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GENERALIZATIONS ON THE RESULTS OF CAO AND ZHANG

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Abstract. We establish some uniqueness results for meromorphic functions when two nonlinear differential polynomials $P(f)\prod_{i=1}^{k}(f^{(i)})^{n_i}$ and $P(g)\prod_{i=1}^{k}(g^{(i)})^{n_i}$ share a nonzero polynomial with certain degree and our results improve and generalize some recent results in Y.-H. Cao, X.-B. Zhang (2012). Also we exhibit two examples to show that the conditions used in the results are sharp.

Keywords: meromorphic function; uniqueness; weighted sharing; differential polynomial

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1. Introduction and preliminary results

In this entire paper we mean by meromorphic functions those complex valued functions which have poles as the only singularities in $\mathbb{C}$. In this paper we use the standard notations of the value distribution theory (see [8]). We define the function $T(r)$ by $T(r)=\max\{T(r,f),T(r,g)\}$. The function $S(r)$ is defined by $S(r)=o(T(r))$ as $r\to\infty$ outside of a possible exceptional set of finite linear measure. If $T(r,a)=S(r,f)$, then we say that $a(z)$ is a small function with respect to $f(z)$. If $f(z_0)=z_0$, then z_0 is called a fixed point of $f(z)$.

Let $k\in\mathbb{N}\cup\{0\}\cup\{\infty\}$. For $a\in\mathbb{C}\cup\{\infty\}$, we denote by $E_k(a;f)$ the set of all a-points of $f(z)$, where an a-point of multiplicity m is counted m times if $m\leqslant k$ and $k+1$ times if $m>k$. If we have for two meromorphic functions $f(z)$ and $g(z)$ that $E_k(a;f)=E_k(a;g)$, then we say that $f(z)$ and $g(z)$ share a with weight k. The IM and CM sharing correspond to the weight 0 and ∞, respectively. If $a(z)$ is a small function we define that $f(z)$ and $g(z)$ share $a(z)$ IM or $a(z)$ CM or with weight l depending on whether $f(z)-a(z)$ and $g(z)-a(z)$ share $(0,0)$ or $(0,\infty)$ or $(0,l)$, respectively.

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The following well known theorem in value distribution theory was posed by Hayman (see [8]) and settled by several authors almost at the same time, see [3]–[5].

Theorem A. *Let $f(z)$ be a transcendental meromorphic function and $n \in \mathbb{N}$. Then $f^n(z)f'(z) = 1$ has infinitely many solutions.*

To investigate the uniqueness result corresponding to Theorem A, both Fang and Hua in [6], and Yang and Hua in [16] obtained the following result.

Theorem B. *Let $f(z)$ and $g(z)$ be two non-constant entire (or meromorphic) functions and $n \in \mathbb{N}$ such that $n \geqslant 6$ (or $n \geqslant 11$, respectively). If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share 1 CM, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, $c, c_1, c_2 \in \mathbb{C}$ such that $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ such that $t^{n+1} = 1$.*

In 2002 Fang and Qiu (see [7]) considered the uniqueness problems of entire or meromorphic functions having fixed points and they obtained the following result.

Theorem C. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic (or entire) functions and $n \in \mathbb{N}$ such that $n \geqslant 11$ (or $n \geqslant 6$, respectively). If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share z CM, then either $f(z) = c_1 e^{cz^2}$ and $g(z) = c_2 e^{-cz^2}$, $c, c_1, c_2 \in \mathbb{C}$ such that $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ such that $t^{n+1} = 1$.*

We now recall the following results due to Xu et al. (see [13]) or Zhang and Li (see [20]), respectively.

Theorem D. *Let $f(z)$ be a transcendental meromorphic function and $k \in \mathbb{N}$, $n \in \mathbb{N} \setminus \{1\}$. Then $f^n(z)f^{(k)}(z)$ takes every finite nonzero value infinitely many times or has infinitely many fixed points.*

Also the following recent results are due to Cao and Zhang, see [4].

Theorem E. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions whose zeros are of multiplicities at least k, where $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that $n > \max\{2k-1, k+4/k+4\}$. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share z CM, $f(z)$ and $g(z)$ share ∞ IM, then one of the following two conclusions holds:*
(i) $f^n(z)f^{(k)}(z) \equiv g^n(z)g^{(k)}(z)$;
(ii) $f(z) = c_1 e^{cz^2}$ *and* $g(z) = c_2 e^{-cz^2}$, *where* $c, c_1, c_2 \in \mathbb{C}$ *such that* $4(c_1c_2)^{n+1}c^2 = -1$.

Theorem F. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions whose zeros are of multiplicities at least k, where $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that $n > \max\{2k-1, k+4/k+4\}$. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share 1 CM, $f(z)$ and $g(z)$ share ∞ IM, then one of the following two conclusions holds:*
(i) $f^n(z)f^{(k)}(z) \equiv g^n(z)g^{(k)}(z)$;
(ii) $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, *where* $c_3, c_4, d \in \mathbb{C}$ *such that* $(-1)^k(c_3c_4)^{n+1}d^{2k} = 1$.

Theorem G. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions whose zeros are of multiplicities at least $k+1$, where $k \in \mathbb{N}$ with $1 \leqslant k \leqslant 5$. Let $n \in \mathbb{N}$ such that $n \geqslant 10$. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share 1 CM, $f^{(k)}(z)$ and $g^{(k)}(z)$ share 0 CM, $f(z)$ and $g(z)$ share ∞ IM, then one of the following two conclusions holds:*

(i) $f(z) \equiv tg(z)$, $t \in \mathbb{C} \setminus \{0\}$ *such that* $t^{n+1} = 1$;

(ii) $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, *where* $c_3, c_4, d \in \mathbb{C}$ *such that* $(-1)^k (c_3 c_4)^{n+1} d^{2k} = 1$.

Now the following questions are inquisitive to any researcher:

Question 1. Is it possible to reduce the lower bound of n in Theorems E–G?

Question 2. Is it possible to weaken more the condition "Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions whose zeros are of multiplicities at least $k+1$, where $k \in \mathbb{N}$" in Theorem G?

Question 3. Does Theorem G hold for $k \geqslant 6$?

Question 4. Can one further deduce generalized forms of Theorems E–G?

2. Main results and some definitions

Throughout this paper, for the sake of simplicity we use the following notations

$$n_i^* = \begin{cases} 0 & \text{if } n_i = 0, \\ 1 & \text{if } n_i \neq 0, \end{cases} \quad \text{and} \quad n_i^{**} = \begin{cases} 0 & \text{if } n_i = 0, \\ n_i & \text{if } n_i \neq 0, \end{cases}$$

where $n_i \in \mathbb{N} \cup \{0\}$ for $i = 1, 2, \ldots, k-1$ and $k, n_k \in \mathbb{N}$. Also we use $t = \sum\limits_{i=1}^{k} n_i^*$, $m = \sum\limits_{i=1}^{k} i n_i^*$, $s = \sum\limits_{i=1}^{k} n_i^{**}$, $m_1 = \sum\limits_{i=1}^{k} i n_i^{**}$ and $n^* = \min\{i \colon i \in \{1, \ldots, k\} \text{ with } n_i \neq 0\}$.

In this paper we use $P(z)$ to denote an arbitrary non-constant polynomial of degree n,

$$P(z) = a_n (z - c_1)^{d_1} (z - c_2)^{d_2} \ldots (z - c_{s_1})^{d_{s_1}}, \tag{2.1}$$

where $a_n \in \mathbb{C} \setminus \{0\}$ and $c_j \in \mathbb{C}$ $(j = 1, 2, \ldots, s_1)$ are distinct; $d_1, d_2, \ldots, d_{s_1}$, $n \in \mathbb{N}$ with $\sum\limits_{i=1}^{s_1} d_i = n$. Let $d = \max\{d_1, d_2, \ldots, d_{s_1}\}$ and c be the corresponding zero of $P(z)$ with multiplicity d. We define

$$P_1(z) = a_n \prod_{\substack{i=1 \\ d_i \neq d}}^{s_1} (z - c_i)^{d_i} = b_{m_2} z^{m_2} + b_{m_2 - 1} z^{m_2 - 1} + \ldots + b_0,$$

where $a_n = b_{m_2}$ and $m_2 = n-d$. Obviously $P(z) = (z-c)^d P_1(z)$. We also use $P_2(z_1)$ as an arbitrary nonzero polynomial defined by

$$P_2(z_1) = a_n \prod_{\substack{i=1 \\ d_i \neq d}}^{s_1} (z_1 + c - c_i)^{d_i} = e_{m_2} z_1^{m_2} + e_{m_2 - 1} z_1^{m_2 - 1} + \ldots + e_0,$$

where $z_1 = z - c$ and $\deg(P_2) = m_2 \geqslant 0$. Obviously $P(z) = z_1^d P_2(z_1)$. Suppose $\Gamma_1 = m_3 + m_4$ and $\Gamma_2 = m_3 + 2m_4$, where m_3 is the number of simple zeros of $P_1(z)$ and m_4 is the number of multiple zeros of $P_1(z)$. We define $k^* \in \mathbb{N}$ as

$$k^* = \begin{cases} k & \text{if } P_2(z_1) \equiv e_i z_1^i \not\equiv 0, \\ k+1 & \text{if } P_2(z_1) \not\equiv e_i z_1^i \not\equiv 0 \end{cases} \tag{2.2}$$

for $i \in \{0, 1, 2, \ldots, m_2\}$. Again we use $p(z)$ to denote a nonzero polynomial defined by

$$p(z) = a(z - z_1)^{l_1}(z - z_2)^{l_2} \ldots (z - z_{t_1})^{l_{t_1}}, \tag{2.3}$$

where $a \in \mathbb{C} \cup \{0\}$, $z_i \in \mathbb{C}$, $i = 1, 2, \ldots, t_1$, are distinct and $l_1, l_2, \ldots, l_{t_1} \in \mathbb{N}$ such that either $\sum_{i=1}^{t_1} l_i \leqslant n + s - 1$ or $l_i \leqslant n - 1$ for all $i = 1, 2, \ldots, t_1$.

Throughout the paper we consider $\mathcal{F}(z) = \prod_{i=1}^{k} (f^{(i)}(z))^{n_i}$ and $\mathcal{F}_1(z) = \prod_{i=1}^{k} (f_1^{(i)}(z))^{n_i}$, where $f_1(z) = f(z) - c$; $\mathcal{G}(z)$ and $\mathcal{G}_1(z)$ are defined similarly.

Henceforth, we obtain the following results, keeping all the possible answers of the above questions, into background, which significantly improves and generalizes Theorems E, F and G.

Theorem 2.1. *Let $f(z)$ be a transcendental meromorphic function such that zeros of $f(z) - c$ are of multiplicities at least k^*, where k^* is defined in* (2.2), *and let $a(z)$ ($\not\equiv 0, \infty$) be a small function of $f(z)$. Also let $n, s, n_k \in \mathbb{N}$ and $n_i, \Gamma_1 \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k-1$. If $n > s + \Gamma_1 + 1/k^*$, then $P(f(z))\mathcal{F}(z) - a(z)$ has infinitely many zeros, where $P(z)$ is defined as in* (2.1).

Theorem 2.2. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that zeros of $f(z) - c$ and $g(z) - c$ are of multiplicities at least k, where $k \in \mathbb{N}$. Let $P(z)$ and $p(z)$ be defined as in* (2.1) *and* (2.3), *respectively, and let $n, m, m_1, k_1, s, t, n_k \in \mathbb{N}$, $n_i, \Gamma_2 \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k-1$, be such that*

$$n \geqslant 4\Gamma_2 + 2m + 2s + 1 + \frac{m_1}{2} + \frac{2}{k^*} \quad \text{and} \quad k_1 = \left[\frac{3 + m_1 - s}{n + s + m_1 - 2m - 1}\right] + 3.$$

If $P(f(z))\mathcal{F}(z)-p(z)$, $P(g(z))\mathcal{G}(z)-p(z)$ *share* $(0,k_1)$ *and* $f(z)$, $g(z)$ *share* $(\infty,0)$, *then one of the following conclusions holds:*

(1) $f(z)-c\equiv t(g(z)-c)$ *with* $t^{d_0}=1$, *where* $d_0=\gcd(d+p\colon\ p\in\{0,1,\dots,m_2\}$ *with* $e_p\neq 0)$,

(2) $P(f(z))\mathcal{F}(z)\equiv P(g(z))\mathcal{G}(z)$.

Theorem 2.3. *Let* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions such that the zeros of* $f(z)-c$ *and* $g(z)-c$ *are of multiplicities at least* k^*, *where* k^* *is defined in* (2.2). *Let* $P(z)$ *and* $p(z)$ *be defined as in* (2.1) *and* (2.3), *respectively, and let* $n,m,m_1,s,t,n_k\in\mathbb{N}$, $n_i,m_2,\Gamma_2\in\mathbb{N}\cup\{0\}$, $i=1,2,\dots,k-1$, *be such that*

$$n\geqslant 4\Gamma_2+2m+2s+1+\frac{m_1}{2}+\frac{2}{k^*}\quad\text{and}\quad k_1=\Big[\frac{3+m_1-s}{n+s+m_1-2m-1}\Big]+3.$$

Suppose $(k-1)s-m_1<0$ *when at least one of* $n_1,n_2,\dots,n_{k-1}$ *is nonzero. If* $P(f(z))\mathcal{F}(z)-p(z)$, $P(g(z))\mathcal{G}(z)-p(z)$ *share* $(0,k_1)$ *and* $f(z)$, $g(z)$ *share* $(\infty,0)$, *then one of the following cases holds:*

(1) *If* $P_2(z_1)\equiv e_iz_1^i\not\equiv 0$ *for some* $i\in\{0,1,2,\dots,m_1\}$ *and* $f^{(n^*)}(z)$, $g^{(n^*)}(z)$ *share* $(0,\infty)$, *then* $f(z)-c\equiv t(g(z)-c)$, *where* $t\in\mathbb{C}\setminus\{0\}$ *such that* $t^{d+s+i}=1$ *for some* $i\in\{0,1,2,\dots,m_1\}$.

(2) *If* $P_2(z_1)\not\equiv e_iz_1^i$ *for* $i\in\{0,1,2,\dots,m_1\}$, $(f^{(i)}(z))^{n_i^*}$, $(g^{(i)}(z))^{n_i^*}$ *share* $(0,\infty)$, *where* $i=1,2,\dots,k$, *and* $f(z)$, $g(z)$ *share* $(c,0)$, *then* $f(z)-c\equiv t(g(z)-c)$ *for* $t\in\mathbb{C}\setminus\{0\}$ *such that* $t^{d+s}=1$.

Remark 2.1. Our results generalise Theorems E, F and G in different directions. For examples we consider $P(f(z))$ instead of $f^n(z)$ and $\mathcal{F}(z)$ instead of $f^{(k)}(z)$.

Remark 2.2. Let us take $d=n$, $c=0$, $P_2(z_1)=1$ and $n^*=k$. Then from Theorem 2.2 we can easily get a theorem which is the improvement of Theorem E and Theorem F.

Remark 2.3. Let us take $d=n$, $c=0$, $P_2(z_1)=1$ and $n^*=k$. Clearly $k^*=k$. Then from Theorem 2.3 we can easily get a theorem which is the improvement of Theorem G. Consequently Theorem G holds when zeros of $f(z)$ and $g(z)$ are of multiplicities at least k, where $k\in\mathbb{N}$.

Remark 2.4. It is easy to see that the condition "Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions having zeros of multiplicities at least $k\in\mathbb{N}$" in Theorem 2.3 is sharp by the following example.

E x a m p l e 2.1. Let $f(z) = c_1 e^{az}$ and $g(z) = c_2 e^{-az}$, where $a, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1^{n+2} = -c_2^{n+2}$ and $n \geqslant 14$. Note that

$$\mathcal{F}(z) = f'(z)f''(z) = c_1^2 a^3 e^{2az} \quad \text{and} \quad \mathcal{G}(z) = g'(z)g''(z) = -c_2^2 a^3 e^{-2az}.$$

Since $f(z)$ and $g(z)$ have no zeros, it follows that the condition "Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions having zeros of multiplicities at least $k \in \mathbb{N}$" does not hold. Here we see that $f(z)$, $g(z)$ share ∞ CM and $f'(z)$, $g'(z)$ share 0 CM. On the other hand we see that

$$f^n(z)f'(z)f''(z) - p(z) = c_1^{n+2} a^3 (e^{a(n+2)z} - 1)$$

and

$$g^n(z)g'(z)g''(z) - p(z) = -c_2^{n+2} a^3 (e^{-a(n+2)z} - 1),$$

where $p(z) = c_1^{n+2} a^3$. Clearly $f^n(z)f'(z)f''(z) - p(z)$ and $g^n(z)g'(z)g''(z) - p(z)$ share $(0, \infty)$, but $f(z) \not\equiv tg(z)$, where $t \in \mathbb{C} \setminus \{0\}$ with $t^{n+2} = 1$.

R e m a r k 2.5. It is easy to see that the conditions "$(f^{(i)}(z))^{n_i^*}$, $(g^{(i)}(z))^{n_i^*}$ share $(0, \infty)$, where $i = 1, 2, \ldots, k$" and "$f(z)$, $g(z)$ share $(c, 0)$" in Theorem 2.3 are sharp by the following example.

E x a m p l e 2.2. Let

$$P(z) = z^n((n+2)z - (n+1)), \quad f(z) = \frac{1 - h^{n+1}(z)}{1 - h^{n+2}(z)} \quad \text{and} \quad g(z) = h(z)\frac{1 - h^{n+1}(z)}{1 - h^{n+2}(z)},$$

where $h(z) = e^z - 1$ and $n \in \mathbb{N}$ with $n \geqslant 10$. Observe that $f(z)$ and $g(z)$ share (∞, ∞) but $f(z)$ and $g(z)$ do not share the value 0. Note that

$$f'(z) = \frac{h^n(z)h'(z)((n+2)h(z) - h^{n+2}(z) - (n+1))}{(1 - h^{n+2}(z))^2}$$

and

$$g'(z) = \frac{h'(z)(1 + (n+1)h^{n+2}(z) - (n+2)h^{n+1}(z))}{(1 - h^{n+2}(z))^2}.$$

This shows that $f'(z)$ and $g'(z)$ do not share the value 0. Also we observe that $f^{n+1}(z)(f(z) - 1) \equiv g^{n+1}(z)(g(z) - 1)$, i.e., $f^n(z)((n+2)f(z) - (n+1))f'(z) \equiv g^n(z)((n+2)g(z) - (n+1))g'(z)$. Therefore $f^n(z)((n+2)f(z) - (n+1))f'(z)$ and $g^n(z)((n+2)g(z) - (n+1))g'(z)$ share $(1, \infty)$, but $f(z) \not\equiv tg(z)$, where $t \in \mathbb{C} \setminus \{0\}$ with $t^{n+2} = 1$.

R e m a r k 2.6. The above example shows that the conclusion (2) in Theorem 2.2 cannot be removed.

We now explain some definitions and notations which are used in the paper.

Definition 2.1 ([12]). Let $p\in\mathbb{N}$ and $a\in\mathbb{C}\cup\{\infty\}$. $N(r,a;f\mid\geqslant p)$ ($\overline{N}(r,a;f\mid\geqslant p)$) denotes the *counting function* (*reduced counting function*) of those a-points of $f(z)$ whose multiplicities are not less than p. $N(r,a;f\mid\leqslant p)$ ($\overline{N}(r,a;f\mid\leqslant p)$) denotes the counting function (reduced counting function) of those a-points of $f(z)$ whose multiplicities are not greater than p.

Definition 2.2. We denote by $\overline{N}(r,a;f\mid=k)$ the *reduced counting function* of those a-points of $f(z)$ whose multiplicities are exactly k, where $k\in\mathbb{N}\setminus\{1\}$.

Definition 2.3 ([18]). For $a\in\mathbb{C}\cup\{\infty\}$ and $p\in\mathbb{N}$, we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geqslant 2)+\ldots+\overline{N}(r,a;f\mid\geqslant p)$. Clearly $N_1(r,a;f)=\overline{N}(r,a;f)$.

Definition 2.4 ([1]). Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions such that $f(z)$ and $g(z)$ share the value 1 IM. Let z_0 be a 1-point of $f(z)$ with multiplicity p, a 1-point of $g(z)$ with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of $f(z)$ and $g(z)$, where $p>q$, and by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of $f(z)$ and $g(z)$, where $p=q\geqslant 2$, and each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,1;g)$ and $\overline{N}_E^{(2}(r,1;g)$.

Definition 2.5 ([10]). Let $f(z)$ and $g(z)$ share the value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of $f(z)$ whose multiplicities differ from the multiplicities of the corresponding a-points of $g(z)$. Clearly $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

3. Lemmas

By the non-constant meromorphic functions $F(z)$ and $G(z)$, we construct the functions

$$H(z)=\Big(\frac{F''(z)}{F'(z)}-\frac{2F'(z)}{F(z)-1}\Big)-\Big(\frac{G''(z)}{G'(z)}-\frac{2G'(z)}{G(z)-1}\Big) \tag{3.1}$$

and

$$\begin{aligned} V(z)&=\Big(\frac{F'(z)}{F(z)-1}-\frac{F'(z)}{F(z)}\Big)-\Big(\frac{G'(z)}{G(z)-1}-\frac{G'(z)}{G(z)}\Big)\\ &=\frac{F'(z)}{F(z)(F(z)-1)}-\frac{G'(z)}{G(z)(G(z)-1)}. \end{aligned} \tag{3.2}$$

Lemma 3.1 ([15]). *Let $f(z)$ be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0), a_{n-1}(z),\ldots,a_0(z)$ be the small functions of $f(z)$. Then $T\Big(r,\sum\limits_{i=0}^{n}a_if^i\Big)=nT(r,f)+S(r,f)$.*

Lemma 3.2 ([19]). *Let $f(z)$ be a non-constant meromorphic function and $k, p \in \mathbb{N}$, then $N_p(r, 0; f^{(k)}) \leqslant N_{p+k}(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f)$.*

Lemma 3.3 ([11]). *If $N(r, 0; f^{(k)} \,|f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}(z)$ which are not the zeros of $f(z)$, where a zero of $f^{(k)}(z)$ is counted according to its multiplicity, then*

$$N(r, 0; f^{(k)} \,|f \neq 0) \leqslant k\overline{N}(r, \infty; f) + N(r, 0; f \,|< k) + k\overline{N}(r, 0; f \,|\geqslant k) + S(r, f).$$

Lemma 3.4 ([17], Theorem 1.24). *Let $f(z)$ be a non-constant meromorphic function and let $k \in \mathbb{N}$. If $f^{(k)}(z) \not\equiv 0$, then*

$$N(r, 0; f^{(k)}) \leqslant N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 3.5 ([17]). *Let $f_j(z)$, $j = 1, 2, 3$, be meromorphic and $f_1(z)$ be non-constant. Suppose that $\sum\limits_{j=1}^{3} f_j(z) \equiv 1$ and $\sum\limits_{j=1}^{3} N(r, 0; f_j) + 2\sum\limits_{j=1}^{3} \overline{N}(r, \infty; f_j) < (\lambda + o(1))T_1(r)$ as $r \to \infty$, $r \in I$, where I is a set of infinite linear measure, $\lambda < 1$ and $T_1(r) = \max\limits_{1 \leqslant j \leqslant 3} T(r, f_j)$. Then $f_2(z) \equiv 1$ or $f_3(z) \equiv 1$.*

Lemma 3.6 ([8]). *Let $f(z)$ be a non-constant meromorphic function and let $a_1(z)$, $a_2(z)$ be two small functions of $f(z)$. Then*

$$T(r, f) \leqslant \overline{N}(r, \infty; f) + \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + S(r, f).$$

Lemma 3.7 ([8]). *Suppose that $f(z)$ is a non-constant meromorphic function and $k \in \mathbb{N} \setminus \{1\}$. If $N(r, \infty; f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S(r, f'/f)$, then $f(z) = \mathrm{e}^{az+b}$, where a $(\neq 0)$, $b \in \mathbb{C}$.*

Lemma 3.8. *Let $f(z)$ be a transcendental meromorphic function and $n, n_k \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$ for $i = 1, 2, \ldots, k-1$. Then $\varphi(z) = P(f(z))\mathcal{F}(z)$ is non-constant, where $P(z)$ is defined by (2.1).*

P r o o f. If possible, let $\varphi(z)$ be constant. Then $\overline{N}(r, 0; P(f)) = S(r, f)$ and $\overline{N}(r, \infty; f) = S(r, f)$. If $s_1 \geqslant 2$, by the second fundamental theorem we arrive at a contradiction.

Next we suppose $s_1 = 1$, i.e., $P(z) = a_n(z - c)^n$. Therefore $\varphi(z) = a_n f_1^n(z)\mathcal{F}_1(z)$. Clearly

$$\frac{1}{f_1^{n+s}(z)} \equiv a_n \frac{\mathcal{F}_1(z)}{f_1^s(z)} \frac{1}{\varphi(z)}.$$

Using Lemma 3.1, we now see that

$$
\begin{aligned}
(n+s)T(r,f_1) &\leqslant T\Big(r,\frac{\mathcal{F}_1}{f_1^s}\Big)+T\Big(r,\frac{1}{\varphi}\Big)+O(1)\leqslant \sum_{i=1}^{k} n_i^{**}T\Big(r,\frac{f_1^{(i)}}{f_1}\Big)+O(1)\\
&\leqslant \sum_{i=1}^{k} n_i^{**}N\Big(r,\infty;\frac{f_1^{(i)}}{f_1}\Big)+S(r,f_1)\\
&\leqslant \sum_{i=1}^{k} n_i^{**}(N_i(r,0;f_1)+i\overline{N}(r,\infty;f_1))+S(r,f_1)=S(r,f_1),
\end{aligned}
$$

which is not possible. Consequently $\varphi(z)$ is non-constant. Thus the proof is complete. □

Lemma 3.9. *Let $f(z)$ be a non-constant meromorphic function and $n, n_k, k \in \mathbb{N}$, $n_i \in \mathbb{N}\cup\{0\}$ for $i = 1, 2, \ldots, k-1$ be such, that $n > s$. If $\varphi(z) = P(f(z))\mathcal{F}(z)$, then*

$$(n-s)T(r,f)\leqslant T(r,\varphi)-sN(r,\infty;f)-N(r,0;\mathcal{F})+S(r,f).$$

P r o o f. Note that

$$N(r,\infty;\varphi)=N(r,\infty;P(f))+sN(r,\infty;f)+m_1\overline{N}(r,\infty;f),$$

i.e.,

$$N(r,\infty;P(f))=N(r,\infty,\varphi)-sN(r,\infty;f)-m_1\overline{N}(r,\infty,f)+S(r,f).$$

Also

$$
\begin{aligned}
m(r,P(f)) &= m\Big(r,\frac{\varphi}{\mathcal{F}}\Big)\leqslant m(r,\varphi)+m\Big(r,\frac{1}{\mathcal{F}}\Big)+S(r,f)\\
&= m(r,\varphi)+T(r,\mathcal{F})-N(r,0;\mathcal{F})+S(r,f)\\
&= m(r,\varphi)+N(r,\infty;\mathcal{F})+m(r,\mathcal{F})-N(r,0;\mathcal{F})+S(r,f)\\
&\leqslant m(r,\varphi)+sN(r,\infty;f)+m_1\overline{N}(r,\infty;f)+m\Big(r,\frac{\mathcal{F}}{f^s}\Big)\\
&\quad +m(r,f^s)-N(r,0;\mathcal{F})+S(r,f)\\
&= m(r,\varphi)+sT(r,f)+m_1\overline{N}(r,\infty;f)-N(r,0;\mathcal{F})+S(r,f).
\end{aligned}
$$

Now

$$
\begin{aligned}
nT(r,f) &= N(r,\infty;P(f))+m(r,P(f))\\
&\leqslant T(r,\varphi)+sT(r,f)-sN(r,\infty;f)-N(r,0;\mathcal{F})+S(r,f),
\end{aligned}
$$

i.e.,

$$(n-s)T(r,f)\leqslant T(r,\varphi)-sN(r,\infty;f)-N(r,0;\mathcal{F})+S(r,f).$$

Thus the proof is complete. □

Lemma 3.10. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions such that zeros of $f(z)-c$ and $g(z)-c$ are of multiplicities at least k^*, where k^* is defined by (2.2). Let $n, n_k \in \mathbb{N}$ and $n_i \in \mathbb{N}\cup\{0\}$, $i=1,2,\ldots,k-1$, be such that $n > 2\Gamma_1 + 2/k^* + s + t + m$. Let $F(z) = P(f(z))\mathcal{F}(z)/p(z)$ and $G(z) = P(g(z))\mathcal{G}(z)/p(z)$, where $p(z)$ is a nonzero polynomial and $P(z)$ is defined by (2.1). If $f(z)$, $g(z)$ share $(\infty, 0)$ and $H(z)\equiv 0$, then one of the following three cases holds:*

(1) *$P(f(z))\mathcal{F}(z)P(g(z))\mathcal{G}(z) \equiv p^2(z)$, where $P(f(z))\mathcal{F}(z)-p(z)$ and $P(g(z))\mathcal{G}(z)-p(z)$ share $(0,\infty)$,*

(2) *$(f(z)-c)\equiv t(g(z)-c)$, $t^{d_0}=1$, where $d_0 = \gcd(d+p\colon\, p\in\{0,1,\ldots,m_2\}$ with $e_p\neq 0)$,*

(3) *$P(f(z))\mathcal{F}(z) \equiv P(g(z))\mathcal{G}(z)$.*

P r o o f. Since $H\equiv 0$, by integration we get

$$\frac{F'(z)}{(F(z)-1)^2} \equiv l\frac{G'(z)}{(G(z)-1)^2},$$

i.e.,

$$\Big(\frac{P(f(z))\mathcal{F}(z)-p(z)}{p(z)}\Big)'\Big(\frac{P(f(z))\mathcal{F}(z)-p(z)}{p(z)}\Big)^{-2} \equiv l\Big(\frac{P(g(z))\mathcal{G}(z)-p(z)}{p(z)}\Big)'\Big(\frac{P(g(z))\mathcal{G}(z)-p(z)}{p(z)}\Big)^{-2}, \quad l\in\mathbb{C}\setminus\{0\}.$$

This shows that

$$\frac{P(f(z))\mathcal{F}(z)-p(z)}{p(z)} \quad \text{and} \quad \frac{P(g(z))\mathcal{G}(z)-p(z)}{p(z)}$$

share $(0,\infty)$. Therefore $P(f(z))\mathcal{F}(z)-p(z)$ and $P(g(z))\mathcal{G}(z)-p(z)$ share $(0,\infty)$. Again by integration we obtain

$$\frac{1}{F(z)-1} \equiv \frac{bG(z)+a-b}{G(z)-1}, \tag{3.3}$$

where $a,b\in\mathbb{C}\setminus\{0\}$ and $a\neq 0$. We now consider the following cases.

Case 1. Let $b\neq 0$ and $a\neq b$. If $b=-1$, then from (3.3) we have $F(z)\equiv -a/(G(z)-a-1)$. Therefore $\overline{N}(r,a+1;G) = \overline{N}(r,\infty;F) \leqslant \overline{N}(r,\infty;f) + S(r,f)$. So in view of Lemma 3.9 and the second fundamental theorem we get

$$\begin{aligned}(n-s)T(r,g) &\leqslant T(r,P(g)\mathcal{G}) - sN(r,\infty;g) - N(r,0;\mathcal{G}) + S(r,g)\\ &\leqslant T(r,G) - sN(r,\infty;g) - N(r,0;\mathcal{G}) + S(r,g)\\ &\leqslant \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,a+1;G)\\ &\quad - sN(r,\infty;g) - N(r,0;\mathcal{G}) + S(r,g)\end{aligned}$$

$$
\begin{aligned}
&\leqslant \overline{N}(r,0;P_1(g)) + \overline{N}(r,0;g-c) + \overline{N}(r,0;\mathcal{G}) \\
&\quad + \overline{N}(r,\infty;f) - N(r,0;\mathcal{G}) + S(r,g) \\
&\leqslant \overline{N}(r,\infty;g) + \Gamma_1 T(r,g) + \frac{1}{k^*}T(r,g) + S(r,g) \\
&\leqslant N(r,\infty;g) + \Big(\Gamma_1 + \frac{1}{k^*}\Big)T(r,g) + S(r,g) \\
&\leqslant \Big(\Gamma_1 + \frac{1}{k^*} + 1\Big)T(r,g) + S(r,g)
\end{aligned}
$$

and it is a contradiction as $n > \Gamma_1 + 1/k^* + s + 1$.

If $b \neq -1$, from (3.3) we obtain that $F(z) - (1+1/b) \equiv -a/(b^2(G(z) + (a-b)/b))$. So $\overline{N}(r,(b-a)/b;G) = \overline{N}(r,\infty;F) \leqslant \overline{N}(r,\infty;f) + S(r,f)$. Using Lemma 3.9 and the same argument as used in the case when $b = -1$ we get a contradiction.

Case 2. Let $b \neq 0$ and $a = b$. If $b = -1$, then from (3.3) we get $F(z)G(z) \equiv 1$, i.e., $P(f(z))\mathcal{F}(z)P(g(z))\mathcal{G}(z) \equiv p^2(z)$.

If $b \neq -1$, from (3.3) we have $1/F(z) \equiv bG(z)/((1+b)G(z)-1)$. Therefore $\overline{N}(r,1/(1+b);G) = \overline{N}(r,0;F)$. So in view of Lemmas 3.2, 3.9 and the second fundamental theorem, we get

$$
\begin{aligned}
(n-s)T(r,g) &\leqslant T(r,G) - sN(r,\infty;g) - N(r,0;\mathcal{G}) + S(r,g) \\
&\leqslant \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\Big(r,\frac{1}{1+b};G\Big) \\
&\quad - sN(r,\infty;g) - N(r,0;\mathcal{G}) + S(r,g) \\
&\leqslant \overline{N}(r,0;P(g)) + \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,0;F) - N(r,0;\mathcal{G}) + S(r,g) \\
&\leqslant \overline{N}(r,0;g-c) + \overline{N}(r,0;P_1(g)) + \overline{N}(r,0;f-c) \\
&\quad + \overline{N}(r,0;P_1(f)) + \overline{N}(r,0;F_1) + S(r,g) \\
&\leqslant \Big(\Gamma_1 + \frac{1}{k^*}\Big)(T(r,f) + T(r,g)) + \sum_{i=1}^{k} n_i^* \overline{N}(r,0;f^{(i)}) + S(r,g) \\
&\leqslant \sum_{i=1}^{k} n_i^* (N_{i+1}(r,0;f) + i\overline{N}(r,\infty;f)) \\
&\quad + \Big(\Gamma_1 + \frac{1}{k^*}\Big)(T(r,f) + T(r,g)) + S(r,f) + S(r,g) \\
&\leqslant \Big(\Gamma_1 + \frac{1}{k^*}\Big)(T(r,f) + T(r,g)) + tT(r,f) + mT(r,f) + S(r,g).
\end{aligned}
$$

We suppose $T(r,f) \leqslant T(r,g)$ for $r \in I$. So for $r \in I$, we have

$$
(n-s)T(r,g) \leqslant \Big(2\Gamma_1 + \frac{2}{k^*} + t + m\Big)T(r,g) + S(r,g),
$$

which is a contradiction since $n > 2\Gamma_1 + 2/k^* + s + t + m$.

Case 3. Let $b = 0$. From (3.3) we obtain $F(z) \equiv (G(z) + a - 1)/a$. If $a \neq 1$, then we obtain $\overline{N}(r, 1-a; G) = \overline{N}(r, 0; F)$. We can deduce a contradiction similarly as in Case 2. Therefore $a = 1$ and so we have $F(z) \equiv G(z)$. This gives

$$f_1^d(z)\Big(\sum_{i=0}^{m_2} e_i f_1^i(z)\Big)\mathcal{F}_1(z) \equiv g_1^d(z)\Big(\sum_{i=0}^{m_2} e_i g_1^i(z)\Big)\mathcal{G}_1(z). \tag{3.4}$$

Let $h(z) = f_1(z)/g_1(z)$. If $h(z)$ is a constant, by putting $f_1(z) = hg_1(z)$ in (3.4) we get

$$e_{m_2} g_1^{d+m_2}(z)(h^{d+m_2}-1) + e_{m_2-1} g_1^{d+m_2-1}(z)(h^{d+m_2-1}-1) + \ldots + e_0 g_1^d(z)(h^d - 1) \equiv 0,$$

which gives $h^{d_0} = 1$, where $d_0 = \gcd(d+p\colon p \in \{0, 1, \ldots, m_2\}$ with $e_p \neq 0)$. Thus $f_1(z) \equiv tg_1(z)$, i.e., $f(z) - c \equiv t(g(z) - c)$, $t^{d_0} = 1$, where $d_0 = \gcd(d+p\colon p \in \{0, 1, \ldots, m_2\}$ with $e_p \neq 0)$.

If $h(z)$ is not constant, then we must have $P(f(z))\mathcal{F}(z) \equiv P(g(z))\mathcal{G}(z)$. Thus the proof is complete. □

Lemma 3.11 ([8], Lemma 3.5)**.** *Suppose that $F(z)$ is meromorphic in a domain D and set $f(z) = F'(z)/F(z)$. Then for $n \in \mathbb{N}$ we have*

$$\begin{aligned}\frac{F^{(n)}(z)}{F(z)} = f^n(z) &+ \frac{n(n-1)}{2} f^{n-2}(z) f'(z) \\ &+ a_n f^{n-3}(z) f''(z) + b_n f^{n-4}(z)(f'(z))^2 + P_{n-3}(f(z)),\end{aligned}$$

where $a_n = \frac{1}{6}n(n-1)(n-2)$, $b_n = \frac{1}{8}n(n-1)(n-2)(n-3)$ and $P_{n-3}(f(z))$ is a differential polynomial with constant coefficients, which vanishes identically for $n \leqslant 3$ and has degree $n-3$ when $n > 3$.

Lemma 3.12. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that the zeros of $f(z) - c$ and $g(z) - c$ are of multiplicities at least k, where $k \in \mathbb{N}$. Let $n, n_k \in \mathbb{N}$ and $n_i \in \mathbb{N} \cup \{0\}$ for $i = 1, 2, \ldots, k-1$. Suppose that $P(f(z))\mathcal{F}(z) - p(z)$ and $P(g(z))\mathcal{G}(z) - p(z)$ share $(0, \infty)$, and $f(z)$, $g(z)$ share $(\infty, 0)$, where $P(z)$ and $p(z)$ are defined in (2.1) and (2.3), respectively. Then $P(f(z))\mathcal{F}(z)P(g(z))\mathcal{G}(z) \not\equiv p^2(z)$.*

P r o o f. Suppose

$$P(f(z))\mathcal{F}(z)P(g(z))\mathcal{G}(z) \equiv p^2(z). \tag{3.5}$$

Since $f(z)$ and $g(z)$ share $(\infty, 0)$, from (3.5) we claim that $f(z)$ and $g(z)$ are transcendental entire functions.

Suppose that $P(z)$ is a non-constant polynomial. For the sake of simplicity we may assume that $P_1(z) = a_n(z - c_{m_2})^{m_2}$, where $d + m_2 = n$. Obviously $c \neq c_{m_2}$. By (3.5), we have $N(r, c; f) = O(\log r)$ and $N(r, c_{m_2}; f) = O(\log r)$. So by the second fundamental theorem we obtain $T(r, f) \leqslant \overline{N}(r, c; f) + \overline{N}(r, c_{m_2}; f) + \overline{N}(r, \infty; f) + S(r, f) = S(r, f)$, which is not possible. Therefore $P(z)$ must be of the form $a_n(z-c)^n$ and so (3.5) reduces to the form

$$(3.6)\quad a_n^2(f(z)-c)^n\mathcal{F}(z)(g(z)-c)^n\mathcal{G}(z) \equiv p^2(z), \text{ i.e., } f_1^n(z)\mathcal{F}_1(z)g_1^n(z)\mathcal{G}_1(z) \equiv p_1^2(z),$$

where $p_1(z) = p(z)/a_n$. We now consider the following two cases.

Case 1. Let $\deg(p_1) \in \mathbb{N}$. Then from (3.6) we see that $N(r, 0; f_1^n) = O(\log r)$ and $N(r, 0; g_1^n) = O(\log r)$. Let

$$(3.7)\qquad F_1(z) = \frac{f_1^n(z)\mathcal{F}_1(z)}{p_1(z)} \quad \text{and} \quad G_1(z) = \frac{g_1^n(z)\mathcal{G}_1(z)}{p_1(z)}.$$

Then (3.6) reduces to

$$(3.8)\qquad F_1(z)G_1(z) \equiv 1.$$

If $F_1(z) \equiv eG_1(z)$, where $e \in \mathbb{C} \setminus \{0\}$, then $F_1(z)$ must be a constant, which is not possible by Lemma 3.8. So $F_1(z) \not\equiv eG_1(z)$. Let

$$(3.9)\qquad \Phi(z) = \frac{f_1^n(z)\mathcal{F}_1(z) - p_1(z)}{g_1^n(z)\mathcal{G}_1(z) - p_1(z)}.$$

Since $f_1(z)$ and $g_1(z)$ are transcendental entire functions, it follows that $f_1^n(z)\mathcal{F}_1(z) - p_1(z) \neq \infty$ and $g_1^n(z)\mathcal{G}_1(z) - p_1(z) \neq \infty$. Also since $f_1^n(z)\mathcal{F}_1(z) - p_1(z)$ and $g_1^n(z)\mathcal{G}_1(z) - p_1(z)$ share $(0, \infty)$, we deduce from (3.9) that

$$(3.10)\qquad \Phi(z) \equiv \mathrm{e}^{\beta^*(z)},$$

where β^* is an entire function. Let $f_{11}(z) = F_1(z)$, $f_{21}(z) = -\mathrm{e}^{\beta^*(z)}G_1(z)$ and $f_{31}(z) = \mathrm{e}^{\beta^*(z)}$, where $f_{11}(z)$ is transcendental. Now from (3.10), we have $f_{11}(z) + f_{21}(z) + f_{31}(z) \equiv 1$. Also, by Lemma 3.4 we get

$$\sum_{j=1}^{3} N(r, 0; f_{j1}) + 2\sum_{j=1}^{3} \overline{N}(r, \infty; f_{j1})$$
$$\leqslant N(r, 0; F_1) + N(r, 0; \mathrm{e}^{\beta^*}G_1) + O(\log r) \leqslant (\lambda + o(1))T_1(r)$$

as $r \to \infty$, $r \in I$, $\lambda < 1$ and $T_1(r) = \max_{1\leqslant j\leqslant 3} T(r, f_{j1})$. So by Lemma 3.5, we obtain either $\mathrm{e}^{\beta^*(z)}G_1(z) \equiv -1$ or $\mathrm{e}^{\beta^*(z)} \equiv 1$. But the only possibility is that

$e^{\beta^*(z)}G_1(z) \equiv -1$ otherwise $F_1(z) \equiv G_1(z)$, which is possible. Then $g_1^n(z)\mathcal{G}_1(z) \equiv -e^{-\beta^*(z)}p_1(z)$. Also from (3.6) we obtain $f_1^n(z)\mathcal{F}_1(z) \equiv -e^{\beta^*(z)}p_1(z)$. Therefore $f_1^n(z)\mathcal{F}_1(z)$ and $g_1^n(z)\mathcal{G}_1(z)$ share $(0,\infty)$. As $f_1(z)$ and $g_1(z)$ have finitely many zeros, we can assume that

$$f_1(z) = h_1(z)e^{\alpha(z)} \quad \text{and} \quad g_1(z) = h_2(z)e^{\beta(z)}, \tag{3.11}$$

where $h_1(z)$, $h_2(z)$ are non-constant polynomials and $\alpha(z)$, $\beta(z)$ are two non-constant entire functions. Let

$$\alpha_1(z) = \frac{f_1'(z)}{f_1(z)} = \alpha'(z) + \frac{h_1'(z)}{h_1(z)} \quad \text{and} \quad \beta_1(z) = \frac{g_1'(z)}{g_1(z)} = \beta'(z) + \frac{h_2'(z)}{h_2(z)}.$$

Now from (3.11) and Lemma 3.11 we have

$$f_1^n(z)\mathcal{F}_1(z) \equiv h_1^n(z)\prod_{i=1}^{k}(h_1(z)(\alpha'(z))^i + P_{i-1}(\alpha'(z), h_1'(z)))^{n_i}e^{(n+s)\alpha(z)} \tag{3.12}$$

and

$$g_1^n(z)\mathcal{G}_1(z) \equiv h_2^n(z)\prod_{i=1}^{k}(h_2(z)(\beta'(z))^i + Q_{i-1}(\beta'(z), h_2'(z)))^{n_i}e^{(n+s)\beta(z)}, \tag{3.13}$$

respectively, where $P_{i-1}(\alpha'(z), h_1'(z))$ and $Q_{i-1}(\beta'(z), h_2'(z))$ are differential polynomials in $\alpha'(z)$, $h_1'(z)$ and $\beta'(z)$, $h_2'(z)$, respectively. We now consider the following two subcases.

Subcase 1.1. Let $k \geqslant 2$. First we suppose that both $\alpha(z)$ and $\beta(z)$ are transcendental entire functions. Clearly both $\alpha_1(z)$ and $\beta_1(z)$ are transcendental meromorphic functions. Note that $S(r,\alpha_1) = S(r, f_1'/f_1)$ and $S(r,\beta_1) = S(r, g_1'/g_1)$. Moreover, from (3.6) we have $N(r,0;f_1^{(k)}) = O(\log r)$ and $N(r,0;g_1^{(k)}) = O(\log r)$. From this and using (3.11), we have

$$N(r,\infty;f_1) + N(r,0;f_1) + N(r,0;f_1^{(k)}) = S(r,\alpha_1) = S\Big(r, \frac{f_1'}{f_1}\Big) \tag{3.14}$$

and

$$N(r,\infty;g_1) + N(r,0;g_1) + N(r,0;g_1^{(k)}) = S(r,\beta_1) = S\Big(r, \frac{g_1'}{g_1}\Big). \tag{3.15}$$

Using (3.14), (3.15) and Lemma 3.7, we get $f_1(z) = e^{a^*z+b^*}$ and $g_1(z) = e^{c^*z+d^*}$, where a^* $(\neq 0)$, b^*, c^* $(\neq 0)$, $d^* \in \mathbb{C}$, which is possible as zeros of $f_1(z)$ and $g_1(z)$ are of multiplicities at least k.

Next we suppose that both $\alpha(z)$ and $\beta(z)$ are non-constant polynomials. Since $f_1^n(z)\mathcal{F}_1(z) \equiv -\mathrm{e}^{\beta^*(z)}p_1(z)$ and $g_1^n(z)\mathcal{G}_1(z) \equiv -\mathrm{e}^{-\beta^*(z)}p_1(z)$, from (3.12) and (3.13) we have

$$f_1^n(z)\mathcal{F}_1(z) \equiv h_1^n(z)\prod_{i=1}^{k}(h_1(z)(\alpha'(z))^i + P_{i-1}(\alpha'(z), h_1'(z)))^{n_i}\mathrm{e}^{(n+s)\alpha(z)} \equiv Ap_1(z)\mathrm{e}^{(n+s)\alpha(z)} \tag{3.16}$$

and

$$g_1^n(z)\mathcal{G}_1(z) \equiv h_2^n(z)\prod_{i=1}^{k}(h_2(z)(\beta'(z))^i + Q_{i-1}(\beta'(z), h_2'(z)))^{n_i}\mathrm{e}^{(n+s)\beta(z)} \equiv Bp_1(z)\mathrm{e}^{(n+s)\beta(z)}, \tag{3.17}$$

respectively, where $A, B \in \mathbb{C}\setminus\{0\}$. Now from (3.6), (3.16) and (3.17) we deduce that $\alpha(z)+\beta(z) \in \mathbb{C}$, i.e., $\alpha'(z) \equiv -\beta'(z)$ and so $\deg(\alpha) = \deg(\beta)$. Note that $\deg(\alpha), \deg(\beta) \in \mathbb{N}$. Since either $\deg(p_1) \leqslant n+s-1$ or zeros of $p_1(z)$ are of multiplicities at most $n-1$ from (3.16) or (3.17) we arrive at a contradiction.

Finally we suppose that one of $\alpha(z)$ and $\beta(z)$ is transcendental and the other one is polynomial. For the sake of simplicity we assume that $\beta(z)$ is a polynomial. In this case we get a contradiction from (3.17).

Subcase 1.2. Let $k=1$. From (3.11) we deduce that

$$f_1^n(z)(f_1'(z))^{n_1} \equiv h_1^n(z)(h_1(z)\alpha'(z) + h_1'(z))^{n_1}\mathrm{e}^{(n+n_1)\alpha(z)} \tag{3.18}$$

and

$$g_1^n(z)(g_1'(z))^{n_1} \equiv h_2^n(z)(h_2(z)\beta'(z) + h_2'(z))^{n_1}\mathrm{e}^{(n+n_1)\beta(z)}. \tag{3.19}$$

First we suppose that both of $\alpha(z)$ and $\beta(z)$ are transcendental. Then from (3.6), (3.18) and (3.19) we get

$$(h_1(z)h_2(z))^n(h_1(z)\alpha'(z) + h_1'(z))^{n_1} \times(h_2(z)\beta'(z) + h_2'(z))^{n_1}\mathrm{e}^{(n+n_1)(\alpha(z)+\beta(z))} \equiv p_1^2(z). \tag{3.20}$$

Let $\alpha(z)+\beta(z) = \gamma(z)$ and $s_2 = n+n_1$. We claim that $\gamma(z) \notin \mathbb{C}$. If not, suppose $\gamma \in \mathbb{C}$. Then $\alpha'(z) \equiv -\beta'(z)$ and so from (3.20) we have

$$H_{2n_1}(z)(\alpha'(z))^{2n_1} + H_{2n_1-1}(z)(\alpha'(z))^{2n_1-1} + \ldots + H_0(z) \equiv 0, \tag{3.21}$$

where $H_0(z), H_1(z), \ldots, H_{2n_1}(z)$ ($\not\equiv 0$) are polynomials. Since a transcendental entire function is non-algebraic, from (3.21) we arrive at a contradiction. Hence $\gamma \notin \mathbb{C}$.

Now (3.20) reduces to

$$(h_1(z)h_2(z))^n(h_1(z)\alpha'(z)+h_1'(z))^{n_1} \times(h_2(z)(\gamma'(z)-\alpha'(z))+h_2'(z))^{n_1}e^{s_2\gamma(z)} \equiv p_1^2(z). \tag{3.22}$$

We have $T(r,\gamma') = m(r,s_2\gamma')+O(1) = m(r,(e^{s_2\gamma})'/e^{s_2\gamma}) = S(r,e^{s_2\gamma})$. Thus from (3.22) we get

$$\begin{aligned} T(r,e^{s_2\gamma}) &\leqslant T\Big(r,\frac{p_1^2}{(h_1h_2)^n(h_1\alpha'+h_1')^{n_1}(h_2(\gamma'-\alpha')+h_2')^{n_1}}\Big)+O(1)\\ &\leqslant n_1T(r,\alpha')+n_1T(r,\gamma'-\alpha')+O(\log r)+O(1)\\ &\leqslant 2n_1T(r,\alpha')+S(r,\alpha')+S(r,e^{s_2\gamma}), \end{aligned}$$

implying that $T(r,e^{s_2\gamma})=O(T(r,\alpha'))$ and so $S(r,e^{s_2\gamma})$ can be replaced by $S(r,\alpha')$. Thus $T(r,\gamma')=S(r,\alpha')$ and so $\gamma'(z)$ is a small function with respect to $\alpha'(z)$. In view of (3.22) and by Lemma 3.6, we get

$$\begin{aligned} T(r,\alpha') &\leqslant \overline{N}(r,\infty;\alpha')+\overline{N}(r,0;h_1\alpha'+h_1')+\overline{N}(r,0;h_2(\gamma'-\alpha')+h_2')+S(r,\alpha')\\ &\leqslant O(\log r)+S(r,\alpha') \end{aligned}$$

and it shows that $\alpha'(z)$ is a polynomial and consequently $\alpha(z)$ is a polynomial. Similarly we can prove that $\beta(z)$ is also a polynomial. This contradicts that $\alpha(z)$ and $\beta(z)$ are both transcendental.

Next suppose that both $\alpha(z)$ and $\beta(z)$ are polynomials. Since $f_1^n(z)\mathcal{F}_1(z) \equiv -e^{\beta^*(z)}p_1(z)$ and $g_1^n(z)\mathcal{G}_1(z)\equiv -e^{-\beta^*(z)}p_1(z)$, from (3.18) and (3.19), we have

$$f_1^n(z)(f_1'(z))^{n_1} \equiv h_1^n(z)(h_1(z)\alpha'(z)+h_1'(z))^{n_1}e^{s_2\alpha(z)} \equiv A_1p_1(z)e^{s_2\alpha(z)} \tag{3.23}$$

and

$$g_1^n(z)(g_1'(z))^{n_1} \equiv h_2^n(z)(h_2(z)\beta'(z)+h_2'(z))^{n_1}e^{s_2\beta(z)} \equiv B_1p_1(z)e^{s_2\beta(z)}, \tag{3.24}$$

respectively, where $A_1,B_1\in\mathbb{C}\setminus\{0\}$. Now from (3.6), (3.23) and (3.24) we deduce that $\alpha(z)+\beta(z)\in\mathbb{C}$, i.e., $\alpha'(z)\equiv-\beta'(z)$ and so $\deg(\alpha)=\deg(\beta)$. Note that $\deg(\alpha),\deg(\beta)\in\mathbb{N}$. Since either $\deg(p_1)\leqslant n+s-1$ or zeros of $p_1(z)$ are of multiplicities at most $n-1$, from (3.23) or (3.24) we arrive at a contradiction.

Finally we suppose that one of $\alpha(z)$ and $\beta(z)$ is transcendental and the other one is polynomial. For the sake of simplicity we assume that $\beta(z)$ is a polynomial. In this case we get a contradiction from (3.24).

Case 2. Let $p_1(z)\equiv b\in\mathbb{C}\setminus\{0\}$. Then (3.6) reduces to $f_1^n(z)\mathcal{F}_1(z)g_1^n(z)\mathcal{G}_1(z)\equiv b^2$. This shows that both $f_1(z)$ and $g_1(z)$ have no zeros. But this is not possible as zeros of $f_1(z)$ and $g_1(z)$ are of multiplicities at least k ($\geqslant 1$). Thus the proof is complete. □

Lemma 3.13 ([9]). *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Suppose that $f(z)$ and $g(z)$ share $(0,\infty)$, (∞,∞); $f^{(k)}(z)$ and $g^{(k)}(z)$ share $(0,\infty)$ for $k=1,2,\dots,6$. Then $f(z)$ and $g(z)$ satisfy one of the following cases:*

(i) $f(z)\equiv tg(z)$, *where* $t\in\mathbb{C}\setminus\{0\}$,
(ii) $f(z)=\mathrm{e}^{az+b}$ *and* $g(z)=\mathrm{e}^{cz+d}$, *where* $a,b,c,d\in\mathbb{C}\setminus\{0\}$ *such that* $ac\neq 0$,
(iii) $f(z)=a/(1-b\mathrm{e}^{\alpha(z)})$ *and* $g(z)=a/(\mathrm{e}^{-\alpha(z)}-b)$, *where* $a,b\in\mathbb{C}\setminus\{0\}$, α *is a non-constant entire function,*
(iv) $f(z)=a(1-b\mathrm{e}^{cz})$ *and* $g(z)=d(\mathrm{e}^{-cz}-b)$, *where* $a,b,c,d\in\mathbb{C}\setminus\{0\}$.

Lemma 3.14. *Let*

$$Q_1(x)=n_1(x-1)(x-2)\dots(x-k+1)+2n_2x(x-2)\dots(x-k+1)+\dots+kn_kx(x-1)\dots(x-k+2)$$

and

$$Q_2(x)=x(x-1)(x-2)\dots(x-k+1),$$

where $n_k\in\mathbb{N}$, $n_i\in\mathbb{N}\cup\{0\}$, $i=1,2,\dots,k-1$, but at least one of $n_1,n_2,\dots,n_{k-1}$ is nonzero. Suppose $(k-1)s-m_1<0$. Then all the roots of the equation $(sx-m_1)\times Q_1(x)-\lambda Q_2(x)=0$, where $\lambda\in\mathbb{R}$, lie in the interval $(-\infty,k-1)$.

P r o o f. By the given conditions we have $k\geqslant 2$ and $1<m_1/s<k$. Also we see that $m_1/s\neq 2,3,\dots,k-1$. Therefore $js-m_1<0$ for $j=1,2,\dots,k-1$ and $ks-m_1>0$. Let $f(x)=x^{n_1}(x-1)^{2n_2}\dots(x-k+1)^{kn_k}$. Then $f'(x)=x^{n_1-1}(x-1)^{2n_2-1}\dots(x-k+1)^{kn_k-1}Q_1(x)$. By Rolle's theorem, we can say that each of the $(k-1)$ intervals $(0,1)$, $(1,2),\dots,(k-2,k-1)$ contains at least one real root of the equation $f'(x)=0$.

Let α_i, $i=1,2,\dots,k-1$, be the roots of the equation $Q_1(x)=0$ such that $i-1<\alpha_i<i$ for $i=1,2,\dots,k-1$. Then $Q_1(x)=m_1(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_{k-1})$.

Let $F(x)=(sx-m_1)Q_1(x)-\lambda Q_2(x)$. Now we consider the following three cases.

Case 1. Let $sm_1-\lambda<0$. We now consider the following two subcases.

Subcase 1.1. Suppose k is an odd positive integer. Note that $F(-\infty)>0$, $F(0)<0$, $F(1)>0$, $F(2)<0$, $F(3)>0,\dots,F(k-2)>0$, $F(k-1)<0$. Therefore each of the intervals $(-\infty,0)$, $(0,1)$, $(1,2),\dots,(k-2,k-1)$ contains a real root of the equation $F(x)=0$. Since the equation is of degree k, all its roots are real and simple. Therefore all the roots of the equation $F(x)=0$ lie in the interval $(-\infty,k-1)$.

Subcase 1.2. Suppose that k is an even positive integer. Note that $F(-\infty)<0$, $F(0)>0$, $F(1)<0$, $F(2)>0$, $F(3)<0,\dots,F(k-2)>0$, $F(k-1)<0$. Therefore each of the intervals $(-\infty,0)$, $(0,1)$, $(1,2),\dots,(k-2,k-1)$ contains a real root of the equation $F(x)=0$. Since the equation is of degree k, all its roots are real and simple. Therefore all the roots of the equation $F(x)=0$ lie in the interval $(-\infty,k-1)$.

Case 2. Let $sm_1 - \lambda > 0$. We omit the proof since it can be carried out in the line of the proof of Case 1.

Case 3. Let $sm_1 - \lambda = 0$. In this case the equation $F(x) = 0$ is of degree $k-1$. Consequently each of the intervals $(0,1)$, $(1,2), \ldots, (k-2, k-1)$ contains a real root of the equation $F(x) = 0$. Since the equation is of degree $k-1$, all its roots are real and simple. Therefore all the roots of the equation $F(x) = 0$ lie in the interval $(0, k-1)$. Thus the proof is complete. □

Lemma 3.15. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that the zeros of $f(z)-c$ and $g(z)-c$ are of multiplicities at least k, where $k \in \mathbb{N}$. Let $n, n_k \in \mathbb{N}$ and $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k-1$. Suppose $(k-1)s - m_1 < 0$ when at least one of $n_1, n_2, \ldots, n_{k-1}$ is nonzero. Also we assume that $f^{(n^*)}(z)$, $g^{(n^*)}(z)$ share $(0,\infty)$ and $f(z)$, $g(z)$ share $(\infty, 0)$. Now when $(f(z)-c)^n \mathcal{F}(z) \equiv (g(z)-c)^n \mathcal{G}(z)$, then $(f(z)-c) \equiv t(g(z)-c)$, where $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+s} = 1$.*

Proof. Suppose that

$$f_1^n(z)\mathcal{F}_1(z) \equiv g_1^n(z)\mathcal{G}_1(z), \quad \text{i.e.,} \quad f_1^n(z)/g_1^n(z) \equiv \mathcal{G}_1(z)/\mathcal{F}_1(z). \tag{3.25}$$

Since $f_1(z)$ and $g_1(z)$ share $(\infty, 0)$, it follows from (3.25) that $f_1(z)$ and $g_1(z)$ share (∞,∞) and so $(f_1^{(i)}(z))^{n_i^*}$ and $(g_1^{(i)}(z))^{n_i^*}$ share (∞,∞), where $i = 1, 2, \ldots, k$. Again since $f^{(n^*)}(z)$ and $g^{(n^*)}(z)$ share $(0,\infty)$, it follows that $f_1^{(n^*)}(z)$ and $g_1^{(n^*)}(z)$ share $(0,\infty)$. Suppose $n^* = k$. Then from (3.25) we have $f_1^n(z)(f_1^{(k)}(z))^{n_k} \equiv g_1^n(z)(g_1^{(k)}(z))^{n_k}$, and so $f_1(z)$ and $g_1(z)$ share $(0,\infty)$. Next we suppose $n^* < k$. For the sake of simplicity we assume that $n^* = 1$. Let z_{11} be a zero of $f_1(z)$ of multiplicity p_{11} $(\geqslant k)$. Then z_{11} is a zero of $f_1'(z)$ of multiplicity $p_{11} - 1$ $(\geqslant 1)$. Since $f_1'(z)$ and $g_1'(z)$ share $(0,\infty)$, it follows that z_{11} is a zero of $g_1'(z)$ of multiplicity $p_{11} - 1$ $(\geqslant 1)$. Clearly z_{11} is a zero of both $f_1^{(i)}(z)$ and $g_1^{(i)}(z)$ of multiplicity $p_{11} - i$, where $i \in \{1, 2, \ldots, k\}$. Consequently z_{11} is a zero of both $\mathcal{F}_1(z)$ and $\mathcal{G}_1(z)$ of multiplicity $p_{11}s - m_1$. Note that z_{11} is a zero of $f_1^n(z)\mathcal{F}_1(z)$ of multiplicity $p_{11}(n+s) - m_1$. Therefore, from (3.25) we see that z_{11} must be a zero of $g_1(z)$ of multiplicity p_{11}. Hence $f_1(z)$ and $g_1(z)$ share $(0,\infty)$. Let $h_1(z) = f_1(z)/g_1(z)$ and $h_2(z) = \mathcal{F}_1(z)/\mathcal{G}_1(z)$. Then $h_1(z)$ and $h_2(z) \neq 0, \infty$. Now (3.25) yields

$$h_1^n(z)h_2(z) \equiv 1. \tag{3.26}$$

First we suppose that $h_1(z)$ is a non-constant entire function. Clearly $h_2(z)$ is also a non-constant entire function. Let $F_1(z) = h_1^n(z)$ and $G_1(z) = h_2(z)$. Also from (3.26), we get

$$F_1(z)G_1(z) \equiv 1. \tag{3.27}$$

Clearly $F_1(z) \not\equiv d_0 G_1(z)$, $d_0 \in \mathbb{C}\setminus\{0\}$, otherwise we have $F_1 \in \mathbb{C}\setminus\{0\}$ from (3.27) and so $h_1 \in \mathbb{C} \setminus \{0\}$. Since $F_1(z)$ and $G_1(z) \neq 0, \infty$, there exist two non-constant entire functions $\alpha(z)$ and $\beta(z)$ such that $F_1(z) = \mathrm{e}^{\alpha(z)}$ and $G_1(z) = \mathrm{e}^{\beta(z)}$. Now from (3.27) we see that $\alpha + \beta \in \mathbb{C}$ and so $\alpha'(z) \equiv -\beta'(z)$. Note that $F_1'(z) = \alpha'(z)\mathrm{e}^{\alpha(z)}$ and $G_1'(z) = \beta'(z)\mathrm{e}^{\beta(z)}$. This shows that $F_1'(z)$ and $G_1'(z)$ share $(0, \infty)$. Note that $F_1(z), G_1(z) \neq 0, \infty$ and $F_1(z) \not\equiv d_0 G_1(z)$, $d_0 \in \mathbb{C} \setminus \{0\}$. Now in view of Lemma 3.13 we get $F_1(z) = c_1 \mathrm{e}^{az}$ and $G_1(z) = c_2 \mathrm{e}^{-az}$, $a, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ with $c_1 c_2 = 1$. Since $(f_1(z)/g_1(z))^n = c_1 \mathrm{e}^{az}$ it follows that

$$f_1(z)/g_1(z) = t_1 \mathrm{e}^{(a/n)z} = t_1 \mathrm{e}^{cz}, \tag{3.28}$$

where $c, t_1 \in \mathbb{C} \setminus \{0\}$ such that $t_1^n = c_1$ and $c = a/n$. Also we have

$$\mathcal{F}_1(z)/\mathcal{G}_1(z) = c_2 \mathrm{e}^{-az}. \tag{3.29}$$

Let

$$\Phi_1(z) = \frac{\mathcal{F}_1'(z)}{\mathcal{F}_1(z)} - \frac{\mathcal{G}_1'(z)}{\mathcal{G}_1(z)}. \tag{3.30}$$

Using (3.29), we deduce that

$$\Phi_1(z) = -a. \tag{3.31}$$

Noting $g_1^{(0)}(z) = g_1(z)$, we calculate from (3.28) that

$$f_1^{(j)}(z) = t_1 \sum_{i=0}^{j} {}^jC_i (\mathrm{e}^{cz})^{(j-i)} g_1^{(i)}(z) = t_1 \mathrm{e}^{cz}(c^j g_1(z) + jc^{j-1} g_1'(z) + \tfrac{1}{2}j(j-1)c^{j-2} g_1''(z) + \ldots + jcg_1^{(j-1)}(z) + g_1^{(j)}(z)).$$

Consequently we have

$$(f_1^{(j)}(z))^{n_j} = t_1^{n_j} \mathrm{e}^{cn_j z}\Big((g_1^{(j)}(z))^{n_j} + jn_j c g_1^{(j-1)}(z)(g_1^{(j)}(z))^{n_j - 1} + \sum_{\lambda} P_{1\lambda} g_1^{p_0^\lambda}(z)(g_1'(z))^{p_1^\lambda} \ldots (g_1^{(j)}(z))^{p_j^\lambda}\Big),$$

where $P_{1\lambda} \in \mathbb{C} \setminus \{0\}$ and $p_0^\lambda, p_1^\lambda, \ldots, p_j^\lambda \in \mathbb{N} \cup \{0\}$ such that $p_i^\lambda \leqslant n_j$, where $i = 0, 1, \ldots, j-1$, $p_j^\lambda < n_j$ and $p_0^\lambda + p_1^\lambda + \ldots + p_j^\lambda = n_j$. Therefore

$$\mathcal{F}_1(z) = t_1^s \mathrm{e}^{csz}\Big(\mathcal{G}_1(z) + c\sum_{j=1}^{k} jn_j g_1^{(j-1)}(z)(g_1^{(j)}(z))^{n_j - 1} \prod_{\substack{i=1 \\ i \neq j}}^{k} (g_1^{(i)}(z))^{n_i} + \sum_{\lambda} Q_{1\lambda} g_1^{q_0^\lambda}(z)(g_1'(z))^{q_1^\lambda} \ldots (g_1^{(k)}(z))^{q_k^\lambda}\Big), \tag{3.32}$$

where $Q_{1\lambda} \in \mathbb{C} \setminus \{0\}$ and $q_0^\lambda, q_1^\lambda, \ldots, q_k^\lambda \in \mathbb{N} \cup \{0\}$ such that $q_i^\lambda \leqslant s$, where $i = 0, 1, \ldots, k-1$, $q_k^\lambda < s$ and $q_0^\lambda + q_1^\lambda + \ldots + q_k^\lambda = s$. It is clear that $0 \leqslant q_1^\lambda + 2q_2^\lambda + \ldots + kq_k^\lambda \leqslant m_1 - 2$. Note that

$$\begin{aligned}\mathcal{F}_1'(z) = t_1^s \mathrm{e}^{csz}\Big(&\mathcal{G}_1'(z) + c(m_1+s)\mathcal{G}_1(z) \\ &+ \sum_\lambda R_{1\lambda} g_1^{r_0^\lambda}(z)(g_1'(z))^{r_1^\lambda}\ldots(g_1^{(k)}(z))^{r_k^\lambda}\Big) \\ &+ cst_1^s\mathrm{e}^{csz}\sum_\lambda Q_{1\lambda} g_1^{q_0^\lambda}(z)(g_1'(z))^{q_1^\lambda}\ldots(g_1^{(k)}(z))^{q_k^\lambda},\end{aligned} \tag{3.33}$$

where $R_{1\lambda} \in \mathbb{C} \setminus \{0\}$ and $r_0^\lambda, r_1^\lambda, \ldots, r_k^\lambda \in \mathbb{N} \cup \{0\}$ such that $r_i^\lambda \leqslant s$, where $i = 0, 1, \ldots, k-1$, $r_k^\lambda < s$ and $r_0^\lambda + r_1^\lambda + \ldots + r_k^\lambda = s$. It is clear that $0 \leqslant r_1^\lambda + 2r_2^\lambda + \ldots + kr_k^\lambda \leqslant m_1 - 1$. Now from (3.30), (3.32) and (3.33) we have

$$\begin{aligned}\Phi_1(z) = \frac{1}{F_3(z) + \mathcal{G}_1^2(z)}\Big(&H_1(z) + c(m_1+s)\mathcal{G}_1^2(z) \\ &- c\sum_{j=1}^k jn_j g_1^{(j-1)}(z)(g_1^{(j)}(z))^{n_j-1}\prod_{\substack{i=1\\ i\neq j}}^k (g_1^{(i)}(z))^{n_i}\mathcal{G}_1'(z)\Big),\end{aligned} \tag{3.34}$$

where $H_1(z) = F_2(z) - G_2(z)$ with

$$\begin{aligned}F_2(z) = \mathcal{G}_1(z)\Big(&\sum_\lambda R_{1\lambda} g_1^{r_0^\lambda}(z)(g_1'(z))^{r_1^\lambda}\ldots(g_1^{(k)}(z))^{r_k^\lambda} \\ &+ cs\sum_\lambda Q_{1\lambda} g_1^{q_0^\lambda}(z)(g_1'(z))^{q_1^\lambda}\ldots(g_1^{(k)}(z))^{q_k^\lambda}\Big), \\ G_2 = \mathcal{G}_1'(z)&\sum_\lambda Q_{1\lambda} g_1^{q_0^\lambda}(z)(g_1'(z))^{q_1^\lambda}\ldots(g_1^{(k)}(z))^{q_k^\lambda}\end{aligned}$$

and

$$\begin{aligned}F_3(z) = \mathcal{G}_1(z)\Big(&c\sum_{j=1}^k jn_j g_1^{(j-1)}(z)(g_1^{(j)(z)})^{n_j-1}\prod_{\substack{i=1\\ i\neq j}}^k (g_1^{(i)}(z))^{n_i} \\ &+ \sum_\lambda Q_{1\lambda} g_1^{q_0^\lambda}(z)(g_1'(z))^{q_1^\lambda}\ldots(g_1^{(k)}(z))^{q_k^\lambda}\Big).\end{aligned}$$

Let z_p be a zero of $g_1(z)$ with multiplicity p ($\geqslant k$). Then the Taylor expansion of $g_1(z)$ about z_p is

$$g_1(z) = a_p(z-z_p)^p + a_{p+1}(z-z_p)^{p+1} + \ldots, \quad a_p \neq 0. \tag{3.35}$$

Therefore $g_1^{(i)}(z) = N_i a_p (z - z_p)^{p-i} + \ldots$, where $N_i = p(p-1)\ldots(p-i+1)$. Consequently $(g_1^{(i)}(z))^{n_i} = N_i^{n_i} a_p^{n_i} (z - z_p)^{pn_i - in_i} + \ldots$ and so

$$\mathcal{G}_1(z) = \left(\prod_{i=1}^{k} N_i^{n_i} a_p^{n_i}\right)(z - z_p)^{ps-m_1} + \ldots$$

Note that

$$\mathcal{G}_1^2(z) = \left(\prod_{i=1}^{k} N_i^{n_i} a_p^{n_i}\right)^2 (z - z_p)^{2ps-2m_1} + \ldots \tag{3.36}$$

and

$$\mathcal{G}_1'(z) = (ps - m_1)\left(\prod_{i=1}^{k} N_i^{n_i} a_p^{n_i}\right)(z - z_p)^{ps-m_1-1} + \ldots \tag{3.37}$$

Also we see that

$$\prod_{\substack{i=1,\\ i\neq j}}^{k} (g_1^{(i)}(z))^{n_i} = \left(\prod_{i=1}^{k} N_i^{n_i} a_p^{n_i}\right) N_j^{-n_j} a_p^{-n_j} (z - z_p)^{ps-m_1-pn_j+jn_j} + \ldots$$

and

$$g_1^{(j-1)}(z)(g_1^{(j)}(z))^{n_j-1} = N_{j-1} a_p N_j^{n_j-1} a_p^{n_j-1} (z - z_p)^{pn_j - jn_j + 1}.$$

Consequently,

$$g_1^{(j-1)}(z)(g_1^{(j)}(z))^{n_j-1} \prod_{\substack{i=1\\ i\neq j}}^{k} (g_1^{(i)}(z))^{n_i} = \frac{1}{p-j+1}\left(\prod_{i=1}^{k} N_i^{n_i} a_p^{n_i}\right)(z - z_p)^{ps-m_1+1} + \ldots$$

and so

$$\begin{aligned} &\sum_{j=1}^{k} j n_j^{**} g_1^{(j-1)} (g_1^{(j)})^{n_j-1} \prod_{\substack{i=1,\\ i\neq j}}^{k} (g_1^{(i)}(z))^{n_i} \mathcal{G}_1'(z) \\ &\qquad = (ps - m_1)\left(\prod_{i=1}^{k} N_i^{n_i} a_p^{n_i}\right)^2 \sum_{j=1}^{k} \frac{j n_j^{**}}{p-j+1} (z - z_p)^{2ps-2m_1} + \ldots \end{aligned} \tag{3.38}$$

Also $F_2(z) = A_1(z - z_p)^{2ps-2m_1+1} + \ldots$, $G_2(z) = A_2(z - z_p)^{2ps-2m_1+1} + \ldots$ and $F_3(z) = A_3(z - z_p)^{2ps-2m_1+1} + \ldots$, where A_1, A_2, A_3 are suitable nonzero constants.

Now from (3.34), (3.36) and (3.38) we have

$$\begin{aligned}(3.39)\qquad \Phi_1(z_p) &= \Big(\prod_{i=1}^{k} N_i^{n_i} a_p^{n_i}\Big)^{-2}\Big(c(m_1+s)\Big(\prod_{i=1}^{k} N_i^{n_i} a_p^{n_i}\Big)^{2} \\ &\quad - c(ps-m_1)\Big(\sum_{j=1}^{k}\frac{jn_j}{p-j+1}\Big)\Big(\prod_{i=1}^{k} N_i^{n_i} a_p^{n_i}\Big)^{2}\Big) \\ &= \frac{a}{n}\Big(m_1+s-(ps-m_1)\sum_{j=1}^{k}\frac{jn_j}{p-j+1}\Big).\end{aligned}$$

We now consider the following two cases.

Case 1. Suppose $n_1 = n_2 = \ldots = n_{k-1} = 0$. Then from (3.39) we get $\Phi_1(z_p) = cn_k(p+1)/(p-k+1)$ and so from (3.31) we arrive at a contradiction.

Case 2. Suppose that at least one of $n_1, n_2, \ldots, n_{k-1}$ is nonzero. Then from (3.31) and (3.39) we have

$$(ps-m_1)\sum_{i=1}^{k}\frac{jn_j}{p-j+1} - (n+s+m_1) = 0,$$

i.e.,

$$(3.40)\qquad (ps-m_1)Q_1(p) - (n+s+m_1)Q_2(p) = 0,$$

where $Q_1(p)$ and $Q_2(z)$ are as in Lemma 3.14. By Lemma 3.14 we see that the roots of the equation $(px-m_1)Q_1(x)-(n+s+m_1)Q_2(x) = 0$ lie in the interval $(-\infty, k-1)$. Therefore the roots of the equation (3.40) also lie in the interval $(-\infty, k-1)$ but this is not possible as z_p is a zero of g_1 with multiplicity $p \geqslant k$. Thus the only possibility is that $g_1(z)$ has no zeros. Since $f_1(z)$ and $g_1(z)$ share $(0, \infty)$, it follows that $f_1(z)$ and $g_1(z)$ have no zeros, which is possible as zeros of $f_1(z)$ and $g_1(z)$ are of multiplicities at least k ($\geqslant 1$). Hence $h_1 \in \mathbb{C}\setminus\{0\}$. Then from (3.25) we get $h_1^{n+s} = 1$ and so $f_1(z) \equiv tg_1(z)$, i.e., $(f(z)-c) \equiv t(g(z)-c)$, $t \in \mathbb{C}\setminus\{0\}$ with $t^{n+s} = 1$. Thus the proof is complete. □

Lemma 3.16. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that the zeros of $f(z)-c$ and $g(z)-c$ are of multiplicities at least k^*, where k^* is defined in* (2.2)*. Let $n, n_k \in \mathbb{N}$ and $n_i \in \mathbb{N}\cup\{0\}$ for $i = 1, 2, \ldots, k-1$. Suppose $(k-1)s - m_1 < 0$ when at least one of $n_1, n_2, \ldots, n_{k-1}$ is nonzero. Also we assume that $f(z)$ and $g(z)$ share $(\infty, 0)$. If $f_1^d(z)P_2(f_1(z))\mathcal{F}_1(z) \equiv g_1^d(z)P_2(g_1(z))\mathcal{G}_1(z)$, then one of the following cases holds:*

1. *If $P_2(z_1) \equiv e_i z_1^i \not\equiv 0$ for some $i \in \{0, 1, 2, \ldots, m_1\}$ and $f^{(n^*)}(z)$, $g^{(n^*)}(z)$ share $(0, \infty)$, then $f(z)-c \equiv t(g(z)-c)$, where $t \in \mathbb{C}\setminus\{0\}$ such that $t^{d+s+i} = 1$ for some $i \in \{0, 1, 2, \ldots, m_1\}$.*

(2) *If* $P_2(z_1) \not\equiv e_i z_1^i$ *for* $i \in \{0,1,2,\dots,m_1\}$, $(f^{(i)}(z))^{n_i^*}$, $(g^{(i)}(z))^{n_i^*}$ *share* $(0,\infty)$, *where* $i = 1,2,\dots,k$, *and* $f(z)$, $g(z)$ *share* $(c,0)$, *then* $f(z) - c \equiv t(g(z) - c)$ *for* $t \in \mathbb{C} \setminus \{0\}$ *such that* $t^{d+s} = 1$.

Proof. Suppose

$$f_1^d(z)P_2(f_1(z))\mathcal{F}_1(z) \equiv g_1^d(z)P_2(g_1(z))\mathcal{G}_1(z), \tag{3.41}$$

i.e.,

$$\frac{P_2(f_1(z))}{P_2(g_1(z))} \equiv \frac{g_1^d(z)\mathcal{G}_1(z)}{f_1^d(z)\mathcal{F}_1(z)}. \tag{3.42}$$

We now consider the following two cases.

Case 1. Suppose $P_2(z_1) \equiv e_i z_1^i \not\equiv 0$ for some $i \in \{0,1,2,\dots,m_2\}$. Then the result follows from Lemma 3.15.

Case 2. Suppose $P_2(z_1) \not\equiv e_i z_1^i$ where $i \in \{0,1,2,\dots,m_2\}$. For the sake of simplicity we assume that $P_2(z_1) = e_{m_2} z_1^{m_2} + e_{m_2-1} z_1^{m_2-1} + \dots + e_1 z_1 + e_0$, $e_{m_2}, e_0 \neq 0$. Since $f_1(z)$ and $g_1(z)$ share $(\infty,0)$, from (3.41) we see that $f_1(z)$ and $g_1(z)$ share (∞,∞). Now we prove that $f_1(z)$ and $g_1(z)$ share $(0,\infty)$. Note that $P_2(0) \neq 0$. Let z_{12} be a zero of $f_1(z)$ of multiplicity r_{12} ($\geqslant k+1$). Since $f_1(z)$ and $g_1(z)$ share $(0,0)$, z_{12} is a zero of $g_1(z)$ of multiplicity q_{12} ($\geqslant k+1$). Clearly z_{12} is a zero of $f_1^{(k)}(z)$ of multiplicity $r_{12} - k$ and a zero of $g_1^{(k)}(z)$ of multiplicity $q_{12} - k$. Since $f_1^{(k)}(z)$ and $g_1^{(k)}(z)$ share $(0,\infty)$, we have $r_{12} = q_{12}$. Therefore $f_1(z)$ and $g_1(z)$ share $(0,\infty)$. Since $f_1(z)$ and $g_1(z)$ share $(0,\infty)$ and (∞,∞), it follows that $f_1(z) = \mathrm{e}^{\gamma(z)} g_1(z)$, where $\gamma(z)$ is an entire function. Let

$$h_1^*(z) = \frac{P_2(f_1(z))}{P_2(g_1(z))} \quad \text{and} \quad h_2^*(z) = \frac{f_1^d(z)\mathcal{F}_1(z)}{g_1^d(z)\mathcal{G}_1(z)}.$$

Since $\mathcal{F}_1(z)$ and $\mathcal{G}_1(z)$ share $(0,\infty)$, we have $h_2(z) \neq 0,\infty$. Also from (3.41) we see that $h_1(z) \neq 0,\infty$ and

$$h_1^*(z)h_2^*(z) \equiv 1. \tag{3.43}$$

We now consider the following two subcases.

Subcase 2.1. Suppose $h_1^* \equiv b \in \mathbb{C} \setminus \{0\}$. Let $b = 1$. Then from (3.41) we have $f_1^d(z)\mathcal{F}_1(z) \equiv g_1^d(z)\mathcal{G}_1(z)$. Then the result follows from Lemma 3.15. Let $b \neq 1$. Then we have

$$\sum_{i=0}^{m_2} e_i f_1^i \equiv b \sum_{i=0}^{m_2} e_i g_1^i. \tag{3.44}$$

Since $f_1(z) = \mathrm{e}^{\gamma(z)} g_1(z)$, from (3.44) we have

$$e_{m_2} g_1^{m_2}(z)(\mathrm{e}^{m_2\gamma(z)} - b) + \ldots + e_1 g_1(z)(\mathrm{e}^{\gamma(z)} - b) \equiv e_0(b-1). \tag{3.45}$$

Note that $g_1(z) \not\equiv d \in \mathbb{C}$. Then from (3.45) we see that $g_1(z)$ has no zero. But this is impossible because zeros of $g_1(z)$ are of multiplicities at least $k+1$.

Subcase 2.2. Suppose $h_1^* \notin \mathbb{C}$. Then $h_2^* \notin \mathbb{C}$. Note that $h_1^*(z) \not\equiv d_0^* h_2^*(z)$, $d_0^* \in \mathbb{C} \setminus \{0\}$. Since $h_1^*(z)$ and $h_2^*(z) \neq 0, \infty$, then there exist two non-constant entire functions $\alpha^*(z)$ and $\beta^*(z)$ such that $h_1^*(z) = \mathrm{e}^{\alpha^*(z)}$ and $h_2^*(z) = \mathrm{e}^{\beta^*(z)}$. Now from (3.43) we see that $\alpha^{*\prime}(z) \equiv -\beta^{*\prime}(z)$. Therefore $h_1^{*\prime}(z)$ and $h_2^{*\prime}(z)$ share $(0, \infty)$. Now in view of Lemma 3.13, we get $h_1^*(z) = c_1^* \mathrm{e}^{az}$ and $h_2^*(z) = c_2^* \mathrm{e}^{-az}$, where $a, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ are such that $c_1 c_2 = 1$. Therefore we have

$$\sum_{i=1}^{m_2} e_i g_1^i(z)(\mathrm{e}^{i\gamma(z)} - c_1^* \mathrm{e}^{az}) \equiv e_0(c_1^* \mathrm{e}^{az} - 1). \tag{3.46}$$

Note that the zeros of $(c_1^* \mathrm{e}^{az} - 1)$ are simple. Also from (3.46) we see that zeros of $g_1(z)$ are the zeros of $(c_1^* \mathrm{e}^{az} - 1)$. Since zeros of $g_1(z)$ are of multiplicities at least $k+1$, from (3.46) we arrive at a contradiction. Thus the proof is complete. $\square$

Lemma 3.17. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that the zeros of $f(z)-c$ and $g(z)-c$ are of multiplicities at least k^*, where k^* is defined by (2.2) and $F(z) = P(f(z))\mathcal{F}(z)/p(z)$ and $G(z) = P(g(z))\mathcal{G}(z)/p(z)$, where $n, n_k \in \mathbb{N}$ and $n_i \in \mathbb{N} \cup \{0\}$ for $i = 1, 2, \ldots, k-1$ are such that $n + s + m_1 > 2m + 1$ and $P(z)$ is defined by (2.1). Suppose $H(z) \not\equiv 0$. If $F(z)$ and $G(z)$ share $(1, k_1)$ except for the zeros of $p(z)$, and $f(z)$ and $g(z)$ share $(\infty, 0)$, then*

$$\begin{aligned}\overline{N}(r, \infty; f) \leqslant\ & \frac{k^* t + k^* \Gamma_1 + 1}{k^*(n + s + m_1 - 2m - 1)}(T(r, f) + T(r, g)) \\ & + \frac{1}{n + s + m_1 - 2m - 1}\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g).\end{aligned}$$

P r o o f. Since $H(z) \not\equiv 0$, it follows that $F \not\equiv G$. First we observe that if ∞ is Picard's exceptional value of $f(z)$, then the result follows immediately. Next we suppose that ∞ is not Picard's exceptional value of $f(z)$. Since $f(z)$ and $g(z)$ share $(\infty, 0)$, it follows that ∞ is not Picard's exceptional value of $g(z)$. We claim that $V(z) \not\equiv 0$. If possible, suppose $V(z) \equiv 0$. Then by integration we obtain $1 - 1/F(z) = A_0(1 - 1/G(z))$, where $A_0 \neq 0, 1$. Let z_{q_0} be a pole of $f(z)$ of multiplicity q_0 such that $p(z_{q_0}) \neq 0$. Since $f(z)$ and $g(z)$ share $(\infty, 0)$, we suppose that z_{q_0} is a pole of $g(z)$ of multiplicity r_0. Therefore $1/F(z_{q_0}) = 0$ and $1/G(z_{q_0}) = 0$

and so $A_0 = 1$, which is not possible. Hence $V(z) \not\equiv 0$. Note that z_{q_0} is a pole of $F(z)$ with multiplicity $(n+s)q_0 + m_1$ and a pole of $G(z)$ with multiplicity $(n+s)r_0 + m_1$. Clearly

$$\frac{F'(z)}{F(z)(F(z)-1)} = O((z - z_{q_0})^{(n+s)q_0+m_1-1})$$

and

$$\frac{G'(z)}{G(z)(G(z)-1)} = O((z - z_{q_0})^{(n+s)r_0+m_1-1}).$$

Consequently we have

$$V(z) = O((z - z_{q_0})^{(n+m)t_0+k-1}),$$

where $t_0 = \min\{q_0, r_0\} \geqslant 1$. This shows that z_{q_0} is a zero of $V(z)$ of multiplicity at least $n + s + m_1 - 1$. Also $m(r,V) = S(r,f) + S(r,g)$. Thus using Lemma 3.1 and Lemma 3.3, we see that

$$\begin{aligned}
&(n+s+m_1-1)\overline{N}(r,\infty;f) \\
&\quad \leqslant N(r,0;V) + O(\log r) \leqslant N(r,\infty;V) + S(r,f) + S(r,g) \\
&\quad \leqslant \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\
&\quad \leqslant \overline{N}(r,0;P_1(f)) + \overline{N}(r,0;f_1) + \sum_{i=1}^{k} n_i^* \overline{N}(r,0;f_1^{(i)} \mid f_1 \neq 0) \\
&\qquad + \overline{N}(r,0;P_1(g)) + \overline{N}(r,0;g_1) + \sum_{i=1}^{k} n_i^* \overline{N}(r,0;g_1^{(i)} \mid g_1 \neq 0) \\
&\qquad + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\
&\quad \leqslant \overline{N}(r,0;P_1(f)) + \overline{N}(r,0;f_1) + \sum_{i=1}^{k} n_i^* (i\overline{N}(r,\infty;f_1) + N_i(r,0;f_1)) \\
&\qquad + \overline{N}(r,0;P_1(g)) + \overline{N}(r,0;g_1) + \sum_{i=1}^{k} n_i^* (i\overline{N}(r,\infty;g_1) + N_i(r,0;g_1)) \\
&\qquad + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\
&\quad \leqslant \overline{N}(r,0;P_1(f)) + \overline{N}(r,0;f_1) + m\overline{N}(r,\infty;f_1) + tN(r,0;f_1) \\
&\qquad + \overline{N}(r,0;P_1(g)) + \overline{N}(r,0;g_1) + m\overline{N}(r,\infty;g_1) + tN(r,0;g_1) \\
&\qquad + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\
&\quad \leqslant \frac{k^*t + k^*\Gamma_1 + 1}{k^*}(T(r,f) + T(r,g)) \\
&\qquad + 2m\overline{N}(r,\infty;f) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g).
\end{aligned}$$

Thus the proof is complete. □

Lemma 3.18 ([2]). *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions such that they share $(1,k_1)$, where $2\leqslant k_1\leqslant\infty$. Then*

$$\overline{N}(r,1;f\,|=2)+2\overline{N}(r,1;f\,|=3)+\ldots+(k_1-1)\overline{N}(r,1;f\,|=k_1)+k_1\overline{N}_L(r,1;f)$$
$$+(k_1+1)\overline{N}_L(r,1;g)+k_1\overline{N}_E^{(k_1+1}(r,1;g)\leqslant N(r,1;g)-\overline{N}(r,1;g).$$

4. Proof of the main theorems

Proof of Theorem 2.1. Let $F(z)=P(f(z))\mathcal{F}(z)$. Now in view of Lemma 3.9 and using the second theorem for small functions (see [14]), we get

$$\begin{aligned}(n-s)T(r,f)&\leqslant T(r,F)-sN(r,\infty;f)-N(r,0;\mathcal{F}_1)+S(r,f)\\&\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,a;F)-sN(r,\infty;f)\\&\quad-N(r,0;\mathcal{F})+(\varepsilon+o(1))T(r,f)\\&\leqslant \overline{N}(r,0;P_1(f))+\overline{N}(r,0;f-c)+\overline{N}(r,a;F)+(\varepsilon+o(1))T(r,f)\\&\leqslant(\Gamma_1+1/k^*)T(r,f)+\overline{N}(r,a;F)+(\varepsilon+o(1))T(r,f)\end{aligned}$$

for all $\varepsilon>0$. Take $\varepsilon<n-s-\Gamma_1-1/k^*$. Since $n>s+\Gamma_1+1/k^*$, one can easily say that $F-a$ has infinitely many zeros. Thus the proof is complete. □

Proof of Theorem 2.3. Let

$$F(z)=\frac{P(f(z))\mathcal{F}(z)}{p(z)}\quad\text{and}\quad G(z)=\frac{P(g(z))\mathcal{G}(z)}{p(z)}.$$

Then $F(z)$ and $G(z)$ share $(1,k_1)$ except for the zeros of $p(z)$ and $f(z)$, $g(z)$ share $(\infty,0)$.

Case 1. Let $H(z)\not\equiv 0$. Now from (3.1) we observe that

$$\begin{aligned}(4.1)\quad N(r,\infty;H)&\leqslant\overline{N}_*(r,\infty;f,g)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F\,|\geqslant 2)+\overline{N}(r,0;G\,|\geqslant 2)\\&\quad+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g),\end{aligned}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of $F'(z)$ which are not the zeros of $F(z)(F(z)-1)$ and $\overline{N}_0(r,0;G')$ is defined similarly. Let z_0 be a simple zero of $F(z)-1$ but $p(z_0)\neq 0$. Then z_0 is a simple zero of $G(z)-1$ and a zero of $H(z)$. Therefore $N(r,1;F\,|=1)\leqslant N(r,0;H)\leqslant N(r,\infty;H)+S(r,f)+S(r,g)$ and so from (4.1) we get

$$\begin{aligned}(4.2)\quad \overline{N}(r,1;F)&\leqslant N(r,1;F\,|=1)+\overline{N}(r,1;F\,|\geqslant 2)\\&\leqslant\overline{N}(r,\infty;f)+\overline{N}(r,0;F\,|\geqslant 2)+\overline{N}(r,0;G\,|\geqslant 2)+\overline{N}_*(r,1;F,G)\\&\quad+\overline{N}(r,1;F\,|\geqslant 2)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g).\end{aligned}$$

Now in view of Lemmas 3.3 and 3.18 we get

$$
\begin{aligned}
(4.3)\quad & \overline{N}_0(r,0;G') + \overline{N}(r,1;F\,|\geqslant 2) + \overline{N}_*(r,1;F,G) \\
& \leqslant \overline{N}_0(r,0;G') + \overline{N}(r,1;F\,|=2) + \overline{N}(r,1;F\,|=3) + \ldots + \overline{N}(r,1;F\,|=k_1) \\
& \quad + \overline{N}_E^{(k_1+1}(r,1;F) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_*(r,1;F,G) \\
& \leqslant \overline{N}_0(r,0;G') + N(r,1;G) - \overline{N}(r,1;G) \\
& \quad - (k_1-2)\overline{N}_L(r,1;F) - (k_1-1)\overline{N}_L(r,1;G) \\
& \leqslant N(r,0;G' \mid G \neq 0) - (k_1-2)\overline{N}_L(r,1;F) - (k_1-1)\overline{N}_L(r,1;G) \\
& \leqslant \overline{N}(r,0;G) + \overline{N}(r,\infty;g) - (k_1-2)\overline{N}_*(r,1;F,G) - \overline{N}_L(r,1;G).
\end{aligned}
$$

Hence using (4.2), (4.3) and Lemma 3.2, we get from the second fundamental theorem that

$$
\begin{aligned}
(4.4)\quad T(r,F) & \leqslant \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) - N_0(r,0;F') + S(r,f) \\
& \leqslant 2\overline{N}(r,\infty,f) + N_2(r,0;F) + \overline{N}(r,0;G\,|\geqslant 2) + \overline{N}(r,1;F\,|\geqslant 2) \\
& \quad + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;G') + S(r,f) + S(r,g) \\
& \leqslant 3\overline{N}(r,\infty;f) + N_2(r,0;F) + N_2(r,0;G) \\
& \quad - (k_1-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\
& \leqslant 3\overline{N}(r,\infty;f) + 2\overline{N}(r,0;f_1) + N_2(r,0;P_1(f)) + N_2(r,0;\mathcal{F}_1) \\
& \quad + 2\overline{N}(r,0;g_1) + N_2(r,0;P_1(g)) + N_2(r,0;\mathcal{G}_1) \\
& \quad - (k_1-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\
& \leqslant 3\overline{N}(r,\infty;f) + (\Gamma_2 + \frac{2}{k^*})(T(r,f)+T(r,g)) + N_2(r,0;\mathcal{F}_1) \\
& \quad + N_2(r,0;\mathcal{G}_1) - (k_1-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\
& \leqslant 3\overline{N}(r,\infty;f) + \left(\Gamma_2 + \frac{2}{k^*}\right)(T(r,f)+T(r,g)) + N_2(r,0;\mathcal{F}_1) \\
& \quad + \sum_{i=1}^{k} n_i^{**} N_2(r,0;g^{(i)}) - (k_1-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\
& \leqslant 3\overline{N}(r,\infty;f) + \left(\Gamma_2 + \frac{2}{k^*}\right)(T(r,f)+T(r,g)) + N_2(r,0;\mathcal{F}_1) \\
& \quad + \sum_{i=1}^{k} n_i^{**} N_{i+2}(r,0;g) + \sum_{i=1}^{k} i n_i^{**} \overline{N}(r,\infty;g) \\
& \quad - (k_1-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\
& \leqslant (3+m_1)\overline{N}(r,\infty;f) + \left(\Gamma_2 + \frac{2}{k^*}\right)(T(r,f)+T(r,g)) + N_2(r,0;\mathcal{F}_1) \\
& \quad + sN(r,0;g) - (k_1-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g).
\end{aligned}
$$

Now using Lemmas 3.17 and 3.9 we get from (4.4)

$$\begin{aligned}
(n-s)T(r,f) &\leqslant T(r,F) - sN(r,\infty;f) - N(r,0;\mathcal{F}_1) + S(r,f)\\
&\leqslant (3+m_1-s)\overline{N}(r,\infty;f) + \Big(\Gamma_2 + \frac{2}{k^*}\Big)T(r,f) + \Big(\Gamma_2 + \frac{2}{k^*}\Big)T(r,g)\\
&\quad + sN(r,0;g) - (k_1-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
&\leqslant 2\frac{(k^*t + k^*\Gamma_1 + 1)(3+m_1-s)}{k^*(n+s+m_1-2m-1)}T(r) + \Big(2\Gamma_2 + \frac{4}{k^*} + s\Big)T(r) + S(r)\\
&\leqslant \Big(2\frac{(k^*t + k^*\Gamma_1 + 1)(3+m_1-s)}{k^*(n+s+m_1-2m-1)} + 2\Gamma_2 + \frac{4}{k^*} + s\Big)T(r) + S(r).
\end{aligned}$$

We obtain a similar inequality for $g(z)$. Combining these inequalities we obtain

$$(n-s)T(r) \leqslant \Big(2\frac{(k^*t + k^*\Gamma_1 + 1)(3+m_1-s)}{k^*(n+s+m_1-2m-1)} + 2\Gamma_2 + \frac{4}{k^*} + s\Big)T(r) + S(r),$$

i.e.,

$$(k^*n^2 - ((2\Gamma_2 + s + 2m + 1 - m_1)k^* + 4)n + A)T(r) \leqslant S(r),$$

where

$$\begin{aligned}
A = k^*(&4m\Gamma_2 + 2\Gamma_2 + 4ms + 2s + 2s\Gamma_1 + 2ts - 2m_1 s - 2s^2 - 2s\Gamma_2\\
&- 2m_1\Gamma_2 - 2m_1\Gamma_1 - 6\Gamma_1 - 6t - 2m_1 t) + 8m - 6m_1 - 2s - 2.
\end{aligned}$$

Therefore

$$(n-K_1)(n-K_2)T(r) \leqslant S(r), \tag{4.5}$$

where

$$K_1 = \frac{(2\Gamma_2 + s + 2m + 1 - m_1)k^* + 4 + \sqrt{L}}{2k^*}$$

and

$$K_2 = \frac{(2\Gamma_2 + s + 2m + 1 - m_1)k^* + 4 - \sqrt{L}}{2k^*},$$

so that $L = ((2\Gamma_2 + s + 2m + 1 - m_1)k^* + 4)^2 - 4k^*A$. Note that

$$\begin{aligned}
L &= ((2\Gamma_2 + s + 2m + 1 - m_1)k^* + 4)^2 - 4k^*A\\
&= (k^*)^2(4\Gamma_2^2 + 4m^2 + 9s^2 + m_1^2 + 1 - 8m\Gamma_2 + 12s\Gamma_2 + 4m_1\Gamma_2\\
&\quad - 4\Gamma_2 + 6sm_1 - 12sm - 8st - 8s\Gamma_1 - 4mm_1 - 2m_1\\
&\quad + 24t + 24\Gamma_1 + 8tm_1 + 8m_1\Gamma_1 + 4m - 6s)\\
&\quad + 4k^*(4\Gamma_2 + 4s - 4m + 4 + 4m_1) + 16\\
&\leqslant (k^*)^2(4\Gamma_2^2 + 4m^2 + 9s^2 + m_1^2 + 1 + 12s\Gamma_2 + 12m_1\Gamma_2 + 20\Gamma_2 - 6s\\
&\quad + 4mm_1 + 6sm_1 + 28m - 8m\Gamma_2 - 12sm - 8st - 8s\Gamma_1 - 2m_1)\\
&\quad + 16(k^*(\Gamma_2 + s - m + 1 + m_1) + 1)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant (k^*)^2(36\Gamma_2^2 + 4m^2 + 9s^2 + 4m_1^2 + 1 + 24m\Gamma_2 + 36s\Gamma_2 + 24m_1\Gamma_2 \\
&\quad + 12\Gamma_2 + 6ms + 8mm_1 + 4m + 6sm_1 + 6s + 4m_1) \\
&\quad + k^*(16\Gamma_2 + 6s - 16m + 16 + 14m_1) + 16 \\
&\quad + (k^*)^2(8\Gamma_2 + 4m - 24m\Gamma_2 - 24s\Gamma_2 - 6ms - 4m_1 \\
&\quad - 32\Gamma_2^2 - 8st - 4s\Gamma_1 - 3m_1^2 - 12m_1\Gamma_2 - 4mm_1 - 6s) \\
&\leqslant (k^*(6\Gamma_2 + 2m + 3s + 2m_1 + 1))^2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
K_1 &< \frac{(2\Gamma_2 + s + 2m + 1 - m_1)k^* + 4 + \sqrt{(k^*(6\Gamma_2 + 2m + 3s + 2m_1 + 1))^2}}{2k^*} \\
&= 4\Gamma_2 + 2m + 2s + 1 + \frac{m_1}{2} + \frac{2}{k^*}.
\end{aligned}
$$

Since $n \geqslant 4\Gamma_2 + 2m + 2s + 1 + m_1/2 + 2/k^*$, (4.5) leads to a contradiction.

Case 2. Let $H(z) \equiv 0$. Now the theorem follows from Lemmas 3.10, 3.12 and 3.16. $\square$

P r o o f of Theorem 2.2. Using Lemmas 3.10 and 3.12, the theorem can be proved in the line of the proof of Theorem 2.3. So we omit the details. $\square$

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