GENERALIZED QUADRATIC OPERATORS AND PERTURBATIONS

Khalid Souilah, Oujda

Abstract. We provide a complete description of the perturbation class and the commuting perturbation class of all generalized quadratic bounded operators with respect to a given idempotent bounded operator in the context of complex Banach spaces. Furthermore, we give simple characterizations of the generalized quadraticity of linear combinations of two generalized quadratic bounded operators with respect to a given idempotent bounded operator.

Keywords: generalized quadratic operator; perturbation classes problem

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1. Introduction

Throughout this paper, X denotes a complex Banach space with $\dim X \geqslant 3$, and $\mathcal{B}(X)$ denotes the algebra of all bounded linear operators acting on X.

Given a subset $\Lambda \subset \mathcal{B}(X)$, the *perturbation class* $\mathcal{P}(\Lambda)$ and the *commuting perturbation class* $\mathcal{P}_{\mathrm{c}}(\Lambda)$ of Λ are, respectively, defined by

$$\mathcal{P}(\Lambda) = \{S \in \mathcal{B}(X) \colon\ S + T \in \Lambda \text{ for every } T \in \Lambda\}$$

and

$$\mathcal{P}_{\mathrm{c}}(\Lambda) = \{S \in \mathcal{B}(X) \colon\ S + T \in \Lambda \text{ for every } T \in \Lambda \text{ commuting with } S\}.$$

The perturbation class problem of Λ concerns the study of the components of $\mathcal{P}(\Lambda)$ and $\mathcal{P}_{\mathrm{c}}(\Lambda)$, and giving a complete characterization of them. Note that in order to check whether an operator S satisfies the definition of a perturbation class, we have to study the properties of the sum $S + T$ for T in a family of operators and not just the action of S, which can be laborious in the general case.

The concept of perturbation classes has been considered in the general situation of Banach algebras. Indeed, it is well-known that the perturbation class of the set of all invertible elements in a Banach algebra is the radical of that algebra; and its commuting perturbation class is the set of all quasinilpotent elements (see [7], [13]). For more details on the perturbation classes problem, we refer the interested readers to [1], [5], [6], [7], [9], [13] and the references therein.

Recall that an operator $T \in \mathcal{B}(X)$ is said to be *algebraic* if there exists a nonzero complex polynomial P such that $P(T) = 0$. In particular, when T is annihilated by a complex polynomial of degree at most two, it is said to be *quadratic*.

One of the interesting results in [8] asserts that the perturbation class of the set of all bounded quadratic operators on X is the one-dimensional subspace of scalar multiples of the identity operator. Afterwards, in [9], the authors establish that the perturbation class of the set of all bounded algebraic operators, acting on an infinite-dimensional complex Hilbert space, consists exactly of finite-rank operators plus scalar multiples of the identity, and its commuting perturbation class is the set itself. However, the perturbation class of such operators in the context of infinite-dimensional Banach spaces is not well known.

Since generalized quadratic operators are algebraic, we propose in this paper to consider the perturbation class problem, as well as the commuting perturbation class problem, of all generalized quadratic operators with respect to a given idempotent operator in the setting of complex Banach spaces. Recall that an operator $T \in \mathcal{B}(X)$ is said to be *generalized quadratic* if there exist an idempotent operator $Q \in \mathcal{B}(X)$ and $a, b \in \mathbb{C}$ such that

$$TQ = QT = T \quad \text{and} \quad (T - aQ)(T - bQ) = 0.$$

In particular, when $Q = I$, such operator T fulfills $(T - aI)(T - bI) = 0$, and hence it becomes a quadratic operator.

Throughout this paper, $P \in \mathcal{B}(X)$ is a nonzero idempotent operator, and $\omega(P)$ is the subset of all generalized quadratic operators in $\mathcal{B}(X)$ with respect to P, that is, the set of all operators $T \in \mathcal{B}(X)$ such that $TP = PT = T$ and $(T - aP)(T - bP) = 0$ for some $a, b \in \mathbb{C}$.

Generalized quadratic operators are becoming useful tools in various areas of applied linear algebra and statistics. Several mathematicians investigated the properties of such operators (see [2], [3], [4]) and paid special attention to the generalized quadraticity of linear combinations of generalized quadratic matrices (see for instance [10], [11], [12]).

Recently, in [10], the authors investigated the generalized quadraticity of linear combinations of two generalized quadratic matrices A and B satisfying $(A - a_1Q) \times (A - a_2Q) = 0$ and $(B - b_1Q)(B - b_2Q) = 0$ for some idempotent matrix Q and

$a_1, a_2, b_1, b_2 \in \mathbb{C}$ with $a_1 \neq a_2$ and $b_1 \neq b_2$. However, they did not treat the case when $a_1 = a_2$ or $b_1 = b_2$. As the second aim of this paper, we generalize and complement their results by considering the infinite-dimensional case and all situations of generalized quadratic operators.

2. Perturbation class and commuting perturbation class of $\omega(P)$

In the following theorem, we completely characterize the perturbation class and the commuting perturbation class of $\omega(P)$.

Theorem 2.1. *The following assertions hold:*
(1) *If* $\dim \operatorname{ran}(P) \geqslant 3$ *then*

$$\mathcal{P}(\omega(P)) = \mathcal{P}_{\mathrm{c}}(\omega(P)) = \mathbb{C}P.$$

(2) *If* $\dim \operatorname{ran}(P) \leqslant 2$ *then*

$$\mathcal{P}(\omega(P)) = \mathcal{P}_{\mathrm{c}}(\omega(P)) = \omega(P).$$

Before proving this theorem, we need to establish some preliminary results. In what follows, $\mathcal{N}_2(P)$ denotes the set of all square-zero operators $N \in \mathcal{B}(X)$ such that $NP = PN = N$, and $\mathcal{I}(P)$ denotes the set of all idempotent operators $Q \in \mathcal{B}(X)$ such that $QP = PQ = Q$.

Proposition 2.2. *We have* $\omega(P) = (\mathbb{C}P + \mathcal{N}_2(P)) \cup (\mathbb{C}P + \mathbb{C}\mathcal{I}(P))$.

P r o o f. Clearly, we have $(\mathbb{C}P + \mathcal{N}_2(P)) \cup (\mathbb{C}P + \mathbb{C}\mathcal{I}(P)) \subseteq \omega(P)$.

Let $T \in \omega(P)$. Then $TP = PT = T$ and $(T - aP)(T - bP) = 0$ for some $a, b \in \mathbb{C}$. If $a = b$, then the operator $N = T - aP$ belongs to $\mathcal{N}_2(P)$ and $T = aP + N \in \mathbb{C}P + \mathcal{N}_2(P)$. Assume that $a \neq b$. Since $TP = T$ and $P^2 = P$, we have

$$(T - aP)^2 - (b - a)(T - aP) = (T - aP)(T - aP - (b - a)P)$$
$$= (T - aP)(T - bP) = 0,$$

and so $(T - aP)^2 = (b - a)(T - aP)$. Thus

$$((b - a)^{-1}(T - aP))^2 = (b - a)^{-1}(T - aP),$$

and hence the operator $Q = (b - a)^{-1}(T - aP)$ belongs to $\mathcal{I}(P)$, and we have $T = aP + (b - a)Q \in \mathbb{C}P + \mathbb{C}\mathcal{I}(P)$. $\qquad \square$

As a consequence of the above proposition, we give a result establishing the close relationship between a generalized quadratic operator in $\omega(P)$ and the idempotent operator P. Hence, we extend the corresponding result of [10], Theorem 1.1 to the infinite-dimensional case.

For an operator $T \in \mathcal{B}(X)$, we write $\ker(T)$ for its kernel and $\mathrm{ran}(T)$ for its range.

Corollary 2.3. *Let $T \in \mathcal{B}(X)$ be such that $T \notin \mathbb{C}P$. Then the following assertions are equivalent:*

(1) $T \in \mathbb{C}P + \mathbb{C}\mathcal{I}(P)$;
(2) $T \in \omega(P)$ *and there exist* $a, b \in \mathbb{C}$ *with* $a \neq b$ *such that* $(T - aP)(T - bP) = 0$;
(3) *there exist two idempotent operators* $A, B \in \mathcal{B}(X)$ *such that* $T = aA + bB$, $A + B = P$ *and* $AB = BA = 0$.

P r o o f. (1) $\Rightarrow$ (2) Suppose that $T \in \mathbb{C}P + \mathbb{C}\mathcal{I}(P)$. Then, from Proposition 2.2 we have $T \in \omega(P)$, and there exist $\alpha, \beta \in \mathbb{C}$ such that $T = \alpha P + \beta Q$ and $QP = PQ = Q$. It follows that $T - \alpha P = \beta Q$ and $T - (\alpha + \beta)P = \beta(Q - P)$, so

$$(T - \alpha P)(T - (\alpha + \beta)P) = \beta^2 Q(Q - P) = 0.$$

Since $T \notin \mathbb{C}P$, we have $\beta \neq 0$, and so it suffices to take $a = \alpha$ and $b = \alpha + \beta$.

(2) $\Rightarrow$ (3) Suppose that $T \in \omega(P)$ and there exist $a, b \in \mathbb{C}$ with $a \neq b$ such that $(T - aP)(T - bP) = 0$. Since $P^2 = P$, we can write $X = \ker(P) \oplus \mathrm{ran}(P)$ where the direct sum is topological. Furthermore, the closed subspaces $\ker(P)$ and $\mathrm{ran}(P)$ are T-invariant because $TP = PT = T$. Now, with respect to the decomposition of X, we have $P = 0 \oplus I$ and $T = 0 \oplus T_1$ where $T_1 = T_{|\mathrm{ran}(P)}$. On other hand, taking the restriction on $\mathrm{ran}(P)$ in $(T - aP)(T - bP) = 0$, we get that $(T_1 - a)(T_1 - b) = 0$, and hence $\mathrm{ran}(P) = \ker(T_1 - a) \oplus \ker(T_1 - b)$. Thus, relatively to the new decomposition $X = \ker(P) \oplus \ker(T_1 - a) \oplus \ker(T_1 - b)$, we have $P = 0 \oplus I \oplus I$ and $T = 0 \oplus aI \oplus bI$. Consider the idempotent operators $A, B \in \mathcal{B}(X)$ given by

$$A = 0 \oplus I \oplus 0 \quad \text{and} \quad B = 0 \oplus 0 \oplus I.$$

Clearly, we have $T = aA + bB$, $A + B = P$ and $AB = BA = 0$.

(3) $\Rightarrow$ (1) Suppose that there exist two idempotent operators $A, B \in \mathcal{B}(X)$ such that $T = aA + bB$, $A + B = P$ and $AB = BA = 0$. Then,

$$T = a(P - B) + bB = aP + (b - a)B.$$

Since $BP = PB = B$, we conclude that $T \in \mathbb{C}P + \mathbb{C}\mathcal{I}(P)$. $\qquad \square$

Let $z \in X$ and let $f \in X^*$ be nonzero, where X^* denotes the topological dual space. The symbol $z \otimes f$ stands for the rank-one operator defined by $(z \otimes f)(x) = f(x)z$ for all $x \in X$. Note that $z \otimes f$ is a quadratic operator, and $\sigma(z \otimes f) = \{0, f(z)\}$. Moreover, $z \otimes f$ is square-zero if and only if $f(z) = 0$.

Lemma 2.4. *Suppose that* $\dim \mathrm{ran}(P) \geqslant 3$. *Let* $N \in \mathcal{N}_2(P)$ *be nonzero. Then there exist two closed N-invariant subspaces X_1 and X_2 such that $\mathrm{ran}(P) = X_1 \oplus X_2$, $\dim X_1 = 3$ or 4, and*

$$
N_{|X_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad N_{|X_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},
$$

respectively.

P r o o f. Note that since $NP = PN = N$, we have $\ker(P) \subseteq \ker(N)$ and $\mathrm{ran}(P)$ is a closed N-invariant subspace. Put $Z = \mathrm{ran}(P)$ and $N_\mathrm{o} = N_{|Z}$. Then N_o is a nonzero square-zero operator.

Consider first the case when N_o has rank one. Set $N_\mathrm{o} = y \otimes f$ where $y \in Z$ and $f \in Z^*$ are nonzero and $f(y) = 0$. Since $\dim Z \geqslant 3$, there exist $x, z \in Z$ such that $f(x) = 1$, $f(z) = 0$ and $\{x, y, z\}$ is a linearly independent set. Furthermore, since $Z = \mathrm{Span}\{x, y, z\} + \ker(f)$, there exists a closed subspace $X_2 \subseteq \ker(f)$ such that $Z = \mathrm{Span}\{x, y, z\} \oplus X_2$. Clearly, we have $N_{|X_2} = 0$. If we put $X_1 = \mathrm{Span}\{x, y, z\}$, we get that

$$
N_{|X_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Now, assume that $\dim \mathrm{ran}(N_\mathrm{o}) \geqslant 2$. Then, there exist $x_1, x_2 \in Z$ such that $N_\mathrm{o}x_1$ and $N_\mathrm{o}x_2$ are linearly independent. Since $N_\mathrm{o}^2 = 0$, the vectors $\{x_1, N_\mathrm{o}x_1, x_2, N_\mathrm{o}x_2\}$ are linearly independent. Let $f_1, f_2 \in Z^*$ be such that

$$
\begin{cases} f_1(x_1) = f_2(x_2) = 0, \\ f_1(N_\mathrm{o}x_1) = f_2(N_\mathrm{o}x_2) = 1, \\ f_i(N_\mathrm{o}^k x_j) = 0 & \text{for } 1 \leqslant i \neq j \leqslant 2 \text{ and } 0 \leqslant k \leqslant 1. \end{cases}
$$

Consider the operator $Q \in \mathcal{B}(Z)$ defined by

$$
Q = x_1 \otimes f_1 N_\mathrm{o} + N_\mathrm{o}x_1 \otimes f_1 + x_2 \otimes f_2 N_\mathrm{o} + N_\mathrm{o}x_2 \otimes f_2.
$$

One can easily check that $Q^2 = Q$ and $N_\mathrm{o}Q = QN_\mathrm{o} = N_\mathrm{o}x_1 \otimes f_1 N_\mathrm{o} + N_\mathrm{o}x_2 \otimes f_2 N_\mathrm{o}$. Thus, $Z = \ker(I - Q) \oplus \ker(Q)$ and $\ker(Q)$ is a closed N_o-invariant subspace. On the other hand, we verify that $\ker(I - Q) = \mathrm{Span}\{x_1, N_\mathrm{o}x_1, x_2, N_\mathrm{o}x_2\}$. Consequently, the desired subspaces are $X_1 = \ker(I - Q)$ and $X_2 = \ker(Q)$. $\qquad\square$

Now we give the proof of Theorem 2.1.

P r o o f of Theorem 2.1. (1) Assume that $\dim \operatorname{ran}(P) \geqslant 3$. It follows immediately from Proposition 2.2 that $\mathbb{C}P \subseteq \mathcal{P}(\omega(P)) \subseteq \mathcal{P}_c(\omega(P))$. Let us establish that $\mathcal{P}_c(\omega(P)) \subseteq \mathbb{C}P$. Let $A \in \mathcal{P}_c(\omega(P))$. Then $T + A \in \omega(P)$ for every $T \in \omega(P)$ commuting with A. In particular, for $T = 0$, we have $A \in \omega(P)$, and so

$$AP = PA = A \quad \text{and} \quad (A - aP)(A - bP) = 0$$

for some $a, b \in \mathbb{C}$. It follows that $\ker(P) \subseteq \ker(A)$ and $\operatorname{ran}(P)$ is a closed A-invariant subspace. Letting $Y = \ker(P)$ and $Z = \operatorname{ran}(P)$, we note that $X = Y \oplus Z$ because $P^2 = P$.

Assume to the contrary that $A \notin \mathbb{C}P$. Hence, if $a = b$, then $(A - aP)^2 = 0$ and $A - aP \neq 0$. Without loss of generality, we may assume that $a = 0$. It follows from Lemma 2.4 that Z is a direct sum of two closed A-invariant subspaces X_1 and X_2 such that $\dim X_1 = 3$ or 4, and

$$A_{|X_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad A_{|X_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

respectively. Note that $X = Y \oplus X_1 \oplus X_2$ and $P = 0 \oplus I \oplus I$ with respect to this decomposition. Consider the operator $T \in \mathcal{B}(X)$ given by $T_{|Y} = 0$, $T_{|X_2} = I$ and

$$T_{|X_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{or} \quad T_{|X_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

respectively. Clearly, we have $TP = PT = T$ and $TA = AT$. Since $(T - P)_{|Y} = 0$ and $(T - P)_{|X_2} = 0$, we have $(T - P)(T - 2P)_{|Y \oplus X_2} = 0$. For the subspace X_1, we have

$$(T - P)_{|X_1}(T - 2P)_{|X_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

or

$$(T - P)_{|X_1}(T - 2P)_{|X_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0,$$

respectively. Thus, $(T - P)(T - 2P) = 0$, and so $T \in \omega(P)$. As above, we get easily that $(T + A - P)^2(T + A - 2P)^2 = 0$, $(T + A - P)^2 \neq 0$, $(T + A - 2P)^2 \neq 0$ and $(T + A - P)(T + A - 2P) \neq 0$. Thus, $T + A \notin \omega(P)$, which is a contradiction.

Suppose that $a \neq b$. From $(A - aP)(A - bP) = 0$, we obtain that

$$(A_o - aI)(A_o - bI) = 0,$$

where $A_o = A_{|Z}$. It follows that $Z = \ker(A_o - aI) \oplus \ker(A_o - bI)$. Without loss of generality, we can assume that $\dim \ker(A_o - aI) \geqslant 2$. Let $\ker(A_o - aI)$ be a direct sum of two closed nontrivial subspaces L_1 and L_2. With respect to the decomposition $X = Y \oplus L_1 \oplus L_2 \oplus \ker(A_o - bI)$, we can write $A = 0 \oplus aI \oplus aI \oplus bI$. Let $S \in \mathcal{B}(X)$ given by

$$S = 0 \oplus 0 \oplus 2(a - b)I \oplus 2(a - b)I.$$

Clearly, $SA = AS$. Since $P = 0 \oplus I \oplus I \oplus I$, we have $S \in \omega(P)$, but

$$S + A = 0 \oplus aI \oplus (3a - 2b)I \oplus (2a - b)I$$

does not belong to $\omega(P)$. This contradiction shows that $A \in \mathbb{C}P$.

(2) Assume that $\dim \operatorname{ran}(P) \leqslant 2$. Clearly, we have $\mathcal{P}(\omega(P)) \subseteq \mathcal{P}_c(\omega(P)) \subseteq \omega(P)$. Let $T, S \in \omega(P)$ be nonzero. Then, $(T + S)P = P(T + S) = T + S$, $\ker(P) \subseteq \ker(T + S)$ and $\operatorname{ran}(P)$ is a closed $(T + S)$-invariant subspace. Since $\dim \operatorname{ran}(P) \leqslant 2$, then $(T + S)_{|\operatorname{ran}(P)}$ is a quadratic matrix. Thus, there exist $\alpha, \beta \in \mathbb{C}$ such that

$$(T + S - \alpha I)(T + S - \beta I)_{|\operatorname{ran}(P)} = 0.$$

It follows from $X = \ker(P) \oplus \operatorname{ran}(P)$ that $(T + S - \alpha P)(T + S - \beta P) = 0$, and hence $T + S \in \omega(P)$. This shows that $\omega(P) \subseteq \mathcal{P}(\omega(P))$. $\qquad \square$

As an immediate consequence of Theorem 2.1, we rediscover the result of [8], Proposition 2.4 establishing the perturbation class of all quadratic operators in $\mathcal{B}(X)$:

Corollary 2.5. *Let $\omega(I)$ be the set of all quadratic operators in $\mathcal{B}(X)$. Then*

$$\mathcal{P}(\omega(I)) = \mathcal{P}_c(\omega(I)) = \mathbb{C}I.$$

3. Generalized quadraticity of linear combinations in $\omega(P)$

We start this section by the following example which shows that $\omega(P)$ is not stable under linear combinations of square-zero operators in $\omega(P)$.

E x a m p l e 3.1. Write $X = \ker(P) \oplus \operatorname{ran}(P)$, and let $\operatorname{ran}(P)$ be a direct sum of two closed subspaces X_1 and X_2 such that $\dim X_1 = 3$. Consider the operators $T, S \in \mathcal{B}(X)$ given by $T_{|\ker(P)} = T_{|X_2} = S_{|\ker(P)} = S_{|X_2} = 0$,

$$T_{|X_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S_{|X_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

with respect to an arbitrary basis of X_1. Clearly, T and S are square-zero operators, $TP = PT = T$, and $SP = PS = S$. Thus, $T, S \in \omega(P)$. However, $(T + S)^3 = 0$ and $(T + S)^2 \neq 0$. Consequently, the operator $T + S$ does not belong to $\omega(P)$.

The following proposition provides necessary and sufficient conditions for the stability of $\omega(P)$ under linear combinations of two square-zero operators in $\omega(P)$.

Proposition 3.2. *Let $N, M \in \mathcal{N}_2(P)$. Then the following assertions are equivalent:*

(1) $\alpha N + \beta M \in \omega(P)$ *for all $\alpha, \beta \in \mathbb{C}$;*
(2) $N + M \in \omega(P)$;
(3) *there exists $c \in \mathbb{C}$ such that $NM + MN = cP$.*

P r o o f. (1) $\Rightarrow$ (2) Is obvious.

(2) $\Rightarrow$ (3) Suppose that $N + M \in \omega(P)$. There is no loss of generality in assuming that $N + M \neq 0$. Then, there exist $a, b \in \mathbb{C}$ such that

$$(N + M - aP)(N + M - bP) = 0.$$

Since $NP = PN = N$ and $MP = PM = M$, it follows that

(3.1) $$(a + b)N + (a + b)M - NM - MN - abP = 0.$$

Hence, it suffices to show that $a + b = 0$. Assume to the contrary that $a + b \neq 0$. Then, left and right multiplication by N in (3.1) gives

(3.2) $$(a + b)NM - NMN - abN = (a + b)MN - NMN - abN = 0.$$

This implies that $(a + b)NM = (a + b)MN$, and hence $NM = MN$. Thus, equation (3.2) becomes $(a + b)MN - abN = 0$. Now, multiplying this equation by M,

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we get that $abNM = 0$. Clearly, these two last equations imply that $NM = 0$, and so $(N + M)^2 = 0$. It follows from (3.1) that $N + M = ab(a + b)^{-1}P$, which leads to a contradiction with $(N + M)^2 = 0$ and $N + M \neq 0$.

(3) $\Rightarrow$ (1) Assume that $NM + MN = cP$ for some $c \in \mathbb{C}$. Let $\alpha, \beta \in \mathbb{C}$, and let $\lambda \in \mathbb{C}$ be such that $\lambda^2 = \alpha\beta c$. One can easily verify that

$$(\alpha N + \beta M - \lambda P)(\alpha N + \beta M + \lambda P) = 0,$$

and so $\alpha N + \beta M \in \omega(P)$. This completes the proof. $\qquad\square$

In the commutative case, we reformulate the previous result as follows.

Corollary 3.3. *Let $N, M \in \mathcal{N}_2(P)$ be such that $NM = MN$. Then the following assertions are equivalent:*

(1) $\alpha N + \beta M \in \omega(P)$ *for all* $\alpha, \beta \in \mathbb{C}$;
(2) $N + M \in \omega(P)$;
(3) $NM = 0$.

P r o o f. (1) $\Rightarrow$ (2), and (3) $\Rightarrow$ (1) follow immediately from Proposition 3.2.

(2) $\Rightarrow$ (3) It follows again from Proposition 3.2 that there exists $c \in \mathbb{C}$ such that $NM + MN = cP$, and so $NM = 2^{-1}cP$ because $NM = MN$. Now, we have

$$4^{-1}c^2 P = (NM)^2 = N^2 M^2 = 0,$$

so $c = 0$. This finishes the proof. $\qquad\square$

In the following proposition, we study the stability of $\omega(P)$ under linear combinations of a square-zero operator and an idempotent operator in $\omega(P)$.

Proposition 3.4. *Let $N \in \mathcal{N}_2(P)$ be nonzero, and let $Q \in \mathcal{I}(P)$ be such that $Q \notin \mathbb{C}P$. Then the following assertions are equivalent:*

(1) $\alpha N + \beta Q \in \omega(P)$ *for all* $\alpha, \beta \in \mathbb{C}$;
(2) $N + Q \in \omega(P)$;
(3) *there exists $c \in \mathbb{C}$ such that $N - NQ - QN = cP$.*

P r o o f. (1) $\Rightarrow$ (2) Is obvious.

(2) $\Rightarrow$ (3) Suppose that $N + Q \in \omega(P)$. Then $N + Q \neq 0$, and there exist $a, b \in \mathbb{C}$ such that $(N + Q - aP)(N + Q - bP) = 0$. Since $N^2 = 0$, $Q^2 = Q$, $NP = PN = N$ and $QP = PQ = Q$, it follows that

(3.3) $\qquad -(a + b)N + (1 - a - b)Q + NQ + QN + abP = 0.$

Assume that $1 - a - b \neq 0$. Then, left and right multiplication by N in (3.3) gives

$$(3.4) \quad (1 - a - b)NQ + NQN + abN = (1 - a - b)QN + NQN + abN = 0.$$

This implies that $(1 - a - b)NQ = (1 - a - b)QN$, and so $NQ = QN$. It follows from (3.3) that $-(a + b)N + (1 - a - b)Q + 2NQ + abP = 0$. Thus,

$$(1 - a - b)Q + abP = (a + b)N - 2NQ.$$

But, as $((a + b)N - 2NQ)^2 = 0$, we get that $((1 - a - b)Q + abP)^2 = 0$. On the other hand, since $QP = PQ = Q$, we have $\ker(P) \subseteq \ker(Q)$ and $\operatorname{ran}(P)$ is a Q-invariant subspace. The assumption $Q \notin \mathbb{C}P$ implies that there exists a nonzero vector $x \in \operatorname{ran}(P)$ such that $Qx = 0$. Therefore, $((1 - a - b)Q + abP)^2 x = (ab)^2 x = 0$, and so $ab = 0$. Hence

$$((1 - a - b)Q + abP)^2 = ((1 - a - b)Q)^2 = (1 - a - b)^2 Q = 0,$$

which implies that $Q = 0$, a contradiction. Hence $a + b = 1$, and so $-N + NQ + QN + abP = 0$.

(3) $\Rightarrow$ (1) Assume that $N - NQ - QN = ac^{-1}P$ for some $a \in \mathbb{C}$, and let $b \in \mathbb{C}$. One can easily verify that $(bN + cQ)^2 - c(bN + cQ) + abP = 0$, and so $bN + cQ \in \omega(P)$. This completes the proof. $\square$

The following surprising corollary shows that $N + Q \notin \omega(P)$ for any commuting $N \in \mathcal{N}_2(P)$ and $Q \in \mathcal{I}(P) \setminus \mathbb{C}P$.

Corollary 3.5. *Let* $N \in \mathcal{N}_2(P)$ *be nonzero, and let* $Q \in \mathcal{I}(P)$ *be such that* $Q \notin \mathbb{C}P$ *and* $NQ = QN$. *Then* $\alpha N + \beta Q \notin \omega(P)$ *for all* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$.

P r o o f. Suppose on the contrary that there exist $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ such that $\alpha N + \beta Q \in \omega(P)$. Then, it follows from Proposition 3.4 that there exists $c \in \mathbb{C}$ such that

$$(3.5) \quad N - 2NQ = cP$$

because $NQ = QN$. Multiplying equation (3.5) by N, we get that $0 = cN$, which implies that $c = 0$. Thus, equation (3.5) becomes $N - 2NQ = 0$. Multiplying this equation by Q, we obtain that $QN = 0$, and thus $N = 0$. This contradiction completes the proof. $\square$

We give in the following proposition necessary and sufficient conditions on two idempotent operators $R, S \in \mathcal{I}(P)$ ensuring that the generalized quadraticity of $aR + bS$ and that of $R + S$ are equivalent.

Proposition 3.6. *Let $R, S \in \mathcal{I}(P)$ be nonzero such that $RS \neq SR$. Then the following assertions are equivalent:*

(1) $aR + bS \in \omega(P)$ *for all $a, b \in \mathbb{C}$;*
(2) $R + S \in \omega(P)$;
(3) *there exists $c \in \mathbb{C}$ such that $RS + SR = R + S + cP$.*

P r o o f. (1) $\Rightarrow$ (2) Is obvious.

(2) $\Rightarrow$ (3) Assume that $R + S \in \omega(P)$. Then, it follows that $(R + S - \alpha P) \times (R + S - \beta P) = 0$, and so

$$(3.6) \qquad (1 - \alpha - \beta)R + (1 - \alpha - \beta)S + RS + SR + \alpha\beta P = 0.$$

Hence, right and left multiplication by R in (3.6) gives

$$(3.7) \qquad (1 - \alpha - \beta + \alpha\beta)R + (2 - \alpha - \beta)SR + RSR$$
$$= (1 - \alpha - \beta + \alpha\beta)R + (2 - \alpha - \beta)RS + RSR = 0.$$

This implies that $(2 - \alpha - \beta)(RS - SR) = 0$, and thus $2 - \alpha - \beta = 0$ because $RS \neq SR$. Then (3.6) becomes $-R - S + RS + SR + \alpha\beta P = 0$, the desired relation.

(3) $\Rightarrow$ (1) Suppose that $RS + SR = R + S + cP$ for some scalar $c \in \mathbb{C}$. For every $a, b \in \mathbb{C}$, we have

$$(aR + bS)^2 = a^2R + ab(RS + SR) + b^2S$$
$$= a^2R + abR + abS + abcP + b^2S$$
$$= (a + b)(aR + bS) + abcP.$$

So $aR + bS \in \omega(P)$. This completes the proof. $\qquad\qquad\square$

In the following proposition, we prove that there are two possible linear combinations of two commuting idempotent operators $R, S \in \mathcal{I}(P)$ guarantee their generalized quadraticity in $\omega(P)$, namely, $R + S$ and $-R + S$.

Proposition 3.7. *Let $R, S \in \mathcal{I}(P) \backslash \mathbb{C}P$ be such that $RS = SR$ and $S \notin \mathbb{C}P + \mathbb{C}R$. Then, for every nonzero $a \in \mathbb{C}$, the following assertions are equivalent:*

(1) $aR + S \in \omega(P)$;
(2) *there exist $c_1, c_2 \in \{0, 1\}$ such that $RS = c_1R + c_2S - c_1c_2P$ and $a = 1 - 2|c_1 - c_2|$.*

P r o o f. $(1) \Rightarrow (2)$ Assume that $aR + S \in \omega(P)$. Then, there exist $\alpha, \beta \in \mathbb{C}$ such that $(aR + S - \alpha P)(aR + S - \beta P) = 0$, and so

(3.8)
$$a(a - \alpha - \beta)R + (1 - \alpha - \beta)S + 2aRS + \alpha\beta P = 0.$$

Since $R(R - P) = 0$, then right multiplication by $R - P$ in (3.8) gives

(3.9)
$$(1 - \alpha - \beta)S(R - P) + \alpha\beta(R - P) = 0.$$

Multiplying this equation by $(S - P)$, we get that $\alpha\beta(R - P)(S - P) = 0$. We have two cases:

Case 1. If $\alpha\beta \neq 0$, then $(R - P)(S - P) = 0$ and so $RS = R + S - P$. That is, $RS = c_1 R + c_2 S - c_1 c_2 P$ with $c_1 = c_2 = 1$. Now, replacing RS by $R + S - P$ in (3.8) we obtain that

$$(a(a - \alpha - \beta) + 2a)R + (1 - \alpha - \beta + 2a)S + (\alpha\beta - 2a)P = 0.$$

Since $S \notin \mathbb{C}P + \mathbb{C}R$ and $R \notin \mathbb{C}P$, then

$$a(a - \alpha - \beta) + 2a = 1 - \alpha - \beta + 2a = \alpha\beta - 2a = 0.$$

One gets easily that $a = 1$.

Case 2. If $\alpha\beta = 0$, then from (3.9) we have $(1 - \alpha - \beta)S(R - P) = 0$. If $1 - \alpha - \beta \neq 0$, then $SR = S$ and (3.8) becomes $a(a - \alpha - \beta)R + (1 - \alpha - \beta + 2a)S = 0$, and so

$$a(a - \alpha - \beta) = 1 - \alpha - \beta + 2a = 0,$$

which implies that $a = -1$.

If $1 - \alpha - \beta = 0$, then (3.8) becomes $a(a - \alpha - \beta)R + 2aRS = 0$, and so $RS = 2^{-1}(1 - a)R$. Hence

$$RS = (RS)^2 = (2^{-1}(1 - a))^2 R = 2^{-1}(1 - a)R.$$

This implies that either $a = 1$ and $RS = 0$, or $a = -1$ and $RS = R$.

$(2) \Rightarrow (1)$ We deal with two cases: If $c_1 \neq c_2$, then $a = -1$, $c_1 + c_2 = 1$ and $c_1 c_2 = 0$. It follows that

$$(-R + S)^2 = R + S - 2RS = R + S - 2(c_1 R + c_2 S)$$
$$= (2c_1 - 1)(-R + S) = (2c_1 - 1)(-R + S)P.$$

This means that $-R + S$ belongs to $\omega(P)$.

If $c_1 = c_2$, then $a = 1$ and $c_1 c_2 = c_1$. It follows that

$$(R + S)^2 = R + S + 2RS = R + S + 2(c_1 R + c_2 S - c_1 c_2 P) = (2c_1 + 1)(R + S) - 2c_1 P.$$

So $(R + S - 2c_1 P)(R + S - P) = 0$. This completes the proof. $\qquad \square$

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Author's address: *Khalid Souilah*, Département Math, Labo Ibn Al-Banna, Faculté des Sciences d'Oujda, Université Mohammed Premier, 60000 Oujda, Maroc, e-mail: `k.souilah@ump.ac.ma`.