ON THE NEHARI MANIFOLD FOR A LOGARITHMIC FRACTIONAL SCHRÖDINGER EQUATION WITH POSSIBLY VANISHING POTENTIALS

Cong Nhan Le, Xuan Truong Le, Ho Chi Minh City

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Abstract. We study a class of logarithmic fractional Schrödinger equations with possibly vanishing potentials. By using the fibrering maps and the Nehari manifold we obtain the existence of at least one nontrivial solution.

Keywords: Nehari manifold; fibrering maps; vanishing potential; logarithmic nonlinearity

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1. Introduction

In this paper, we are concerned with the following Schrödinger equations involving fractional p-Laplacian operators in the whole space:

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f_\lambda(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $s \in (0,1)$, $p > 1$, $N > sp$ and $(-\Delta)_p^s$ is the fractional p-Laplacian defined by

$$(-\Delta)_p^s u(x) := 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} \,\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

which is a generalization of the linear fractional Laplacian $(-\Delta)^s$ when $p = 2$, and $V\colon \mathbb{R}^N \to \mathbb{R}$ is a possibly vanishing potential in the sense that $V(x) \to 0$ as $|x| \to \infty$.

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Some motivations that have led to the study of this kind of operators can be found in Caffarelli (see [10]). The nonlinearity term in (1.1) $f_\lambda(x,u)$ is of logarithmic perturbation of p-linear growth

$$f_\lambda(x,u) = \lambda K(x)|u|^{p-2}u + \mu Q(x)|u|^{q-2}u\log|u|, \quad \lambda > 0, \tag{1.2}$$

where $\lambda, \mu > 0$ and q satisfies

$$p < q < p_s^* := \frac{Np}{N-ps} \quad \text{as } ps < N.$$

In the particular case of $p = 2$, equation (1.1) reduces to the so-called fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f_\lambda(x,u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

which arises in the study of the nonlinear fractional Schrödinger equation

$$\mathrm{i}\frac{\partial\Phi}{\partial t} = (-\Delta)^s\Phi + V(x)\Phi - f_\lambda(x,\Phi), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{1.4}$$

The applications of these equations can be found in [7] for the classical case $s = 1$ and in the papers of Laskin (see [26], [27]) for the general case.

Literature on logarithmic Schrödinger equations seems not to be very extensive, we refer the interested readers to [12], [13], [18], [24], [32] and their applications can be found in [8], [23], [36]. In [32], Squassina and Szulkin studied the logarithmic Schrödinger equations with periodic potential

$$-\Delta u + V(x)u = Q(x)u\log u^2, \quad x \in \mathbb{R}^N, \tag{1.5}$$

where the potential $Q \in C^1(\mathbb{R}^N)$ such that $\inf_{\mathbb{R}^N} Q(x) > 0$ and $\inf_{\mathbb{R}^N}(V(x) + Q(x)) > 0$. The case of $Q(x) = 1$ was also studied by Ji and Szulkin (see [24]). In the two papers, Szulkin et al. studied the multiplicity results for solutions to problem (1.5) through the critical points of the functional $J\colon H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + (V(x) + Q(x))u^2)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} u^2\log u^2\,\mathrm{d}x.$$

This functional is not smooth in $H^1(\mathbb{R}^N)$, however, it can be decomposed into the sum of a C^1 function and a convex lower semicontinuous functional. By using the critical point theory developed by Szulkin (see [33]), they obtained the existence of infinitely many solutions. In the case of constant potentials V and Q, in [18] the authors considered the functional J on radial spaces $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and applied the nonsmooth critical point theory of [11], [17], [19] to obtain a similar result on the existence of infinitely many weak solutions to problem (1.5). Another approach to treat this kind of problems comes from the paper [12], in which Cazenave found a suitable Banach space X endowed with a Luxemburg-type norm where the functional $J\colon X \to \mathbb{R}$ is well defined and smooth. Following the ideas of Cazenave, in [5] the author studied the case of fractional Schrödinger equations with logarithmic nonlinearity.

Motivated by these results, in this paper we study the logarithmic fractional Schrödinger equation (1.1) which extends the logarithmic nonlinearities in [12], [18], [24], [32]. By the standard variational method, we search for the critical points of the functional

$$
\begin{aligned}
J(u) = {} & \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u(x)|^p\,\mathrm{d}x \\
& - \frac{\lambda}{p}\int_{\mathbb{R}^N} K(x)|u(x)|^p\,\mathrm{d}x + \frac{\mu}{q^2}\int_{\mathbb{R}^N} Q(x)|u(x)|^q\,\mathrm{d}x \\
& - \frac{\mu}{q}\int_{\mathbb{R}^N} Q(x)|u(x)|^q \log|u(x)|\,\mathrm{d}x.
\end{aligned} \tag{1.6}
$$

As many problems proposed in the whole space, due to the lack of compactness in the embedding $W^{p,s}(\mathbb{R}^N)$ into Lebesgue spaces or potential Lebesgue spaces, the functional J is not smooth in $W^{p,s}(\mathbb{R}^N)$ neither it satisfies the Palais-Smale conditions. Using similar ideas as in [12], we find a suitable Banach space in which we not only have the compactness but also the C^1 smoothness of the functional J, and it is convenient to consider the potential spaces

$$
X = W^{s,p}_V(\mathbb{R}^N) := \Big\{u \in W^{s,p}(\mathbb{R}^N)\colon \int_{\mathbb{R}^N} V(x)|u(x)|^p\,\mathrm{d}x < \infty\Big\}. \tag{1.7}
$$

There are various types of conditions on V to gain compactness, for example, we refer the interested readers to [1], [2], [4], [6], [7], [22] for classical Schrödinger equations, [14], [16], [25], [29], [30], [31], [34] for fractional Schrödinger equations, and [28], [35] for fractional p-Laplacian equations.

In our setting, we assume that the potentials V, K and Q are positive continuous functions on $\mathbb{R}^N$ and satisfy the following assumptions:

(A1) K is bounded on $\mathbb{R}^N$ and

$$
\lim_{|x|\to\infty} \frac{K(x)}{V(x)} = 0,
$$

(A2) Q is bounded on $\mathbb{R}^N$ and for $q \in (p, p_s^*)$, we assume that

$$\lim_{|x|\to\infty} \frac{Q(x)}{V^{\sigma(q-\beta)}(x)} = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \frac{Q(x)}{V^{\sigma(q+\beta)}(x)} = 0,$$

where $0 < \beta < \min\{\frac{1}{2}(q-p), \frac{1}{2}(p_s^* - q)\}$ and

$$\sigma(r) := \frac{p_s^* - r}{p_s^* - p} = \frac{Np - r(N - sp)}{sp^2} > 0.$$

In this paper, by a solution to (1.1), we mean a function $u \in X$ such that $Q(x)|u|^q \log|u| \in L^1(\mathbb{R}^N)$ and $\psi_r(z) = |z|^{r-2}z$,

$$\begin{aligned} \iint_{\mathbb{R}^{2N}} \frac{\psi_p(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \, \mathrm{d}x \, \mathrm{d}y + \int_{\mathbb{R}^N} V(x)\psi_p(u)v \, \mathrm{d}x \\ = \lambda \int_{\mathbb{R}^N} K(x)\psi_p(u)v \, \mathrm{d}x + \mu \int_{\mathbb{R}^N} Q(x)\psi_q(u) \log|u| v \, \mathrm{d}x \end{aligned} \tag{1.8}$$

for all $v \in X$.

We are now in the position to state our main result.

Theorem 1.1. *Assume that* (A_1) *and* (A_2) *hold. Then, for each* $\mu > 0$*, problem* (1.1) *has at least one nontrivial positive solution provided that* $\lambda < \lambda_1$*.*

Here we denote by λ_1 the first eigenvalue of the eigenvalue problem

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = \lambda K(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

which, due to Proposition 2.2, is characterized by the variational formula (see [28] for more details)

$$\lambda_1 = \inf_{u \in X\setminus\{0\}} \frac{\|u\|^p}{\|u\|_{p,K}^p} > 0,$$

where $\|\cdot\|$ and $\|\cdot\|_{p,K}$ are defined as in (2.2) and (2.3).

It is worth noting that under the assumptions (A_1) and (A_2), the functional J is C^1 smooth but not coercive on X. Hence it is natural to search for its critical points on the Nehari manifold where we apply the fibrering method due to Drábek and Pohozaev (see [21]). For this reason, the paper is organized as follows. In the next section we give some preliminary results. In Section 3, we study the Nehari manifold associated with (1.1) through its fibrering maps and in the last section we give the proof of Theorem 1.1.

Let us first recall that the so-called fractional Sobolev spaces $W^{s,p}(\mathbb{R}^N)$, $s \in (0,1)$ and $p \in (1,\infty)$ are Banach spaces with the usual norm

$$\|u\|_{s,p}^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} |u(x)|^p\,\mathrm{d}x.$$

We also recall the fractional Sobolev theorem.

Theorem 2.1 ([20])**.** *Let $s \in (0,1)$ and $p \in [1,\infty)$ be such that $sp < N$. Then there exists a positive constant $C = C(N,p,s)$ such that*

$$\left(\int_{\mathbb{R}^N} |u(x)|^{p_s^*}\,\mathrm{d}x\right)^{p/p_s^*} \leqslant C \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y, \tag{2.1}$$

where $p_s^* := Np/(N-sp)$ is the so-called "fractional critical exponent". Consequently, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p,p_s^*]$. Moreover, the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q_{\mathrm{loc}}(\mathbb{R}^N)$ is compact for $q \in [p,p_s^*)$.

We then consider the reflexive Banach space

$$X := \left\{u \in W^{s,p}(\mathbb{R}^N)\colon \int_{\mathbb{R}^N} V(x)|u(x)|^p\,\mathrm{d}x < \infty\right\},$$

endowed with the norm

$$\|u\| := \|u\|_X = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u(x)|^p\,\mathrm{d}x\right)^{1/p}. \tag{2.2}$$

On the other hand, for every $r \in [p,p_s^*)$, we denote by $L^q_w(\mathbb{R}^N)$ the weighted Lebesgue spaces

$$L^r_w(\mathbb{R}^N) := \left\{u\colon \mathbb{R}^N \to \mathbb{R} \text{ such that } \int_{\mathbb{R}^N} w(x)|u(x)|^r\,\mathrm{d}x < \infty\right\},$$

normed by

$$\|u\|_{r,w} = \left(\int_{\mathbb{R}^N} w(x)|u(x)|^r\,\mathrm{d}x\right)^{1/r}. \tag{2.3}$$

Then we have the following compactness result.

Proposition 2.2. *Assume that* (A_1) *and* (A_2) *hold. Then we have:*

(i) *X is continuously embedded in $L^{p_s^*}(\mathbb{R}^N)$ and compactly embedded into the spaces $L^r_{\mathrm{loc}}(\mathbb{R}^N)$ for any $r \in [p,p_s^*)$.*

(ii) *X is compactly embedded into the spaces $L^p_K(\mathbb{R}^N)$ and $L^q_Q(\mathbb{R}^N)$ for $q \in (p,p_s^*)$.*

P r o o f. The conclusion of (i) follows from Theorem 2.1 and the fact that $W_V^{s,p}(B_R) = W^{s,p}(B_R)$ for all $R > 0$. By following the ideas in the proof of Proposition 1.1 in [28], we merely need to verify that

$$\lim_{|x|\to\infty} \frac{Q(x)}{V^{\sigma(q)}(x)} = 0.$$

Indeed, by assumption (A$_2$), since $\sigma(q) = \frac{1}{2}\sigma(q-\beta) + \frac{1}{2}\sigma(q+\beta)$, we have

$$\begin{aligned}\lim_{|x|\to\infty} \frac{Q(x)}{V^{\sigma(q)}(x)} &= \lim_{|x|\to\infty} \frac{Q(x)}{V^{\sigma(q-\beta)/2}(x)V^{\sigma(q+\beta)/2}(x)} \\ &= \lim_{|x|\to\infty} \Big(\frac{Q(x)}{V^{\sigma(q-\beta)}(x)}\Big)^{1/2}\Big(\frac{Q(x)}{V^{\sigma(q+\beta)}(x)}\Big)^{1/2} = 0.\end{aligned}$$

Thus, the proof is complete. □

We next give a lemma that will be essential to pass the limit in the logarithmic nonlinearity.

Lemma 2.3. *Suppose that V and Q satisfy* (A$_2$). *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence such that $u_n \to u$ weakly in X. Then we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} Q(x)|u_n(x)|^q \log|u_n(x)| \,\mathrm{d}x = \int_{\mathbb{R}^N} Q(x)|u(x)|^q \log|u(x)| \,\mathrm{d}x. \tag{2.4}$$

P r o o f. By direct calculation, we have

$$|\log t| = |\log t|\chi_{[0,1]}(t) + |\log t|\chi_{[1,\infty)}(t) \leqslant C(t^{-\beta} + t^{\beta}) \quad \forall\, t > 0,$$

where β is a positive constant given by assumption (A$_2$) and we denote by C a general constant which can be varied from line to line. So we have

$$Q(x)|t^q \log t| \leqslant CQ(x)t^{q-\beta} + CQ(x)t^{q+\beta} \quad \forall\, x \in \mathbb{R}^N,\ t > 0. \tag{2.5}$$

For $x \in \mathbb{R}^N$ fixed and $r \in (p, p_s^*)$, consider the function $t \mapsto V(x)t^{p-r} + t^{p_s^*-r}$, $t > 0$. We have

$$C(r)V^{(p_s^*-r)/(p_s^*-p)}(x) \leqslant V(x)t^{p-r} + t^{p_s^*-r} \quad \forall\, t > 0,$$

where $C(r)$ is a positive constant given by

$$C(r) := \Big(\frac{r-p}{p_s^*-r}\Big)^{(p-r)/(p_s^*-p)} + \Big(\frac{r-p}{p_s^*-r}\Big)^{(p_s^*-r)/(p_s^*-p)}, \quad p < r < p_s^*.$$

Assume further that

$$\lim_{|x|\to\infty} \frac{Q(x)}{V^{\sigma(r)}(x)} = 0,$$

then, for given $\varepsilon > 0$, there is $R > 0$ large enough such that

$$Q(x) \leqslant \varepsilon C(r) V^{(p_s^*-r)/(p_s^*-p)}(x) \leqslant \varepsilon(V(x)t^{p-r} + t^{p_s^*-r}) \quad \forall\, t > 0,\ |x| \geqslant R,$$

which implies

$$Q(x)t^r \leqslant \varepsilon(V(x)t^p + t^{p_s^*}) \quad \forall\, t > 0,\ |x| \geqslant R.$$

By assumption (A$_2$), applying the estimate above for $r = q - \beta$ and $r = q + \beta$, we derive from (2.5) that

$$Q(x)|t^q \log t| \leqslant \varepsilon C(V(x)t^p + t^{p_s^*}) \quad \forall\, t > 0,\ |x| \geqslant R.$$

As a consequence, we have

$$\text{(2.6)} \quad \int_{B_R^c(0)} Q(x)|u_n(x)|^q |\log(|u_n(x)|)| \,\mathrm{d}x \leqslant \varepsilon C \int_{\mathbb{R}^N} (V(x)|u_n(x)|^p + |u_n(x)|^{p_s^*}) \,\mathrm{d}x.$$

On the other hand, since $\{u_n\}_{n=1}^{\infty}$ is bounded in X, by Proposition 2.2, there is $M > 0$ such that

$$\int_{\mathbb{R}^N} V(x)|u_n(x)|^p \,\mathrm{d}x \leqslant M, \quad \text{and} \quad \int_{\mathbb{R}^N} |u_n(x)|^{p_s^*} \,\mathrm{d}x \leqslant M.$$

Thus, from (2.6) we derive

$$\text{(2.7)} \qquad \int_{B_R^c} Q(x)|u_n(x)|^q |\log(|u_n(x)|)| \,\mathrm{d}x \leqslant 2MC\varepsilon.$$

Using similar arguments, we see that the estimate (2.7) holds also for u. Hence, it remains to show that

$$\text{(2.8)} \qquad \lim_{n\to\infty} \int_{B_R} Q(x)|u_n(x)|^q \log(|u_n(x)|) \,\mathrm{d}x = \int_{B_R} Q(x)|u(x)|^q \log(|u(x)|) \,\mathrm{d}x.$$

Indeed, since $Q \in L^\infty(\mathbb{R}^N)$, we have

$$\text{(2.9)} \qquad \int_{B_R} |Q(x)|u_n(x)|^q \log(|u_n(x)|) - Q(x)|u(x)|^q \log(|u(x)|)| \,\mathrm{d}x$$
$$\leqslant \|Q\|_{L^\infty(\mathbb{R}^N)} \int_{B_R} ||u_n(x)|^q \log(|u_n(x)|) - |u(x)|^q \log(|u(x)|)| \,\mathrm{d}x.$$

On the other hand, since $q \in (p, p_s^*)$, we have

$$\frac{t^q \log(|t|)}{t^{p_s^*}} \to 0 \quad \text{as } t \to \infty.$$

By Proposition 2.2, we also know that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N} |u_n(x)|^{p_s^*} \,\mathrm{d}x < \infty, \tag{2.10}$$

$$|u_n(x)|^q \log(|u_n(x)|) \to |u(x)|^q \log(|u(x)|) \quad \text{a.e. in } \mathbb{R}^N. \tag{2.11}$$

Applying the compactness lemma of Strauss (see [7], Theorem A.I, page 338) with a bounded set $B_R(0)$ and $Q(s) = s^{p_s^*}$ and $P(s) = s^q \log(|s|)$, we obtain (2.8). $\square$

The next lemma shows that the functional J is differentiable on the potential space X.

Lemma 2.4. *The functional J is smooth, that is, $J \in C^1(X, \mathbb{R})$.*

Proof. We first prove $J(u)$ to be well-defined for $u \in X$. By Proposition 2.2 it suffices to prove that $Q|u|^q \log|u| \in L^1(\mathbb{R}^N)$ for $u \in X$. Using similar arguments as in the proof of Lemma 2.3 we have that

$$Q(x)|t^q \log t| \leqslant CQ(x)t^{q-\gamma} + CQ(x)t^{q+\gamma} \quad \forall\, x \in \mathbb{R}^N,\ t > 0,\ \gamma > 0. \tag{2.12}$$

Since $q \in (p, p_s^*)$ we can choose γ small enough so that $p < q - \gamma < q + \gamma < p_s^*$. By virtue of Proposition 2.2 (ii) we imply that $Q|u|^q \log|u| \in L^1(\mathbb{R}^N)$.

It is not difficult to see that J is Gateaux differentiable and

$$\begin{aligned} \langle J'(u), v\rangle &= \iint_{\mathbb{R}^{2N}} \frac{\psi_p(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V(x)\psi_p(u)v \,\mathrm{d}x \\ &\quad - \lambda \int_{\mathbb{R}^N} K(x)\psi_p(u)v \,\mathrm{d}x - \mu \int_{\mathbb{R}^N} Q(x)\psi_q(u)\log|u|v \,\mathrm{d}x \\ &= \langle H_1(u), v\rangle - \langle H_2(u), v\rangle \quad \forall\, u, v \in X, \end{aligned} \tag{2.13}$$

where

$$\langle H_1(u), v\rangle = \iint_{\mathbb{R}^{2N}} \frac{\psi_p(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V(x)\psi_p(u)v \,\mathrm{d}x, \tag{2.14}$$

$$\langle H_2(u), v\rangle = \lambda \int_{\mathbb{R}^N} K(x)\psi_p(u)v \,\mathrm{d}x + \mu \int_{\mathbb{R}^N} Q(x)\psi_q(u)\log|u|v \,\mathrm{d}x. \tag{2.15}$$

So it remains to show that $J'\colon X \to X'$ is continuous. To this aim, it suffices to prove that

$$J(u_n) \text{ converges in } X' \text{ to } J(u)$$

for every sequence $\{u_n\}_{n=1}^{\infty} \in X$ converging to u strongly in X.

In fact, by using Proposition 2.2 and Lemma 2.3 we can see easily that

$$H_2(u_n) \text{ converges in } X' \text{ to } H_2(u), \tag{2.16}$$

and for any $v \in X \setminus \{0\}$, we have

$$\lim_{n\to\infty} \frac{1}{\|v\|} \int_{\mathbb{R}^N} V(x)|\psi_p(u_n(x)) - \psi_p(u(x))|v \,\mathrm{d}x = 0. \tag{2.17}$$

Next, for $n \in \{1, 2, \ldots\}$, put

$$w_n(x,y) = \frac{\psi_p(u_n(x) - u_n(y))}{|x-y|^{N/p'+s(p-1)}} \quad \text{and} \quad w(x,y) = \frac{\psi_p(u(x) - u(y))}{|x-y|^{N/p'+s(p-1)}}.$$

Since $u_n \to u$ in X, the sequence $\{w_n\}_{n=1}^{\infty}$ is bounded in $L^{p'}(\mathbb{R}^{2N})$ and moreover it converges almost everywhere to the function w. This implies

$$w_n \to w \text{ weakly in } L^{p'}(\mathbb{R}^{2N}).$$

Therefore, if $v \in X \setminus \{0\}$, then

$$\lim_{n\to\infty} \|v\|^{-1} \left| \iint_{\mathbb{R}^{2N}} (w_n(x,y) - w(x,y)) \frac{(v(x) - v(y))}{|x-y|^{N/p+s}} \,\mathrm{d}x\,\mathrm{d}y \right| = 0$$

thanks to the fact that the function

$$(x,y) \mapsto \frac{(v(x) - v(y))}{|x-y|^{N/p+s}}$$

belongs to $L^p(\mathbb{R}^{2N})$. So we can conclude

$$H_1(u_n) \to H_1(u) \quad \text{in } X'.$$

The proof of Lemma 2.4 is completed. □

3. Nehari manifold and fibrering maps

The Euler-Lagrange functional $J\colon X \to \mathbb{R}$ associated to problem (1.1) is defined as follows:

$$J(u) = \frac{1}{p}(\|u\|^p - \lambda\|u\|_{p,K}^p) + \frac{\mu}{q^2}\|u\|_{q,Q}^q - \frac{\mu}{q}\int_{\mathbb{R}^N} Q(x)|u(x)|^q \log|u(x)|\,\mathrm{d}x.$$

It is worth noting that since $\lim_{s\to 0}|s|^q\log|s| = 0$, the function $f(s) = |s|^q\log|s|$ can be extended continuously but still denoted by the same notation such that its value at 0 equals 0. Using this convention, we have $J(0) = 0$. Moreover, $J \in C^1(X,\mathbb{R})$ and its Gateaux derivative is

$$\begin{aligned}\langle J'(u), v\rangle = &\iint_{\mathbb{R}^{2N}} \frac{\psi_p(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V(x)\psi_p(u)v\,\mathrm{d}x\\ &- \lambda\int_{\mathbb{R}^N} K(x)\psi_p(u)v\,\mathrm{d}x - \mu\int_{\mathbb{R}^N} Q(x)\psi_q(u)\log|u|v\,\mathrm{d}x \quad \forall\, u,v\in X.\end{aligned}$$

Notice that the weak solutions of problem (1.1) correspond to critical points of J. As many other problems J is not bounded below on X. It is appropriate to consider J on the Nehari manifold which is defined by

$$\mathcal{N} := \{u \in X\setminus\{0\}\colon\ \langle J'(u), u\rangle = 0\}.$$

Clearly, all critical points of J must lie on $\mathcal{N}$ and further $u\in\mathcal{N}$ if and only if

$$I(u) := \|u\|^p - \lambda\|u\|_{p,K}^p - \mu\int_{\mathbb{R}^N} Q(x)|u(x)|^q\log|u(x)|\,\mathrm{d}x = 0.$$

We analyse $\mathcal{N}$ in terms of the stationary points of fibrering maps $\varphi_u\colon\ \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\varphi_u(t) = J(tu), \quad t\in\mathbb{R}^+.$$

Such maps were introduced by Drábek and Pohozaev in [21]. Then we have

$$\begin{aligned}\varphi_u'(t) = {}& t^{p-1}(\|u\|^p - \lambda\|u\|_{p,K}^p)\\ &- \mu t^{q-1}\int_{\mathbb{R}^N} Q(x)|u(x)|^q\log|u(x)|\,\mathrm{d}x - \mu(t^{q-1}\log t)\|u\|_{q,Q}^q,\end{aligned}$$

and

$$\begin{aligned}\varphi_u''(t) = {}&(p-1)t^{p-2}(\|u\|^p - \lambda\|u\|_{p,K}^p) - \mu((q-1)t^{q-2}\log t + t^{q-2})\|u\|_{q,Q}^q\\ &- \mu(q-1)t^{q-2}\int_{\mathbb{R}^N} Q(x)|u(x)|^q\log|u(x)|\,\mathrm{d}x.\end{aligned}$$

Lemma 3.1. *Let $u\in X\setminus\{0\}$ and $t>0$. Then $tu\in\mathcal{N}$ if and only if $\varphi_u'(t) = 0$.*

P r o o f. The conclusion of the lemma can be directly implied from the definition of $\mathcal{N}$ and φ_u. □

From Lemma 3.1 it follows that $u \in \mathcal{N}$ if and only if $\varphi_u'(1) = 0$. We shall split $\mathcal{N}$ into three subsets $\mathcal{N}^+$, $\mathcal{N}^-$, and $\mathcal{N}^0$ which correspond to local minima, local maxima and points of inflection of fibrering maps, that is,

$$\begin{aligned}
\mathcal{N}^+ &= \{u \in \mathcal{N} \colon\ \varphi_u''(1) > 0\} = \{tu \in X \colon\ \varphi_u'(t) = 0,\ \varphi_u''(t) > 0\},\\
\mathcal{N}^- &= \{u \in \mathcal{N} \colon\ \varphi_u''(1) < 0\} = \{tu \in X \colon\ \varphi_u'(t) = 0,\ \varphi_u''(t) < 0\},\\
\mathcal{N}^0 &= \{u \in \mathcal{N} \colon\ \varphi_u''(1) = 0\} = \{tu \in X \colon\ \varphi_u'(t) = 0,\ \varphi_u''(t) = 0\}.
\end{aligned}$$

The existence of solutions to problem (1.1) can be studied by considering the existence of minimizers to the functional J on the manifold $\mathcal{N}$. As in Brown-Zhang [9], Theorem 2.3 or Chen-Deng [15], Lemma 2.2 we see that such local minimizers are usually critical points of J. More precisely, we have the following lemma whose proof is classical and can be followed step by step as in [3], Proposition 6.7, so we omit it.

Lemma 3.2. *$J \in C^2(X, \mathbb{R})$ and if u_0 is a local minimizer of J on $\mathcal{N}$ and $u_0 \notin \mathcal{N}^0$, then it is a critical point of J.*

We now investigate the functional J on the Nehari manifold $\mathcal{N}$. By the definition of $\mathcal{N}$, for each $u \in \mathcal{N}$, we have

$$J(u) = \Big(\frac{1}{p} - \frac{1}{q}\Big)(\|u\|^p - \lambda\|u\|_{p,K}^p) + \frac{\mu}{q^2}\|u\|_{q,Q}^q. \tag{3.1}$$

Moreover, it is easy to show that

$$\begin{aligned}
\mathcal{N}^+ &= \{u \in \mathcal{N} \colon\ (q-p)A(u) + \mu C(u) < 0\},\\
\mathcal{N}^- &= \{u \in \mathcal{N} \colon\ (q-p)A(u) + \mu C(u) > 0\},\\
\mathcal{N}^0 &= \{u \in \mathcal{N} \colon\ (q-p)A(u) + \mu C(u) = 0\},
\end{aligned}$$

where

$$A(u) := \|u\|^p - \lambda\|u\|_{p,K}^p,$$
$$B(u) := \int_{\mathbb{R}^N} Q(x)|u(x)|^q \log(|u(x)|)\,\mathrm{d}x, \quad \text{and} \quad C(u) := \int_{\mathbb{R}^N} Q(x)|u(x)|^q\,\mathrm{d}x.$$

The corresponding fibrering maps φ_u can be rewritten as

$$\varphi_u(t) = \frac{t^p}{p}A(u) + \frac{\mu}{q^2}t^q(1 - q\log t)C(u) - \frac{\mu}{q}t^q B(u),$$

and

$$\varphi_u'(t) = t^{p-1}\beta_u(t), \quad \text{where } \beta_u(t) := A(u) - \mu B(u)t^{q-p} - \mu C(u)t^{q-p}\log t.$$

In order to find critical points of $\varphi_u(t)$, we need to search zero points of $\beta_u(t)$. We have

$$\beta'_u(t) = 0 \Leftrightarrow t_0 := t_0(u) = \exp\Big(-\frac{1}{q-p}\Big)\exp\Big(-\frac{B(u)}{C(u)}\Big).$$

It is easy to see that the behavior of $\varphi_u(t)$ depends on the signs of $A(u)$ because $C(u)$ is always positive. Setting

$$A^+ = \{u \in X\colon\ A(u) > 0\},$$

we have the following remark which shows the properties of A^+.

Remark 3.3. If $u \in A^+$, then $\varphi_u(t) > 0$ for small $t > 0$ and $\varphi_u(t) \to -\infty$ as $t \to \infty$. Moreover, in this case $\beta_u(t)$ attains its maximum at $t_0(u)$, strictly increases on $(0, t_0(u))$ and strictly decreases on $(t_0(u), \infty)$, and

$$0 < A(u) = \lim_{t\to 0^+} \beta_u(t) = \beta_u(0^+) < \beta_u(t_0) \quad \text{and} \quad \lim_{t\to\infty} \beta_u(t) = -\infty.$$

This implies $\beta_u(t)$ has a unique zero point $t_1(u) > t_0(u) > 0$ and therefore $\varphi_u(t)$ has a unique (maximum) stationary point at $t_1(u) > t_0(u)$ such that $t_1(u)u \in \mathcal{N}^-$.

4. The proof of the main result

Throughout this section we assume $\lambda < \lambda_1$ and (A_1) and (A_2) hold. By the definition of λ_1 we have

$$(4.1) \qquad A(u) = \|u\|^p - \lambda\|u\|^p_{p,K} \geqslant \Big(1 - \frac{\lambda}{\lambda_1}\Big)\|u\|^p > 0 \quad \forall\, u \in X \setminus \{0\}.$$

This implies $u \in A^+$ for all $u \in X \setminus \{0\}$. On the other hand, since $q > p$ and $C(u)$ is positive, by definition we have $\mathcal{N}^+ = \mathcal{N}^0 = \emptyset$. In addition, by Remark 3.3 there is a unique $t_1(u) > 0$ such that $t_1(u)u \in \mathcal{N}^-$ and therefore the Nehari manifold $\mathcal{N}$ can be expressed as follows:

$$\mathcal{N} = \mathcal{N}^- = \{t_1(u)u\colon\ u \in A^+\}.$$

Using this fact, we merely need to investigate the behavior of J on $\mathcal{N}^-$. It is easy to see that J is coercive on $\mathcal{N}^-$ due to $0 < \lambda < \lambda_1$ and

$$(4.2) \qquad \begin{aligned} J(u) &= \Big(\frac{1}{p} - \frac{1}{q}\Big)(\|u\|^p - \lambda\|u\|^p_{p,K}) + \frac{\mu}{q^2}\|u\|^q_{q,Q} \\ &\geqslant \Big(\frac{1}{p} - \frac{1}{q}\Big)\Big(1 - \frac{\lambda}{\lambda_1}\Big)\|u\|^p + \frac{\mu}{q^2}\|u\|^q_{q,Q} \quad \forall\, u \in \mathcal{N}^-. \end{aligned}$$

The next lemma shows that its infimum is positive.

Proposition 4.1. *We have* $\inf_{u\in\mathcal{N}^-} J(u) > 0$.

P r o o f. Assume by contradiction that $\inf_{u\in\mathcal{N}^-} J(u) = 0$ and let $\{u_m\}_{m=1}^\infty \in \mathcal{N}^-$ be a minimizing sequence, that is, we have

$$\lim_{m\to\infty} J(u_m) = 0.$$

Since J is coercive on $\mathcal{N}^-$, it follows that $\{u_m\}_{m=1}^\infty$ is bounded on X. By virtue of Proposition 2.2, this sequence has a subsequence still denoted by $\{u_m\}_{m=1}^\infty$ such that

$$u_m \to u_\infty \text{ weakly in } X \text{ and strongly in } L^p_K(\mathbb{R}^N)\cap L^q_Q(\mathbb{R}^N), \text{ and a.e. in } \mathbb{R}^N.$$

As a consequence, we have

$$\int_{\mathbb{R}^N} K(x)|u_m(x)|^p\,\mathrm{d}x \to \int_{\mathbb{R}^N} K(x)|u_\infty(x)|^p\,\mathrm{d}x, \tag{4.3}$$

$$\int_{\mathbb{R}^N} Q(x)|u_m(x)|^q\,\mathrm{d}x \to \int_{\mathbb{R}^N} Q(x)|u_\infty(x)|^q\,\mathrm{d}x. \tag{4.4}$$

We next show that u_∞ must be zero. Indeed, if this is not the case, then it follows from (4.2) and (4.4) that

$$0 < \frac{\mu}{q^2}\|u_\infty\|^q_{q,Q} = \lim_{m\to\infty}\frac{\mu}{q^2}\|u_m\|^q_{q,Q} \leqslant \lim_{m\to\infty} J(u_m) = 0,$$

which is a contradiction. Moreover, $u_m \to 0$ strongly in X. If not, then we have $0 = \|u_\infty\| < \liminf_{n\to\infty}\|u_m\|$ and therefore we get another contradiction

$$0 = J(u_\infty) < \liminf_{m\to\infty}\Big(\Big(\frac{1}{p}-\frac{1}{q}\Big)(\|u_m\|^p - \lambda\|u_m\|^p_{p,K}) + \frac{\mu}{q^2}\|u_m\|^q_{q,Q}\Big) = \lim_{m\to\infty} J(u_m) = 0.$$

Hence, we get $u_m \to 0$ strongly in X. By setting $v_m = u_m/\|u_m\|$, we may assume that

$$v_m \to v_\infty \text{ weakly in } X \text{ and strongly in } L^p_K(\mathbb{R}^N)\cap L^q_Q(\mathbb{R}^N), \text{ and a.e. in } \mathbb{R}^N. \tag{4.5}$$

In other words, we have

$$\int_{\mathbb{R}^N} K(x)|v_m(x)|^p\,\mathrm{d}x \to \int_{\mathbb{R}^N} K(x)|v_\infty(x)|^p\,\mathrm{d}x, \tag{4.6}$$

$$\int_{\mathbb{R}^N} Q(x)|v_m(x)|^q\,\mathrm{d}x \to \int_{\mathbb{R}^N} Q(x)|v_\infty(x)|^q\,\mathrm{d}x. \tag{4.7}$$

By virtue of Lemma 2.3, we get

$$\lim_{m\to\infty}\int_{\mathbb{R}^N} Q(x)|v_m(x)|^q \log|v_m(x)|\,\mathrm{d}x = \int_{\mathbb{R}^N} Q(x)|v_\infty(x)|^q \log|v_\infty(x)|\,\mathrm{d}x. \tag{4.8}$$

On the other hand, since $u_m \in \mathcal{N}$, we get

$$\|u_m\|^p - \lambda\|u_m\|_{p,K}^p - \mu\int_{\mathbb{R}^N} Q(x)|u_m(x)|^q \log|u_m(x)|\,\mathrm{d}x = 0.$$

Dividing both sides by $\|u_m\|^p$, we obtain

$$\begin{aligned}\|v_m\|^p - \lambda\|v_m\|_{p,K}^p = \mu\|u_m\|^{q-p}&\int_{\mathbb{R}^N} Q(x)|v_m(x)|^q \log|v_m(x)|\,\mathrm{d}x\\ &+ \mu\|u_m\|^{q-p}\log(\|u_m\|)\int_{\mathbb{R}^N} Q(x)|v_m(x)|^q\,\mathrm{d}x.\end{aligned}$$

Let $m\to\infty$. Since $q\in(p,p_s^*)$, it follows from (4.7), (4.8) and the fact that $u_m\to 0$ strongly in X, that

$$\lim_{m\to\infty}(\|v_m\|^p - \lambda\|v_m\|_{p,K}^p) = 0. \tag{4.9}$$

We next show that $v_m\to v_\infty$ strongly in X. Otherwise, we have $\|v_\infty\| < \liminf\limits_{m\to\infty}\|v_m\|$ and by using (4.6), we get a contradiction

$$\|v_\infty\|^p - \lambda\|v_\infty\|_{p,K}^p < \liminf_{m\to\infty}(\|v_m\|^p - \lambda\|v_m\|_{p,K}^p) = 0.$$

Hence, $v_m\to v_\infty$ strongly in X which implies that

$$\Bigl(1-\frac{\lambda}{\lambda_1}\Bigr)\|v_\infty\|^p \leqslant \|v_\infty\|^p - \lambda\|v_\infty\|_{p,K}^p = \lim_{m\to\infty}(\|v_m\|^p - \lambda\|v_m\|_{p,K}^p) = 0.$$

Hence, $v_\infty = 0$ which contradicts $\|v_\infty\| = 1$. The proof is complete. □

We are now in the position to give the proof of the main theorem.

Proof of Theorem 1.1. Let $\{u_m\}_{m=1}^\infty$ be a minimizing sequence of J on $\mathcal{N}^-$ such that

$$J(u_m)\to \inf_{u\in\mathcal{N}^-} J(u) > 0 \quad \text{as } m\to\infty. \tag{4.10}$$

By coerciveness of $J(u)$ on $\mathcal{N}^-$ and Proposition 2.2, we may assume that $u_m\to u_0$ weakly in X and strongly in $L_K^p(\mathbb{R}^N)\cap L_Q^q(\mathbb{R}^N)$, and a.e. in $\mathbb{R}^N$, which implies

$$\int_{\mathbb{R}^N} K(x)|u_m(x)|^p\,\mathrm{d}x \to \int_{\mathbb{R}^N} K(x)|u_0(x)|^p\,\mathrm{d}x, \tag{4.11}$$

$$\int_{\mathbb{R}^N} Q(x)|u_m(x)|^q\,\mathrm{d}x \to \int_{\mathbb{R}^N} Q(x)|u_0(x)|^q\,\mathrm{d}x. \tag{4.12}$$

By virtue of Lemma 2.3, we have

$$\lim_{m\to\infty}\int_{\mathbb{R}^N} Q(x)|u_m(x)|^q \log|u_m(x)|\,\mathrm{d}x = \int_{\mathbb{R}^N} Q(x)|u_0(x)|^q \log|u_0(x)|\,\mathrm{d}x. \tag{4.13}$$

We next show that $u_m \to u_0$ strongly in X. Suppose otherwise, we have $\|u_0\| < \liminf_{m\to\infty} \|u_m\|$. Since $u_m \in \mathcal{N}^- = \mathcal{N}$, we have

$$\begin{aligned}
A(u_0) - \mu B(u_0) &= \|u_0\|^p - \lambda\|u_0\|_{p,K}^p - \mu\int_{\mathbb{R}^N} Q(x)|u_0(x)|^q \log(|u_0(x)|)\,\mathrm{d}x \\
&< \liminf_{m\to\infty}\left(\|u_m\|^p - \lambda\|u_m\|_{p,K}^p - \mu\int_{\mathbb{R}^N} Q(x)|u_m(x)|^q \log(|u_m(x)|)\,\mathrm{d}x\right) \\
&= 0,
\end{aligned}$$

that is, $\beta_{u_0}(1) = A(u_0) - \mu B(u_0) < 0$ which immediately implies that $u_0 \neq 0$, while $\beta_{u_0}(t_0) > 0$. The analysis of the fibrering maps $\varphi_u(t)$ shows that there exists a unique $t_1(u_0) \in (t_0(u_0), 1)$ such that $t_1(u_0)u_0 \in \mathcal{N}^-$. Hence we have

$$\inf_{u\in\mathcal{N}^-} J(u) \leqslant J(t_1(u_0)u_0). \tag{4.14}$$

On the other hand, since $u_m \to u_0$ weakly in X, one has

$$t_1(u_0)u_m \to t_1(u_0)u_0 \text{ weakly in } X,$$

which leads to a contradiction to (4.14):

$$J(t_1(u_0)u_0) < \liminf_{m\to\infty} J(t_1(u_0)u_m) \leqslant \liminf_{m\to\infty} J(u_m) = \inf_{u\in\mathcal{N}^-} J(u). \tag{4.15}$$

Here we use the fact that $u_m \in \mathcal{N}^-$ and therefore $J(tu_m)$ attains its unique maximum at $t = 1$.

Hence $u_m \to u_0$ strongly in X. Finally, we prove that $u_0 \in \mathcal{N}^-$. Indeed, notice that $u_0 \neq 0$ and $u_m \in \mathcal{N}^- = \mathcal{N}$, we have

$$\|u_m\|^p - \lambda\|u_m\|_{p,K}^p - \mu\int_{\mathbb{R}^N} Q(x)|u_m(x)|^q \log|u_m(x)|\,\mathrm{d}x = 0.$$

Let $m \to \infty$, we get $u_0 \in \mathcal{N} = \mathcal{N}^-$. The proof is complete. □

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Authors' addresses: *Cong Nhan Le* (corresponding author), Faculty of Applied Sciences, Ho Chi Minh City University of Technology and Education, No 1 Vo Van Ngan Street, Linh Chieu Ward, Thu Duc District, Ho Chi Minh City, Vietnam, e-mail: `nhanlc@hcmute.edu.vn`; *Xuan Truong Le*, Department of Mathematics and Statistics, University of Economics Ho Chi Minh City, 59C Nguyen Dinh Chieu Street, District 3, Ho Chi Minh City, Vietnam, e-mail: `lxuantruong@ueh.edu.vn`.