improve our presentation.

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