# ON THE RADIUS OF SPATIAL ANALYTICITY FOR THE HIGHER ORDER NONLINEAR DISPERSIVE EQUATION

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Received May 22, 2020. Published online March 16, 2021.
Communicated by Ondřej Kreml

*Abstract.* In this work, using bilinear estimates in Bourgain type spaces, we prove the local existence of a solution to a higher order nonlinear dispersive equation on the line for analytic initial data $u_0$. The analytic initial data can be extended as holomorphic functions in a strip around the $x$-axis. By Gevrey approximate conservation law, we prove the existence of the global solutions, which improve earlier results of Z. Zhang, Z. Liu, M. Sun, S. Li, (2019).

*Keywords*: higher order nonlinear dispersive equation; radius of spatial analyticity; approximate conservation law

*MSC 2020*: 35E15, 35Q53, 35B65, 35C07

## 1. Introduction

In this paper we consider a higher order nonlinear dispersive equation:

$$\begin{cases} \partial_t u + \lambda_1 \partial_x^7 u + \lambda_2 \partial_x^5 u + \lambda_3 \partial_x^3 u + \lambda_4 \partial_x u + u \partial_x u = 0, & (x,t) \in \mathbb{R}^2, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $\lambda_1 \neq 0$, $\lambda_2$, $\lambda_3$, $\lambda_4$ are real numbers. For more details to the higher order nonlinear dispersive equation which arises in the study of water waves with surface tension and arises as mathematical models for the weakly nonlinear propagation of long waves, see [15] and [8].

Well-posedness of the Cauchy problem for the higher order nonlinear dispersive equation in (1.1) has been studied by Zhang et al. (see [15]) in Sobolev spaces. By using the Fourier restriction norm, the authors showed that (1.1) is locally well-posed

DOI: 10.21136/MB.2021.0096-20 

in $H^s(\mathbb{R})$ for $s \geqslant -\frac{5}{8}$ and global well-posedness for $s = 0$. In [4] it has been shown that the Cauchy problem for fifth-order Kadomtsev-Petviashvili I equation is locally well-posed in $G^{\delta,s_1,s_2}$ for $s_1, s_2 \geqslant 0$, see also [3], [2].

The main novelty in this paper is the study of the question of global well-posedness for initial data $u_0(x)$ that is analytic on the line and can be extended as holomorphic functions in a strip around the $x$-axis. A class of analytic functions suitable for our analysis is the analytic Gevrey class $G^{\delta,s}(\mathbb{R})$ introduced in [6], which is defined as

$$G^{\delta,s}(\mathbb{R}) = \{u_0 \in L^2(\mathbb{R})\colon\ \|u_0\|_{G^{\delta,s}(\mathbb{R})} < \infty\}, \tag{1.2}$$

where

$$\|u_0\|^2_{G^{\delta,s}(\mathbb{R})} = \int_{\mathbb{R}} \mathrm{e}^{2\delta|\xi|}\langle\xi\rangle^{2s}|\widehat{u_0}(\xi)|^2\,\mathrm{d}\xi$$

for $s \in \mathbb{R}$, $\delta \geqslant 0$ and $\langle\cdot\rangle := (1 + |\cdot|)$. If $\delta = 0$, the space $G^{\delta,s}$ coincides with the standard Sobolev space $H^s$.

We note the following embedding property of the Gevrey spaces: for all $0 < \delta' < \delta$ and $s, s' \in \mathbb{R}$ we have

$$G^{\delta,s}(\mathbb{R}) \subset G^{\delta',s'}(\mathbb{R}), \quad \text{i.e. } \|u_0\|_{G^{\delta',s'}(\mathbb{R})} \leqslant c_{s,s',\delta,\delta'}\|u_0\|_{G^{\delta,s}(\mathbb{R})}. \tag{1.3}$$

The interest in these spaces is due to the following fact, for which a discussion can be found in [9].

**Proposition 1.1** (Paley-Wiener theorem)**.** *Let $\delta > 0$, $s \in \mathbb{R}$. Then $f \in G^{\delta,s}$ if and only if it is the restriction to the real line of a function $F$ which is holomorphic in the strip*

$$\{x + \mathrm{i}y\colon\ x, y \in \mathbb{R},\ |y| < \delta\}$$

*and satisfies*

$$\sup_{|y|<\delta} \|F(x + \mathrm{i}y)\|_{H^s_x} < \infty.$$

In the view of the Paley-Wiener theorem, it is natural to take initial data in $G^{\delta,s}$ and obtain a better understanding of the behavior of solution as we try to extend it to be global in time. It means that given $u_0 \in G^{\delta,s}$ for some initial radius $\delta > 0$ we want to estimate the behavior of the radius of analyticity $\delta(T)$ as time $T$ goes to $\infty$. This is our second novelty and main goal in this paper.

**Theorem 1.2.** *Assume that $\lambda_1\lambda_2 < 0$ and $\lambda_3 > 0$, let $\delta > 0$ and $s > -\frac{5}{8}$. Then for any $u_0 \in G^{\delta,s}$ there exists $T = T(\|u_0\|_{G^{\delta,s}}) > 0$ and a unique solution $u$ of* (1.1) *on the time interval* $(0, T)$ *such that*

$$u \in C([0, T], G^{\delta,s}).$$

*Moreover, the solution depends continuously on the data* $u_0$. *Here we have*

$$T = \frac{c_0}{(1 + \|u_0\|_{G^{\delta,s}})^\beta} \tag{1.4}$$

*for some constants* $c_0 > 0$ *and* $\beta > 1$ *depending only on* $s$. *Furthermore, the solution* $u$ *satisfies the bound*

$$\|u\|_{X^T_{\delta,s,b}} \leqslant 2C\|u_0\|_{G^{\delta,s}}, \quad \frac{1}{2} < b < 1 \tag{1.5}$$

*with a constant* $C > 0$ *depending only on* $s$ *and* $b$.

Thus, this result shows that for local-in-time the radius of analyticity remains constant. Our next main result for the higher order nonlinear dispersive equation yields an estimate on how the width of the strip of the radius of the spatial analyticity decay with time.

**Theorem 1.3.** *Let* $s > -\frac{5}{8}$ *and* $\delta_0 > 0$, *and assume* $u_0 \in G^{\delta_0,s}$. *Then the solution given by Theorem* 1.2 *extends globally in time and for any* $T' > 0$ *we have*

$$u \in C([0,T'], G^{\delta(T'),s}) \quad \textit{with } \delta(T') = \min\{\delta_0, C_1 T'^{-(8/5+\sigma_0)}\},$$

*where* $\sigma_0 > 0$ *can be taken arbitrarily small and* $C_1 > 0$ *is a constant depending on* $u_0$, $\delta_0$, $s$ *and* $\sigma_0$.

The method used here for proving lower bounds on the radius of analyticity was introduced in [12] in the context of the 1D Dirac-Klein-Gordon equations. It was applied to the modified Kawahara equation (see [10]) and the non-periodic KdV equation in [11] improving an earlier result of Bona et al. (see [1]), to the dispersion-generalized periodic KdV equation in [7] and to the quartic generalized KdV equation on the line in [13].

The rest of the paper is organized as follows. In Section 2, we define the function spaces, linear estimates and bilinear estimates. In Section 3 we prove Theorem 1.2 using the bilinear estimate and the linear estimate together with contraction mapping principle. Section 4 proves the existence of a fundamental approximate conservation law. In the final section, Theorem 1.3 will be proven, using the approximate conservation law.

## 2. Preliminary estimates and function spaces

**Function spaces.** Now we introduce the Bourgain space $X_{s,b}(\mathbb{R}^2) = X_{s,b}$ associated to the higher order nonlinear dispersive equation with respect to the norm

$$\|u\|_{X_{s,b}} = \left( \int_{\mathbb{R}^2} \langle \tau - \varphi(\xi) \rangle^{2b} \langle \xi \rangle^{2s} |\widehat{u}(\xi,\tau)|^2 \,\mathrm{d}\xi \,\mathrm{d}\tau \right)^{1/2}, \tag{2.1}$$

where $\varphi(\xi) = \lambda_1 \xi^7 - \lambda_2 \xi^5 + \lambda_3 \xi^3 - \lambda_4 \xi$.

In addition, we also need the Grevey-Bourgain space, denoted $X_{\delta,s,b}(\mathbb{R}^2) = X_{\delta,s,b}$, defined by

$$\|u\|_{X_{\delta,s,b}} = \|Au\|_{X_{s,b}} = \left( \int_{\mathbb{R}^2} \mathrm{e}^{2\delta|\xi|} \langle \tau - \varphi(\xi) \rangle^{2b} \langle \xi \rangle^{2s} |\widehat{u}(\xi,\tau)|^2 \,\mathrm{d}\xi \,\mathrm{d}\tau \right)^{1/2}, \tag{2.2}$$

where

$$\widehat{Au}^{x}(\xi,t) = \mathrm{e}^{\delta|\xi|} \widehat{u}^{x}(\xi,t). \tag{2.3}$$

For $\delta = 0$, the space $X_{0,s,b}$ coincides with the Bourgain spaces $X_{s,b}$ .

Finally, we will need the restrictions of $X_{s,b}$ and $X_{\delta,s,b}$ to a time slab $\mathbb{R} \times (0,T)$, $T > 0$. These spaces are denoted by $X^T_{s,b}$ and $X^T_{\delta,s,b}$, respectively, and are Banach spaces when equipped with the norms

$$\begin{aligned} \|u\|_{X^T_{s,b}} &= \inf\{\|v\|_{X_{s,b}} \colon\ v = u \text{ on } \mathbb{R} \times (0,T)\}, \\ \|u\|_{X^T_{\delta,s,b}} &= \inf\{\|v\|_{X_{\delta,s,b}} \colon\ v = u \text{ on } \mathbb{R} \times (0,T)\}. \end{aligned}$$

**Linear and bilinear estimates.** To prove our main results we have a need of some multilinear estimate in the analytic Gevrey-Bourgain spaces. Note that the spaces $X_{\delta,s,b}$ are continuously embedded in $C(\mathbb{R}, G^{\delta,s}(\mathbb{R}))$ provided that $b > \frac{1}{2}$. We start with the following useful lemma.

**Lemma 2.1.** *Let $b > \frac{1}{2}$, $s \in \mathbb{R}$ and $\delta \geqslant 0$. Then $X_{\delta,s,b} \subset C(\mathbb{R}, G^{\delta,s}(\mathbb{R}))$ and*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{G^{\delta,s}} \leqslant C\|u(t)\|_{X_{\delta,s,b}}, \tag{2.4}$$

*where $C$ depends only on $b$.*

P r o o f. First, we observe that the operator $A$ satisfies

$$\|u\|_{X_{\delta,s,b}} = \|Au\|_{X_{s,b}} \quad \text{and} \quad \|u\|_{G^{\delta,s}} = \|Au\|_{H^s}, \tag{2.5}$$

where $X_{s,b}$ is introduced in [15]. We observe that $Au$ belongs to $C(\mathbb{R}, H^s)$ and for some $C > 0$ we have

$$\|Au\|_{C(\mathbb{R},H^s)} \leqslant C\|Au\|_{X_{s,b}}. \tag{2.6}$$

Thus, it follows that $u \in C([0,T], G^{\delta,s})$ and

$$\|u\|_{C(\mathbb{R},G^{\delta,s})} \leqslant C\|u\|_{X_{\delta,s,b}}. \tag{2.7}$$

□

**Lemma 2.2.** *Let $s \in \mathbb{R}$, $\delta \geqslant 0$ and $-\frac{1}{2} < b \leqslant b' < \frac{1}{2}$. Then for any $T > 0$ we have*

$$\|u\|_{X^T_{\delta,s,b}} \leqslant CT^{b'-b}\|u\|_{X^T_{\delta,s,b'}}, \tag{2.8}$$

*where $C$ depends only on $b$ and $b'$.*

**Lemma 2.3.** *Let $s \in \mathbb{R}$, $\delta \geqslant 0$, $-\frac{1}{2} < b < \frac{1}{2}$ and $T > 0$. Then for any time interval $I \subset [0,T]$ we have*

$$\|\chi_I(t)u\|_{X_{\delta,s,b}} \leqslant C\|u\|_{X^T_{\delta,s,b}}, \tag{2.9}$$

*where $\chi_I(t)$ is the characteristic function of $I$ and $C$ depends only on $b$.*

P r o o f. The proofs of Lemma 2.2 and Lemma 2.3 for $\delta = 0$ can be found in Section 2.6 of [14] and in Lemma 3.1 of [5], respectively. These inequalities clearly remain valid for $\delta > 0$, as one merely has to replace $u$ by $Au$. □

Next, consider the linear Cauchy problem for given $F = \partial_x u^2$ and $u_0$,

$$\begin{cases} \partial_t u + \lambda_1 \partial_x^7 u + \lambda_2 \partial_x^5 u + \lambda_3 \partial_x^3 u + \lambda_4 \partial_x u = F, \\ u(x,0) = u_0(x). \end{cases} \tag{2.10}$$

By Duhamel's principle the solution can be then written as

$$u(t) = S(t)u_0 - \frac{1}{2}\int_0^t S(t-t')(F(t'))\,\mathrm{d}t', \tag{2.11}$$

where

$$S(t)u_0 = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}(x\xi + t\varphi(\xi))}\widehat{u_0}(\xi)\,\mathrm{d}\xi.$$

**Lemma 2.4.** *Let $s \in \mathbb{R}$, $\frac{1}{2} < b \leqslant 1$, $\delta \geqslant 0$ and $0 < T \leqslant 1$. There is a constant $C > 0$ depending only on $b$ such that*

$$\|S(t)u_0\|_{X^T_{\delta,s,b}} \leqslant C\|u_0\|_{G^{\delta,s}}, \tag{2.12}$$

$$\Big\|\int_0^t S(t-t')F(t')\,\mathrm{d}t'\Big\|_{X^T_{\delta,s,b}} \leqslant C\|F\|_{X^T_{\delta,s,b-1}}. \tag{2.13}$$

Proof. The proofs of (2.12) and (2.13) for $\delta = 0$ can be found in Lemmas 5.1, 5.2 of [15], respectively. These inequalities clearly remain valid for $\delta > 0$, as one merely has to replace $u_0$ by $Au_0$, $F$ by $AF$. □

The author in [15] assumed $\lambda_1, \lambda_2 < 0$, $\lambda_3 > 0$ to prove the bilinear estimate in Sobolev spaces $H^s$. In this paper we used the bilinear estimate in Sobolev spaces $H^s$ and the operator $A$ in order to prove bilinear estimate in analytic Gevrey spaces $G^{\delta,s}$.

**Lemma 2.5.** *Assume that $\lambda_1\lambda_2 < 0$ and $\lambda_3 > 0$. Let $b'$ be close enough to $\frac{1}{2}$ satisfying $b' > \frac{1}{2}$. For $b > \frac{1}{2}$, $\delta \geqslant 0$ and $s > -\frac{5}{8}$ we have*

$$\|\partial_x(u_1u_2)\|_{X_{\delta,s,b'-1}} \leqslant C\|u_1\|_{X_{\delta,s,b}}\|u_2\|_{X_{\delta,s,b}}. \tag{2.14}$$

Proof. We observe, by considering the operator $A$ in (2.3), that

$$\begin{aligned}
\mathrm{e}^{\delta|\xi|}\widehat{u_1u_2} &= (2\pi)^{-2}\mathrm{e}^{\delta|\xi|}\widehat{u_1} * \widehat{u_2} \\
&\leqslant (2\pi)^{-2}\int_{\mathbb{R}^2} \mathrm{e}^{\delta|\xi-\eta_1|}\widehat{u_1}(\xi-\eta_1,\tau-\varrho_1)\mathrm{e}^{\delta|\eta_1|}\widehat{u_2}(\eta_1,\varrho_1)\,\mathrm{d}\eta_1\,\mathrm{d}\varrho_1 \\
&= (\widehat{Au_1Au_2}),
\end{aligned}$$

since $\delta|\xi| \leqslant \delta|\xi-\eta_1| + \delta|\eta_2|$. Thus, we have

$$\|\partial_x(u_1u_2)\|_{X_{\delta,s,b'-1}} \leqslant \|\partial_x(Au_1Au_2)\|_{X_{s,b'-1}}.$$

Thanks to Lemma 4.1 in [15], we have

$$\|\partial_x(Au_1Au_2)\|_{X_{s,b'-1}} \leqslant C\|Au_1\|_{X_{s,b}}\|Au_2\|_{X_{s,b}} = C\|u_1\|_{X_{\delta,s,b}}\|u_2\|_{X_{\delta,s,b}}.$$

□

## 3. Proof of Theorem 1.2

**Existence of solution.** Fix $\delta > 0$, $s > \frac{5}{8}$, and $u_0 \in G^{\delta,s}$. To construct the local solution u to (1.1), we proceed by an iteration argument in the space $X^T_{\delta,s,b}$. Let $\{u^{(n)}\}_{n=0}^{\infty}$ be the sequence defined by

$$\begin{cases} \partial_t u^{(0)} + \lambda_1 \partial_x^7 u^{(0)} + \lambda_2 \partial_x^5 u^{(0)} + \lambda_3 \partial_x^3 u^{(0)} + \lambda_4 \partial_x u^{(0)} = 0, \\ u^{(0)}(0) = u_0, \end{cases}$$

and for $n \in \{1, 2, \ldots\}$ by

$$\begin{cases} \partial_t u^{(n)} + \lambda_1 \partial_x^7 u^{(n)} + \lambda_2 \partial_x^5 u^{(n)} + \lambda_3 \partial_x^3 u^{(n)} + \lambda_4 \partial_x u^{(n)} = -\frac{1}{2}\partial_x (u^{(n-1)} u^{(n-1)}), \\ u^{(n)}(0) = u_0. \end{cases}$$

Based on the comments preceding Lemma 2.4, we may write

$$u^{(0)}(x,t) = S(t)u_0(x),$$
$$u^{(n)}(x,t) = S(t)u_0(x) - \frac{1}{2}\int_0^t S(t-t')\partial_x (u^{(n-1)}(x,t')u^{(n-1)}(x,t'))\,\mathrm{d}t'.$$

It then follows from Lemmas 2.2, 2.5 and 2.4 that

$$\begin{aligned} \|u^{(0)}\|_{X^T_{\delta,s,b}} &\leqslant C\|u_0\|_{G^{\delta,s}}, \\ \|u^{(n)}\|_{X^T_{\delta,s,b}} &\leqslant C\|u_0\|_{G^{\delta,s}} + CT^{b'-b}\|\partial_x (u^{(n-1)}u^{(n-1)})\|_{X^T_{\delta,s,b'-1}} \\ &\leqslant C\|u_0\|_{G^{\delta,s}} + CT^{b'-b}\|u^{(n-1)}\|^2_{X^T_{\delta,s,b}} \end{aligned} \tag{3.1}$$

with $\frac{1}{2} < b < b' < 1$. By induction, it follows that

$$\|u^{(n)}\|_{X^T_{\delta,s,b}} \leqslant 2C\|u_0\|_{G^{\delta,s}} \tag{3.2}$$

for all $n$ if $T \in (0,1]$ is chosen so small that

$$T \leqslant \frac{1}{(8C^2\|u_0\|_{G^{\delta,s}})^{1/(b'-b)}}. \tag{3.3}$$

Using Lemma 2.5 together with (3.2) and (3.1) in that order, we therefore get

$$\begin{aligned} \|u^{(n)} - u^{(n-1)}\|_{X^T_{\delta,s,b}} &\leqslant CT^{b'-b}\|\partial_x (u^{(n-1)}u^{(n-1)} - u^{(n-2)}u^{(n-2)})\|_{X^T_{\delta,s,b'-1}} \\ &\leqslant CT^{b'-b}(\|u^{(n-1)}\|_{X^T_{\delta,s,b}} + \|u^{(n-2)}\|_{X^T_{\delta,s,b}}) \\ &\quad \times \|u^{(n-1)} - u^{(n-2)}\|_{X^T_{\delta,s,b}} \\ &\leqslant \frac{1}{2}\|u^{(n-1)} - u^{(n-2)}\|_{X^T_{\delta,s,b}}. \end{aligned}$$

It follows that the sequence converges to a solution u verifying the bound (3.2).

**Continuous dependence on the initial data.** Now assume that $u$ and $v$ are solutions to the Cauchy problem (1.1) for initial data $u_0$ and $v_0$, respectively. Then similarly as above, again with the same choice of $T$ and for any $T'$ such that $0 < T' < T$, we have

$$\|u - v\|_{X^{T'}_{\delta,s,b}} \leqslant C\|u_0 - v_0\|_{G^{\delta,s}} + \frac{1}{2}\|u - v\|_{X^{T'}_{\delta,s,b}}$$

provided thus $\|u_0 - v_0\|_{G^{\delta,s}}$ is sufficiently small. This proves continuous dependence.

**The uniqueness.** Uniqueness of the solution in $C([0,T], G^{\delta,s})$ can be proved by the following standard argument. Suppose that $u, v \in C([0,T], G^{\delta,s})$ are solutions to (1.1) with $u(\cdot,0) = v(\cdot,0)$ in $G^{\delta,s}$. Setting $w = u - v$, we see that $w$ solves the Cauchy problem

$$\partial_t w + \lambda_1 \partial_x^7 w + \lambda_2 \partial_x^5 w + \lambda_3 \partial_x^3 + \lambda_4 \partial_x w + \frac{1}{2}\partial_x w(u+v) = 0, \qquad w(0) = 0.$$

Multiplying both sides by $w$ and integrating in space yield

$$w\partial_t w + \lambda_1 w\partial_x^7 w + \lambda_2 w\partial_x^5 w + \lambda_3 w\partial_x^3 + \lambda_4 w\partial_x w + \frac{1}{2} w\partial_x w(u+v) = 0.$$

Thus, we have

$$\begin{aligned}(3.4) \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t,\cdot)\|^2_{L^2} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} w^2(t,x)\,\mathrm{d}x = \int_{\mathbb{R}} w(t,x)\partial_t w(t,x)\,\mathrm{d}x \\ &= -\frac{1}{2}\int_{\mathbb{R}} w(t,x)\partial_x(u^2 - v^2)\,\mathrm{d}x = 0\end{aligned}$$

since we have

$$\begin{aligned}\int_{\mathbb{R}} w(t,x)\partial_x^7 w(t,x)\,\mathrm{d}x &= \int_{\mathbb{R}} w(t,x)\partial_x^5 w(t,x)\,\mathrm{d}x = \int_{\mathbb{R}} w(t,x)\partial_x^3 w(t,x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} w(t,x)\partial_x w(t,x)\,\mathrm{d}x = 0.\end{aligned}$$

We may here assume that $w$ and its all spatial derivatives decay to zero as $|x| \to \infty$ (see the argument in [11], page 10). Thanks to equation (3.4) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t,\cdot)\|^2_{L^2} = -\int_{\mathbb{R}} w(t,x)\partial_x(u^2 - v^2)\,\mathrm{d}x = -\int_{\mathbb{R}} w(t,x)\partial_x(f(t,x)w(t,x))\,\mathrm{d}x,$$

where $f = u + v$. Integrating the last integral by parts we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t,\cdot)\|^2_{L^2} = \int_{\mathbb{R}} \partial_x f(t,x) w^2(t,x)\,\mathrm{d}x,$$

from which we deduce the inequality

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\|w(t,\cdot)\|_{L^2}^2\right| = \|\partial_x f\|_{L^\infty}\|w(t)\|_{L^2}^2. \tag{3.5}$$

Since $u, v \in C([0,T], G^{\delta,s})$, we have that $u$ and $v$ are continuous in $t$ on the compact set $[0,T]$ and are $G^{\delta,s}$ in $x$. Thus, we can conclude that

$$\|\partial_x f\|_{L^\infty} \leqslant c < \infty. \tag{3.6}$$

Therefore, from (3.5) and (3.6) we obtain the differential inequality

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\|w(t,\cdot)\|_{L^2}^2\right| \leqslant c\|w(t)\|_{L^2}^2, \quad 0 \leqslant t \leqslant T.$$

Solving it gives

$$\|w(t)\|_{L^2}^2 \leqslant \mathrm{e}^c\|w(0)\|_{L^2}^2, \quad 0 \leqslant t \leqslant T. \tag{3.7}$$

Since $\|w(0)\|_{L^2}^2 = 0$, from (3.7) we obtain that $w(t) = 0$, $0 \leqslant t \leqslant T$ or $u = v$.

## 4. Approximate conservation law

Our goal in this section is to establish an approximate conservation law for a solution to (1.1) based on the conservation of the $L^2(\mathbb{R})$ norm of solutions of the equation. Explicitly, we aim at proving Theorem 4.1.

**Theorem 4.1.** *Let $\kappa \in [0, \frac{5}{8})$ and $T$ be as in Theorem 1.2. There exist $b \in (\frac{1}{2}, 1)$ and $C > 0$ such that for any $\delta > 0$ and any solution $u \in X^T_{\delta,0,b}$ to the Cauchy problem (1.1) on the time interval $[0,T]$, we have the estimate*

$$\sup_{t\in[0,T]} \|u(t)\|_{G^{\delta,0}}^2 \leqslant \|u(0)\|_{G^{\delta,0}}^2 + C\delta^\kappa\|u\|_{X^T_{\delta,0,b}}^3. \tag{4.1}$$

*Moreover, we have*

$$\sup_{t\in[0,T]} \|u(t)\|_{G^{\delta,0}}^2 \leqslant \|u(0)\|_{G^{\delta,0}}^2 + C\delta^\kappa\|u(0)\|_{G^{\delta,0}}^3. \tag{4.2}$$

For the proof of Theorem 4.1 we require the following preliminary estimate.

**Lemma 4.2.** *Given $\kappa \in [0, \frac{5}{8})$, there exist $b \in (\frac{1}{2}, 1)$ and $C > 0$ such that for all $T > 0$ and $u \in X_{\delta,0,b}$ we have*

$$\|G\|_{X_{0,b-1}} \leqslant C\delta^\kappa\|u\|_{X_{\delta,0,b}}^2, \tag{4.3}$$

*where $G = \frac{1}{2}\partial_x((Au)^2 - A(u)^2)$ and the operator $A$ is given by (2.3).*

P r o o f. Let $G=\frac{1}{2}\partial_x((Au)^2-A(u)^2)$.Then

$$\begin{aligned}\|G\|_{X_{0,b-1}} &= \frac{1}{2}\bigg\|\frac{\xi}{\langle\tau-\varphi(\xi)\rangle^{1-b}}\int_{\mathbb{R}^2}(\mathrm{e}^{\delta|\xi_1|}\widehat{u}(\xi_1,\tau_1)\mathrm{e}^{\delta|\xi-\xi_1|}\widehat{u}(\xi-\xi_1,\tau-\tau_1)\\ &\qquad -\mathrm{e}^{\delta|\xi|}\widehat{u}(\xi_1,\tau_1)\widehat{u}(\xi-\xi_1,\tau-\tau_1))\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\bigg\|_{L^2_{\xi,\tau}}\\ &= \frac{1}{2}\bigg\|\frac{\xi}{\langle\tau-\varphi(\xi)\rangle^{1-b}}\int_{\mathbb{R}^2}(\mathrm{e}^{\delta|\xi_1|}\mathrm{e}^{\delta|\xi-\xi_1|}-\mathrm{e}^{\delta|\xi|})\widehat{u}(\xi_1,\tau_1)\\ &\qquad\times\widehat{u}(\xi-\xi_1,\tau-\tau_1)\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\bigg\|_{L^2_{\xi,\tau}}.\end{aligned}$$

Using this and the following estimate (see [11])

$$\mathrm{e}^{\delta|\alpha|}\mathrm{e}^{\delta|\beta|}-\mathrm{e}^{\delta|\alpha+\beta|}\leqslant(2\delta\min(|\alpha|,|\beta|))^{\theta}\mathrm{e}^{\delta|\alpha|}\mathrm{e}^{\delta|\beta|},\quad \theta\in[0,1],$$

and

$$\min(|\xi_1|,|\xi-\xi_1|)\leqslant 2\frac{\langle\xi_1\rangle\langle\xi-\xi_1\rangle}{\langle\xi\rangle}.$$

For $\kappa\in[0,\frac{5}{8})\subset[0,1]$ one can see that

$$\begin{aligned}\|G\|_{X_{0,b-1}} &\leqslant \frac{1}{2}\bigg\|\frac{\xi}{\langle\tau-\varphi(\xi)\rangle^{1-b}}\int_{\mathbb{R}^2}(2\delta\min(|\xi_1|,|\xi-\xi_1|))^{\kappa}\\ &\qquad\times\mathrm{e}^{\delta|\xi_1|}\mathrm{e}^{\delta|\xi-\xi_1|}\widehat{u}(\xi_1,\tau_1)\widehat{u}(\xi-\xi_1,\tau-\tau_1)\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\bigg\|_{L^2_{\xi,\tau}}\\ &\leqslant \frac{1}{2}(2\delta)^{\kappa}\bigg\|\frac{\xi\langle\xi\rangle^{-\kappa}}{\langle\tau-\varphi(\xi)\rangle^{1-b}}\int_{\mathbb{R}^2}\mathrm{e}^{\delta|\xi_1|}\langle\xi_1\rangle^{\kappa}\widehat{u}(\xi_1,\tau_1)\\ &\qquad\times\mathrm{e}^{\delta|\xi-\xi_1|}\langle\xi-\xi_1\rangle^{\kappa}\widehat{u}(\xi-\xi_1,\tau-\tau_1)\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\bigg\|_{L^2_{\xi,\tau}}.\end{aligned}$$

Now by taking $s=-\kappa\in(-\frac{5}{8},0]$ we obtain

$$\begin{aligned}\|G\|_{X_{0,b-1}} &\leqslant \frac{1}{2}(2\delta)^{\kappa}\bigg\|\frac{\xi\langle\xi\rangle^{s}}{\langle\tau-\varphi(\xi)\rangle^{1-b}}\int_{\mathbb{R}^2}\frac{\mathrm{e}^{\delta|\xi_1|}\widehat{u}(\xi_1,\tau_1)}{\langle\xi_1\rangle^{s}}\\ &\qquad\times\frac{\mathrm{e}^{\delta|\xi-\xi_1|}\widehat{u}(\xi-\xi_1,\tau-\tau_1)}{\langle\xi-\xi_1\rangle^{s}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\bigg\|_{L^2_{\xi,\tau}}.\end{aligned}$$

Setting $v=Au$ and $f(\tau,\xi)=\langle\tau-\varphi(\xi)\rangle^{b}\widehat{v}(\tau,\xi)$ we have $\mathrm{e}^{\delta|\xi|}\widehat{u}(\tau,\xi)=\widehat{v}(\tau,\xi)=f(\tau,\xi)\langle\tau-\varphi(\xi)\rangle^{-b}$ and therefore we can write

$$\begin{aligned}\|G\|_{X_{0,b-1}} &\leqslant \frac{1}{2}(2\delta)^{\kappa}\bigg\|\frac{\xi\langle\xi\rangle^{s}}{\langle\tau-\varphi(\xi)\rangle^{1-b}}\int_{\mathbb{R}^2}\frac{f(\xi_1,\tau_1)}{\langle\xi_1\rangle^{s}\langle\tau_1-\varphi(\xi_1)\rangle^{b}}\\ &\qquad\times\frac{f(\xi-\xi_1,\tau-\tau_1)}{\langle\xi-\xi_1\rangle^{s}\langle\tau-\tau_1-\varphi(\xi-\xi_1)\rangle^{b}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\bigg\|_{L^2_{\xi,\tau}}.\end{aligned}$$

By Remark 9 in [11] we get

$$\|G\|_{X_{0,b-1}} \leqslant C\delta^{\kappa}\|f\|_{L^2_{\xi,\tau}}^2 = C\delta^{\kappa}\|v\|_{X_{0,b}}^2 = C\delta^{\kappa}\|u\|_{X_{\delta,0,b}}^2,$$

and the result is proven. □

Now we prove Theorem 4.1.

P r o o f of Theorem 4.1. Let $V(t,x) = Au(t,x)$, which is real-valued since the multiplier $A$ is even and u is real-valued. Applying $A$ to (1.1) we obtain

$$\partial_t V + \lambda_1\partial_x^7 V + \lambda_2\partial_x^5 V + \lambda_3\partial_x^3 V + \lambda_4\partial_x V + V\partial_x V = G, \tag{4.4}$$

where

$$G = \frac{1}{2}\partial_x((Au)^2 - A(u)^2).$$

Multiplying (4.4) by $V$ and integrating in space we obtain

$$\int_{\mathbb{R}} V\partial_t V\,\mathrm{d}x + \lambda_1\int_{\mathbb{R}} V\partial_x^7 V\,\mathrm{d}x + \lambda_2\int_{\mathbb{R}} V\partial_x^5 V\,\mathrm{d}x + \lambda_3\int_{\mathbb{R}} V\partial_x^3 V\,\mathrm{d}x + \lambda_4\int_{\mathbb{R}} V\partial_x V\,\mathrm{d}x + \int_{\mathbb{R}} V^2\partial_x V\,\mathrm{d}x = \int_{\mathbb{R}} VG\,\mathrm{d}x.$$

By noticing that $\partial_x^j V(x,t) \to 0$ as $|x| \to \infty$ (see [11]) we can use integration by parts obtaining

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} V^2\,\mathrm{d}x = \int_{\mathbb{R}} VG\,\mathrm{d}x.$$

Now integrating the last equality with respect to $t \in [0,T]$ we obtain

$$\int_{\mathbb{R}} V^2(T,x)\,\mathrm{d}x = \int_{\mathbb{R}} V^2(0,x)\,\mathrm{d}x + 2\left|\int_{\mathbb{R}^2} \chi_{[0,T]}(t)VG\,\mathrm{d}x\,\mathrm{d}t\right|.$$

Thus,

$$\|u(T)\|_{G^{\delta,0}}^2 = \|u(0)\|_{G^{\delta,0}}^2 + 2\left|\int_{\mathbb{R}^2} \chi_{[0,T]}(t)VG\,\mathrm{d}x\,\mathrm{d}t\right|.$$

We now use Plancherel, Hölder, Lemmas 2.3, 4.2 and the fact that $1-b<b$ since $b>\frac{1}{2}$ and we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \chi_{[0,T]}(t)VG\,\mathrm{d}x\,\mathrm{d}t\right| &\leqslant \|\chi_{[0,T]}(t)V\|_{X_{0,1-b}}\|\chi_{[0,T]}(t)G\|_{X_{0,b-1}} \\ &\leqslant \|V\|_{X^T_{0,1-b}}\|G\|_{X^T_{0,b-1}} \leqslant C\delta^{\kappa}\|u\|^3_{X^T_{\delta,0,b}}.\end{aligned} \tag{4.5}$$

Finally, by using condition (3.2) we conclude that

$$\sup_{t\in[0,T]}\|u(t)\|_{G^{\delta,0}}^2 \leqslant \|u(0)\|_{G^{\delta,0}}^2 + C\delta^{\kappa}\|u(0)\|_{G^{\delta,0}}^3.$$

The proof is now complete. □

## 5. Proof of Theorem 1.3

Fix $\delta_0 > 0$, $s > -\frac{5}{8}$, $\kappa \in (0, \frac{5}{8})$, and $u_0 \in G^{\delta_0, s}$. It suffices to prove that the solution $u$ to (1.1) satisfies

$$u \in C([0, T'], G^{\delta(T'), s}),$$

where

$$\delta(T') = \min\{\delta_0, C_1 T'^{-1/\kappa}\}$$

for all $T' > 0$, and $C_1 > 0$ is a constant depending on $u_0$, $\delta_0$, $s$, and $\kappa$. By Theorem 1.2, there is a maximal time $T^* = T^*(u_0, \delta_0, s) \in (0, \infty]$ such that

$$u \in C([0, T^*], G^{\delta_0, s}).$$

If $T^* = \infty$, we are done. If $T^* < \infty$, as we assume henceforth, it remains to prove

$$u \in C([0, T'], G^{C_1 T'^{-1/\kappa}, s}) \quad \text{for all } T' \geqslant T^*. \tag{5.1}$$

**The case $s = 0$.** Fix $T' \geqslant T^*$ and we will show that for $\delta > 0$ sufficiently small

$$\sup_{t \in [0, T']} \|u(t)\|^2_{G^{\delta,0}} \leqslant 2\|u(0)\|^2_{G^{\delta_0,0}}. \tag{5.2}$$

To prove this, we will use repeatedly Theorems 1.2 and 4.1 with the time step

$$T = \frac{c_0}{(1 + 2\|u(0)\|_{G^{\delta_0,0}})^{\beta}}. \tag{5.3}$$

The smallness conditions on $\delta$ will be

$$\delta \leqslant \delta_0 \quad \text{and} \quad \frac{2T'}{T} C \delta^{\kappa} 2^{3/2} \|u(0)\|_{G^{\delta_0,0}} \leqslant 1, \tag{5.4}$$

where $C > 0$ is the constant in Theorems 4.1. Proceeding by induction, we will verify that

$$\sup_{t \in [0, nT]} \|u(t)\|^2_{G^{\delta,0}} \leqslant \|u(0)\|^2_{G^{\delta,0}} + nC\delta^{\kappa} 2^{3/2} \|u(0)\|^3_{G^{\delta_0,0}}, \tag{5.5}$$

$$\sup_{t \in [0, nT]} \|u(t)\|^2_{G^{\delta,0}} \leqslant 2\|u(0)\|^2_{G^{\delta_0,0}} \tag{5.6}$$

for $n \in \{1, \dots, m+1\}$, where $m \in \mathbb{N}$ is chosen so that $T' \in [mT, (m+1)T)$. This $m$ does exist, since by Theorem 1.2 and the definition of $T^*$, we have

$$T < \frac{c_0}{(1 + \|u(0)\|_{G^{\delta_0,0}})^{\beta}} < T^*, \quad \text{hence } T < T'.$$

In the first step, we cover the interval $[0,T]$, and by Theorem 4.1, we have

$$\sup_{t\in[0,T]} \|u(t)\|^2_{G^{\delta,0}} \leqslant \|u(0)\|^2_{G^{\delta,0}} + C\delta^\kappa \|u(0)\|^3_{G^{\delta,0}} \leqslant \|u(0)\|^2_{G^{\delta,0}} + C\delta^\kappa \|u(0)\|^3_{G^{\delta_0,0}},$$

where we used that $\|u(0)\|_{G^{\delta,0}} \leqslant \|u(0)\|_{G^{\delta_0,0}}$, since $\delta \leqslant \delta_0$. This verifies (5.5) for $n = 1$ and now, (5.6) follows using again $\|u(0)\|_{G^{\delta,0}} \leqslant \|u(0)\|_{G^{\delta_0,0}}$ as well as $C\delta^\kappa\|u(0)\|_{G^{\delta_0,0}} \leqslant 1$. Next, assuming that (5.5) and (5.6) hold for some $n \in \{1,\dots,m\}$, we will prove that they hold for $n+1$. We estimate

$$\begin{aligned}\sup_{t\in[nT,(n+1)T]} \|u(t)\|^2_{G^{\delta,0}} &\leqslant \|u(nT)\|^2_{G^{\delta,0}} + C\delta^\kappa \|u(nT)\|^3_{G^{\delta,0}} \\ &\leqslant \|u(nT)\|^2_{G^{\delta,0}} + C\delta^\kappa 2^{3/2} \|u(0)\|^3_{G^{\delta_0,0}} \\ &\leqslant \|u(0)\|^2_{G^{\delta,0}} + nC\delta^\kappa 2^{3/2} \|u(0)\|^3_{G^{\delta_0,0}} + C\delta^\kappa 2^{3/2} \|u(0)\|^3_{G^{\delta_0,0}},\end{aligned}$$

verifying (5.5) with $n$ replaced by $n+1$. To get (5.6) with $n$ replaced by $n+1$, it is then enough to have

$$(n+1)C\delta^\kappa 2^{3/2}\|u(0)\|_{G^{\delta_0,0}} \leqslant 1,$$

but this holds by (5.4), since $n+1 \leqslant m+1 \leqslant T'/T + 1 < 2T'/T$. Finally, condition (5.4) is satisfied for $\delta \in (0,\delta_0)$ such that

$$\frac{2T'}{T} C\delta^\kappa 2^{3/2}\|u(0)\|_{G^{\delta_0,0}} = 1.$$

Thus, $\delta = C_1 T'^{-1/\kappa}$, where $C_1 = (c_0/C2^{5/2}\|u(0)\|_{G^{\delta_0,0}}(1+2\|u(0)\|_{G^{\delta_0,0}})^\beta)^{1/\kappa}$.

**The general case.** For general $s$, we use the embedding (1.3) to get $u_0 \in G^{\delta_0,s} \subset G^{\delta_0/2,0}$. The case $s=0$ already being proved, we know that there is a $T_1 > 0$ such that

$$u \in C([0,T_1), G^{\delta_0/2,0})$$

and

$$u \in C([0,T'], G^{2\sigma T'^{-1/\kappa},0}) \quad \text{for } T' \geqslant T_1,$$

where $\sigma > 0$ depends on $u_0$, $\delta_0$ and $\kappa$. Applying again embedding (1.3), we now conclude that

$$u \in C([0,T_1), G^{\delta_0/4,s})$$

and

$$u \in C([0,T'], G^{\sigma T'^{-1/\kappa},s}) \quad \text{for } T' \geqslant T_1,$$

and these together imply (5.1). The proof of Theorem 1.3 is now completed. □

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