WEAK SOLUTION FOR NONLINEAR DEGENERATE ELLIPTIC PROBLEM WITH DIRICHLET-TYPE BOUNDARY CONDITION IN WEIGHTED SOBOLEV SPACES

Abdelali Sabri, Ahmed Jamea, Hamad Talibi Alaoui, El Jadida

Received January 11, 2020. Published online April 16, 2021.
Communicated by Dagmar Medková

Abstract. In the present paper, we prove the existence and uniqueness of weak solution to a class of nonlinear degenerate elliptic p-Laplacian problem with Dirichlet-type boundary condition, the main tool used here is the variational method combined with the theory of weighted Sobolev spaces.

Keywords: degenerate elliptic problem; existence; uniqueness; weak solution; weighted Sobolev space

MSC 2020: 35A15, 35J60, 35J65, 74G30

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ be an open bounded domain and $p \in (1, \infty)$. Our aim is to prove the existence and uniqueness of weak solutions for the nonlinear degenerate elliptic problem

$$(1.1)\qquad \begin{cases} -\operatorname{div}(\omega|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))) + \alpha(u) = f & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ω is a measurable positive function defined on Ω, α is a nondecreasing continuous real function defined on $\mathbb{R}$ and Θ is a continuous function defined from $\mathbb{R}$ to $\mathbb{R}^N$, the datum f is in L^∞.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e. equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [3], [4], [8], [9], [11], [14]).

DOI: 10.21136/MB.2021.0004-20

In this paper, using the variational method, we prove in the first part the existence of weak solutions to problem (1.1); we assume that Θ is a Lipschitz function with Lipschitz constant satisfying a suitable condition (see assumption (H_4) below). In the second part, we will prove two lemmas which will be used in the proof of uniqueness part. In the particular case when $\Theta = 0$, the existence and uniqueness of weak solutions to problem (1.1) are treated by several authors (see for example [2], [3], [4], [5]).

In recent years, the study of partial differential equations and variational problems has received considerable attention in many models coming from various branches of mathematical physics, such as elastic mechanics, electrorheological fluid dynamics and image processing, etc. Degenerate phenomena appear in the area of oceanography, turbulent fluid flows, induction heating and electrochemical problems (see for example [6], [10], [13]). As models examples of applications for problem (1.1), we state the following two models:

Model 1. Filtration in a porous medium. The filtration phenomena of fluids in porous media are modeled by equation

$$\frac{\partial c(p)}{\partial t} = \nabla a[k(c(p))(\nabla p + e)], \tag{1.2}$$

where p is the unknown pressure, c volumetric moisture content, k the hydraulic conductivity of the porous medium, a the heterogeneity matrix and $-e$ is the direction of gravity.

Model 2. Fluid flow through porous media. This model is governed by equation

$$\frac{\partial \theta}{\partial t} - \operatorname{div}(|\nabla\varphi(\theta) - K(\theta)e|^{p-2}(\nabla\varphi(\theta) - K(\theta)e)) = 0, \tag{1.3}$$

where θ is the volumetric content of moisture, $K(\theta)$ the hydraulic conductivity, $\varphi(\theta)$ the hydrostatic potential and e is the unit vector in the vertical direction.

Our paper is divided into three sections, organized as follows: In Section 2, we present some preliminaries on weighted Sobolev spaces and some basic tools to prove Theorem 3.2. In Section 3, we introduce the assumptions and we give the definition of weak solution of problem (1.1), we finish this section by proving the main result.

2. Preliminaries and notations

In this section, we give some notations and definitions and state some results which will be used in this work.

Let ω be a measurable positive and a.e. finite function defined on $\mathbb{R}^N$. Further, we suppose that the following integrability conditions are satisfied:

(H_1) $\omega \in L^1_{\mathrm{loc}}(\Omega)$ and $\omega^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega)$,
(H_2) $\omega^{-s} \in L^1_{\mathrm{loc}}(\Omega)$, where $s \in (N/p, \infty) \cap (1/(p-1), \infty]$.

We define the weighted Lebesgue space $L^p(\Omega,\omega)$ by

$$L^p(\Omega,\omega) = \left\{u\colon\ \Omega \to \mathbb{R}\colon\ u \text{ is measurable and } \int_\Omega |u|^p \omega(x)\,\mathrm{d}x < \infty\right\}$$

endowed with the norm

$$\|u\|_{p,\omega} := \|u\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |u|^p \omega(x)\,\mathrm{d}x\right)^{1/p}.$$

The weighted Sobolev space is defined by

$$W^{1,p}(\Omega,\omega) = \{u \in L^p \text{ and } |\nabla u| \in L^p(\Omega,\omega)\}$$

with the norm

$$\|u\|_{1,p,\omega} = \|u\|_p + \|\nabla u\|_{p,\omega} \quad \forall\ u \in W^{1,p}(\Omega,\omega).$$

In the following, the space $W^{1,p}_0(\Omega,\omega)$ denotes the closure of C_0^∞ in $W^{1,p}(\Omega,\omega)$ endowed by the norm

$$\|u\|_{W^{1,p}_0(\Omega,\omega)} = \left(\int_\Omega |\nabla u|^p \omega(x)\,\mathrm{d}x\right)^{1/p}.$$

Let s be a real number such that s satisfies hypothesis (H_2). We define the following critical exponents:

$$p^* = \frac{Np}{N-p} \quad \text{for } p < N, \qquad p_s = \frac{ps}{1+s} < p, \qquad p_s^* = \begin{cases} \dfrac{ps}{(1+s)N - ps} & \text{if } N > p_s,\\ \infty & \text{if } N \leqslant p_s \end{cases}$$

for almost all $x \in \Omega$.

Proposition 2.1 ([9]). *Let $\Omega \subset \mathbb{R}^N$ be an open set of $\mathbb{R}^N$, and let hypothesis* (H_1) *be satisfied. Then we have*

$$L^p(\Omega,\omega) \hookrightarrow L^1_{\mathrm{loc}}(\Omega).$$

Proposition 2.2 ([9]). *Let hypothesis* (H_1) *be satisfied. The space*

$$(W^{1,p}(\Omega,\omega)\|u\|_{1,p,\omega})$$

is a separable and reflexive Banach space.

Proposition 2.3 ([9]). *Assume that hypotheses* (H_1) *and* (H_2) *hold. Then we have the continuous embedding*

$$W^{1,p}(\Omega,\omega) \hookrightarrow W^{1,p_s}(\Omega,\omega).$$

Morover, we have the compact embedding

$$W^{1,p}(\Omega,\omega) \hookrightarrow\hookrightarrow L^r(\Omega),$$

where $1 \leqslant r < p_s^*$ *for all* $x \in \Omega$.

Proposition 2.4 (Hardy-type inequality, see [9]). *There exist a weight function* ω *on* Ω *and a parameter* q, $1 < q < \infty$, *such that the inequality*

$$\left(\int_\Omega \omega(x)|u(x)|^q \,\mathrm{d}x\right)^{1/q} \leqslant C\left(\int_\Omega \omega(x)|\nabla u|^p \,\mathrm{d}x\right)^{1/p} \tag{2.1}$$

holds for every $u \in W_0^{1,p}(\Omega,\omega)$ *with a constant* $C > 0$ *inependent of* u *and, moreover, the embedding*

$$W_0^{1,p}(\Omega,\omega) \hookrightarrow L^q(\Omega,\omega)$$

expressed by inequality (2.1) *is compact.*

Given a constant $k > 0$, we define the cut function $T_k\colon \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \min(k, \max(s,-k)) = \begin{cases} s & \text{if } |s| \leqslant k, \\ k & \text{if } s > k, \\ -k & \text{if } s < -k. \end{cases}$$

For a function $u = u(x)$ defined on Ω, we define the truncated function $T_k u$ as follows: For every $x \in \Omega$, the value of $(T_k u)$ at x is just $T_k(u(x))$.

Definition 2.5 ([12]). Let Y be a reflexive Banach space and let P be an operator from Y to its dual Y'. We say that P is *monotone* if

$$\langle Pu - Pv, u - v\rangle \geqslant 0 \quad \forall\, u, v \in Y.$$

Theorem 2.6 ([12]). *Let* Y *be a reflexive real Banach space and* $P\colon Y \to Y'$ *be a bounded operator, hemi-continuous, coercive and monotone on space* Y. *Then the equation* $Pu = h$ *has at least one solution* $u \in Y$ *for each* $h \in Y'$.

Lemma 2.7 ([1]). *For $\xi, \eta \in \mathbb{R}^N$ and $1 < p < \infty$ we have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leqslant |\xi|^{p-2}\xi(\xi - \eta).$$

Lemma 2.8. *For $a \geqslant 0$, $b \geqslant 0$ and $1 \leqslant p < \infty$ we have*

$$(a + b)^p \leqslant 2^{p-1}(a^p + b^p).$$

Lemma 2.9 ([7]). *Let p, p' be two real numbers such that $p > 1$, $p' > 1$ and $1/p + 1/p' = 1$. We have*

$$||\xi|^{p-2}\xi - |\eta|^{p-2}\eta|^{p'} \leqslant C((\xi - \eta)(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta))^{\beta/2}(|\xi|^p + |\eta|^p)^{1-\beta/2} \quad \forall\, \xi, \eta \in \mathbb{R}^N,$$

where $\beta = 2$ if $1 < p \leqslant 2$ and $\beta = p'$ if $p \geqslant 2$.

Remark 2.10. Hereinafter, C_i, $i \in \{1, 2, \ldots\}$ is a positive constant.

3. Assumptions and main result

In this section, we will introduce the concept of weak solution to problem (1.1) and we will state the existence and uniqueness results for this type of solutions, firstly and in addition to hypotheses (H_1) and (H_2) listed earlier, we suppose the following assumptions:

(H_3) α is a nondecreasing continuous real function defined on $\mathbb{R}$, surjective and such that $\alpha(0) = 0$ and there exists a positive constant λ_1 such that $|\alpha(x)| \leqslant \lambda_1 |x|^{p-1}$ for all $x \in \mathbb{R}$.

(H_4) Θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^N$ such that $\Theta(0) = 0$ and for all real numbers x, y we have $|\Theta(x) - \Theta(y)| \leqslant \lambda_2 |x - y|$, where λ_2 is a real constant such that $0 < \lambda_2 < \frac{1}{2}C^{-1}$ and C is the constant given in Proposition 2.4.

(H_5) $f \in L^\infty(\Omega)$.

Definition 3.1. A function $u \in W_0^{1,p}(\Omega, \omega)$ is called a *weak solution to problem* (1.1) if

$$\int_\Omega \omega |\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))\nabla\varphi \,\mathrm{d}x + \int_\Omega \alpha(u)\varphi \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x \tag{3.1}$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega)$.

Our main result of this work is the following theorem.

Theorem 3.2. *Let hypotheses* (H_1), (H_2), (H_3), (H_4) *and* (H_5) *be satisfied. Then problem* (1.1) *has a unique weak solution.*

Proof. *Existence part.* Let the operator $T\colon W_0^{1,p}(\Omega,\omega) \to (W_0^{1,p}(\Omega,\omega))'$ (where $(W_0^{1,p}(\Omega,\omega))'$ is the dual space of $W_0^{1,p}(\Omega,\omega)$) and let $T = A - L$, where for all $u, \varphi \in W_0^{1,p}(\Omega,\omega)$

$$\langle Au, \varphi\rangle = \int_\Omega \omega|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))\nabla\varphi \,\mathrm{d}x + \int_\Omega \alpha(u)\varphi\,\mathrm{d}x := \langle A_1 u, \varphi\rangle + \langle A_2 u, \varphi\rangle$$

and

$$\langle L, \varphi\rangle = \int_\Omega f\varphi\,\mathrm{d}x.$$

The proof of the existence part of Theorem 3.2 is divided into several steps.

Step 1. The operator T is bounded. One hand, we use Hölder's inequality, hypothesis (H_4), Lemma 2.8 and Proposition 2.4. We have for any $u, \varphi \in W_0^{1,p}(\Omega,\omega)$,

$$\begin{aligned}
|\langle A_1 u, \varphi\rangle| &\leqslant \int_\Omega \omega|\nabla u - \Theta(u)|^{p-1}|\nabla\varphi|\,\mathrm{d}x \\
&\leqslant 2^{p-2}\int_\Omega \omega(|\nabla u|^{p-1} + |\Theta(u)|^{p-1})|\nabla\varphi|\,\mathrm{d}x \\
&\leqslant 2^{p-2}\int_\Omega \omega(|\nabla u|^{p-1} + \lambda_2^{p-1}|u|^{p-1})|\nabla\varphi|\,\mathrm{d}x \\
&\leqslant 2^{p-2}\int_\Omega \omega(|\nabla u|^{p-1} + \lambda_2^{p-1}C^{p-1}|\nabla u|^{p-1})|\nabla\varphi|\,\mathrm{d}x \\
&\leqslant 2^{p-2}(\lambda_2^{p-1}C^{p-1} + 1)\int_\Omega \omega|\nabla u|^{p-1}|\nabla\varphi|\,\mathrm{d}x \\
&\leqslant C_0\Big(\int_\Omega \omega|\nabla u|^p\,\mathrm{d}x\Big)^{(p-1)/p}\Big(\int_\Omega \omega|\nabla\varphi|^p\,\mathrm{d}x\Big)^{1/p} \\
&\leqslant C_0\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega)}\|\varphi\|_{W_0^{1,p}(\Omega,\omega)},
\end{aligned}$$

where $C_0 = 2^{p-2}(\lambda_2^{p-1}C^{p-1} + 1)$. This implies that A_1 is bounded.

One the other hand, using again Hölder's inequality, hypothesis (H_3) and Proposition 2.4, we get

$$|\langle A_2 u, \varphi\rangle| \leqslant \lambda_1\int_\Omega |u|^{p-1}|\varphi|\,\mathrm{d}x \leqslant \lambda_1\|u\|_p^{p-1}\|\varphi\|_p \leqslant \lambda_1 C_1 C_2\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega)}\|\varphi\|_{W_0^{1,p}(\Omega,\omega)},$$

where C_1, C_2 are two constants of compact embedding given by Proposition 2.3. This allows us to deduce that A is bounded. Finally, by Hölder's inequality, we get immediately the boundedness of L. Hence, T is bounded.

Step 2. The operator T is hemi-continuous. Let $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega,\omega)$ and $u \in W_0^{1,p}(\Omega,\omega)$ such that $u_n \to u$ strongly in $W_0^{1,p}(\Omega,\omega)$. Firstly, we will prove that A_1 is continuous on $W_0^{1,p}(\Omega,\omega)$. Indeed, we have for $\varphi \in W_0^{1,p}(\Omega,\omega)$,

$$\begin{aligned}\langle A_1u_n - A_1u, \varphi\rangle = \int_\Omega \omega(&|\nabla u_n - \Theta(u_n)|^{p-2}(\nabla u_n - \Theta(u_n)) \\ &- |\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u)))\nabla\varphi \,\mathrm{d}x.\end{aligned}$$

Set

$$F_{\theta,n} = |\nabla u_n - \Theta(u_n)|^{p-2}(\nabla u_n - \Theta(u_n)) \in L^{p'}(\Omega,\omega)^N,$$
$$F_\theta = |\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u)) \in L^{p'}(\Omega,\omega)^N,$$

where $1/p + 1/p' = 1$. Then, we have by Hölder's inequality

$$\langle A_1u_n - A_1u, \varphi\rangle \leqslant \|F_{\theta,n} - F_\theta\|_{W_0^{1,p}(\Omega,\omega)}\|\varphi\|_{W_0^{1,p}(\Omega,\omega)}.$$

This implies that

$$\|A_1u_n - A_1u\|_{(W_0^{1,p}(\Omega,\omega))'} = \sup_{\|\varphi\|_{L^p(\Omega,\omega)}\leqslant 1} |\langle A_1u_n - A_1u, \varphi\rangle| \leqslant \|F_{\theta,n} - F_\theta\|_{L^{p'}(\Omega,\omega)}.$$

Since $u_n \to u$ strongly in $W_0^{1,p}(\Omega,\omega)$, then

$$F_{\theta,n} \to F_\theta \quad \text{in } L^{p'}(\Omega,\omega)^N.$$

Consequently,

$$A_1u_n \to A_1u \quad \text{in } (W_0^{1,p}(\Omega,\omega))'.$$

This implies that A_1 is continuous on $W_0^{1,p}(\Omega,\omega)$. Secondly, applying hypothesis (H_3), we get immediately the continuity of A_2. Therefore T is hemi-continuous on $W_0^{1,p}(\Omega,\omega)$.

Step 3. The operator T is coercive. For any $u \in W_0^{1,p}(\Omega,\omega)$ we have

$$\langle Tu, u\rangle = \int_\Omega \omega|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))\nabla u \,\mathrm{d}x + \int_\Omega \alpha(u)u \,\mathrm{d}x - \int_\Omega fu \,\mathrm{d}x.$$

On the one hand, we have by application of hypothesis (H_3) that

$$\int_\Omega \alpha(u)u \,\mathrm{d}x \geqslant 0.$$

And, by Hölder's inequality and Proposition 2.3, there exists a positive constant C_3 such that

$$\int_\Omega fu \,\mathrm{d}x \leqslant C_3\|f\|_{p'}\|u\|_{W_0^{1,p}(\Omega,\omega)}.$$

This implies that

$$\langle Tu, u\rangle \geqslant \langle A_1u, u\rangle - C_3\|f\|_{p'}\|u\|_{W_0^{1,p}(\Omega,\omega)}. \tag{3.2}$$

On the other hand, using Lemma 2.7, we obtain that

$$\begin{aligned}\langle A_1 u, u\rangle &= \int_\Omega \omega|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))\nabla u \,\mathrm{d}x\\ &\geqslant \frac{1}{p}\int_\Omega \omega|\nabla u - \Theta(u)|^p \,\mathrm{d}x - \frac{1}{p}\int_\Omega \omega|\Theta(u)|^p \,\mathrm{d}x.\end{aligned}$$

Lemma 2.8 allows us to deduce that

$$\frac{1}{2^{p-1}}|\nabla u|^p = \frac{1}{2^{p-1}}|\nabla u - \Theta(u) + \Theta(u)|^p \leqslant |\nabla u - \Theta(u)|^p + |\Theta(u)|^p.$$

Then

$$\frac{1}{2^{p-1}}|\nabla u|^p - |\Theta(u)|^p \leqslant |\nabla u - \Theta(u)|^p.$$

Consequently,

$$\begin{aligned}\langle A_1 u, u\rangle &\geqslant \int_\Omega \frac{1}{p}\omega\Big(\frac{1}{2^{p-1}}|\nabla u|^p - 2|\Theta(u)|^p\Big) \,\mathrm{d}x\\ &\geqslant \frac{1}{p}\frac{1}{2^{p-1}}\int_\Omega \omega|\nabla u|^p \,\mathrm{d}x - \frac{2\lambda_2^p}{p}\int_\Omega \omega|u|^p \,\mathrm{d}x\\ &\geqslant \frac{1}{p}\frac{1}{2^{p-1}}\int_\Omega \omega|\nabla u|^p \,\mathrm{d}x - \frac{2\lambda_2^p}{p}C^p\int_\Omega \omega|\nabla u|^p \,\mathrm{d}x\\ &\geqslant \frac{1}{p}\Big(\frac{1}{2^{p-1}} - 2\lambda_2^p C^p\Big)\|u\|^p_{W_0^{1,p}(\Omega,\omega)}.\end{aligned}$$

So, the choice of constant λ_2 in (H_4) gives the existence of a positive constant C_4 such that

$$\langle A_1 u, u\rangle \geqslant C_4\|u\|^p_{W_0^{1,p}(\Omega,\omega)}. \tag{3.3}$$

Then, inequality (3.2) becomes

$$\langle Tu, u\rangle \geqslant C_4\|u\|^p_{W_0^{1,p}(\Omega,\omega)} - C_3\|f\|_{p'}\|u\|_{W_0^{1,p}(\Omega,\omega)}.$$

Therefore

$$\frac{\langle Tu, u\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega)}} \to \infty \quad \text{as } \|u\|_{W_0^{1,p}(\Omega,\omega)} \to \infty.$$

This allows us to conclude that T is coercive.

Step 4. The operator T is monotone. In this step, it suffices to prove that A is monotone. Firstly, we have by application of hypothesis (H_3) that

$$\langle A_2 u - A_2 v, u - v\rangle = \int_\Omega (\alpha(u) - \alpha(v))(u - v) \,\mathrm{d}x \geqslant 0 \quad \forall\, u, v \in W_0^{1,p}(\Omega,\omega).$$

It remains to show that $\langle A_1 u - A_1 v, u-v\rangle \geqslant 0$. Indeed, we have

$$\begin{aligned}\langle A_1 u - A_1 v, u-v\rangle &= \langle A_1 u, u\rangle + \langle A_1 v, v\rangle - \langle A_1 u, v\rangle - \langle A_1 v, u\rangle \\ &\geqslant C_4 \Upsilon_1(u,v) - C_0 \Upsilon_2(u,v) \\ &\geqslant \min(C_0, C_4)(\Upsilon_1(u,v) - \Upsilon_2(u,v)),\end{aligned}$$

where C_0 and C_4 are the two constants got in the proof of boundedness and coerciveness of operator T and

$$\begin{aligned}\Upsilon_1(u,v) &= \|u\|^p_{W_0^{1,p}(\Omega,\omega)} + \|v\|^p_{W_0^{1,p}(\Omega,\omega)}, \\ \Upsilon_2(u,v) &= \|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega)}\|v\|_{W_0^{1,p}(\Omega,\omega)} + \|v\|^{p-1}_{W_0^{1,p}(\Omega,\omega)}\|u\|_{W_0^{1,p}(\Omega,\omega)}.\end{aligned}$$

Then $\langle A_1 u - A_1 v, u-v\rangle \geqslant \min(C_0, C_4)[(\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega)} - \|v\|^{p-1}_{W_0^{1,p}(\Omega,\omega)})(\|u\|_{W_0^{1,p}(\Omega,\omega)} - \|v\|_{W_0^{1,p}(\Omega,\omega)})] \geqslant 0$. This implies that A_1 is monotone. Therefore T is monotone. Hence, by Theorem 2.6, there exists a weak solution to problem (1.1).

Uniqueness part. We firstly need the following two lemmas.

Lemma 3.3. *Let hypotheses* (H_1), (H_2), (H_3), (H_4) *and* (H_5) *be satisfied. If u is a weak solution of problem* (1.1), *then there exists a positive constant β such that for all $k>0$ we have*

$$\operatorname{meas}\{|u|>k\} \leqslant \frac{M}{\beta k^{p-1}},$$

where $M = \|f\|_{L^\infty(\Omega)}$.

P r o o f. Choosing $\varphi = T_k(u)$ in equality (3.1), we obtain

$$\int_\Omega \omega|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))\nabla T_k(u)\,\mathrm{d}x + \int_\Omega \alpha(u)T_k(u)\,\mathrm{d}x = \int_\Omega fT_k(u)\,\mathrm{d}x.$$

Since $\int_\Omega \alpha(u)T_k(u)\,\mathrm{d}x \geqslant 0$, then

$$\begin{aligned}&\int_\Omega \omega|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))\nabla T_k(u)\,\mathrm{d}x \\ &\qquad = \int_{|u|\leqslant k} \omega|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))\nabla u\,\mathrm{d}x \leqslant k\|f\|_{L^\infty(\Omega)}.\end{aligned}$$

It may be obtained similarly as (3.3), there exists a constant $\beta>0$ such that

$$\int_{|u|\leqslant k} \omega|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))\nabla u\,\mathrm{d}x \geqslant \beta\int_{|u|\leqslant k}\omega|\nabla u|^p\,\mathrm{d}x.$$

Therefore

$$\int_{|u|\leqslant k}\omega|\nabla u|^p\,\mathrm{d}x \leqslant \frac{k}{\beta}\|f\|_{L^\infty(\Omega)}.$$

As $\nabla T_k(u) = \nabla u \chi_{\{|u|\leqslant k\}}$, then

$$\int_\Omega \omega|\nabla T_k(u)|^p \,\mathrm{d}x \leqslant \frac{k}{\beta}\|f\|_{L^\infty(\Omega)},$$

where χ_B is the characteristic function of the measurable set $B \subset \mathbb{R}^N$. This implies that for all $k > 0$,

$$\frac{1}{k}\int_\Omega \omega|\nabla T_k(u)|^p \,\mathrm{d}x \leqslant \frac{M}{\beta},$$

where $M = \|f\|_{L^\infty(\Omega)}$. Noting that $\{|u| > k\} = \{|T_k(u)| > k\}$, by Markov inequality we have

$$\operatorname{meas}\{|u| > k\} \leqslant \Big(\frac{\|T_k(u)\|_{W_0^{1,p}(\Omega,\omega)}}{k}\Big)^p \leqslant \frac{M}{\beta k^{p-1}}.$$

This completes the proof. □

Lemma 3.4. *Let hypotheses* (H$_1$), (H$_2$), (H$_3$), (H$_4$) *and* (H$_5$) *be satisfied. If u is a weak solution of problem* (1.1), *then*

(1) $$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}} \omega|\nabla u - \Theta(u)|^{p-2}|\nabla u - \Theta(u)|\nabla u \,\mathrm{d}x = 0,$$

(2) $$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}} \omega|\nabla u|^p \,\mathrm{d}x = 0,$$

(3) $$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}} \omega|\nabla u - \Theta(u)|^p \,\mathrm{d}x = 0.$$

Proof. (1) Let k and h be two real numbers such that $0 < k < h$. Taking $\varphi = T_k(u - T_h(u))$ in equality (3.1), we get

(3.4) $$\int_\Omega \omega(|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u)))\nabla T_k(u - T_h(u)) \,\mathrm{d}x + \int_\Omega \alpha(u) T_k(u - T_h(u)) \,\mathrm{d}x = \int_\Omega f T_k(u - T_h(u)) \,\mathrm{d}x.$$

Firstly, we have

$$\int_\Omega \alpha(u) T_k(u - T_h(u)) \,\mathrm{d}x = \int_{\{|u|>h\}} \alpha(u) T_k(u - h\operatorname{sign}(u)) \,\mathrm{d}x,$$

and

$$\operatorname{sign}(u)\chi_{\{|u|>h\}} = \operatorname{sign}(u - h\operatorname{sign}(u))\chi_{\{|u|>h\}} = \operatorname{sign}(T_k(u - h\operatorname{sign}(u)))\chi_{\{|u|>h\}},$$

then

$$\int_\Omega \alpha(u) T_k(u - T_h(u)) \,\mathrm{d}x \geqslant 0.$$

Therefore, equality (3.4) becomes

$$\int_{\{h\leqslant|u|\leqslant h+k\}} \omega(|\nabla u-\Theta(u)|^{p-2}(\nabla u-\Theta(u)))\nabla T_k(u-T_h(u))\,\mathrm{d}x \leqslant k\int_{\{|u|>h\}} |f|\,\mathrm{d}x.$$

By Lemma 3.3, we deduce that meas$\{|u|>h\}$ tends to zero as h goes to infinity, so

$$\lim_{h\to\infty}\int_{\{|u|>h\}} |f|\,\mathrm{d}x = 0.$$

This implies that

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}} \omega|\nabla u-\Theta(u)|^{p-2}|\nabla u-\Theta(u)|\nabla u\,\mathrm{d}x = 0.$$

(2) By using Lemma 2.7 and Lemma 2.8, we have

$$\frac{1}{p}|\nabla u|^p - \frac{2}{p}|\Theta(u)|^p \leqslant |\nabla u-\Theta(u)|^{p-2}|\nabla u-\Theta(u)|\nabla u.$$

We use hypothesis (H_4) and we get

$$\frac{1}{p2^{p-1}}|\nabla u|^p - \frac{2\lambda_2^p}{p}|u|^p \leqslant |\nabla u-\Theta(u)|^{p-2}|\nabla u-\Theta(u)|\nabla u.$$

This implies that

$$\frac{1}{p2^{p-1}}\int_{\Omega_k^h} \omega|\nabla u|^p\,\mathrm{d}x - \frac{2\lambda_2^p}{p}\int_{\Omega_k^h} \omega|u|^p\,\mathrm{d}x \leqslant \int_{\Omega_k^h} \omega|\nabla u-\Theta(u)|^{p-2}|\nabla u-\Theta(u)|\nabla u\,\mathrm{d}x,$$

where $\Omega_k^h=\{h\leqslant|u|\leqslant h+k\}$, and by using Proposition 2.4, we obtain

$$\frac{1}{p2^{p-1}}\int_{\Omega_k^h} \omega|\nabla u|^p\,\mathrm{d}x - \frac{2\lambda_2^p C^p}{p}\int_{\Omega_k^h} \omega|\nabla u|^p\,\mathrm{d}x \\ \leqslant \int_{\Omega_k^h} \omega|\nabla u-\Theta(u)|^{p-2}|\nabla u-\Theta(u)|\nabla u\,\mathrm{d}x.$$

Then, by hypothesis (H_4), there exists a positive constant C_5 such that

$$\int_{\Omega_k^h} \omega|\nabla u|^p\,\mathrm{d}x \leqslant C_5\int_{\Omega_k^h} \omega|\nabla u-\Theta(u)|^{p-2}|\nabla u-\Theta(u)|\nabla u\,\mathrm{d}x.$$

This allows us to deduce that

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}} \omega|\nabla u|^p\,\mathrm{d}x = 0.$$

(3) We have by Lemma 2.7

$$\frac{1}{p}|\nabla u-\Theta(u)|^p-\frac{1}{p}|\Theta(u)|^p \leqslant |\nabla u-\Theta(u)|^{p-2}|\nabla u-\Theta(u)|\nabla u.$$

Then

$$\begin{aligned}
&\frac{1}{p}\int_{\Omega_k^h}\omega|\nabla u-\Theta(u)|^p\,\mathrm{d}x\\
&\qquad\leqslant \int_{\Omega_k^h}\omega|\nabla u-\Theta(u)|^{p-2}|\nabla u-\Theta(u)|\nabla u\,\mathrm{d}x+\frac{1}{p}\int_{\Omega_k^h}\omega|\Theta(u)|^p\,\mathrm{d}x\\
&\qquad\leqslant \int_{\Omega_k^h}\omega|\nabla u-\Theta(u)|^{p-2}|\nabla u-\Theta(u)|\nabla u\,\mathrm{d}x+\frac{\lambda_2^p}{p}\int_{\Omega_k^h}\omega|u|^p\,\mathrm{d}x\\
&\qquad\leqslant \int_{\Omega_k^h}\omega|\nabla u-\Theta(u)|^{p-2}|\nabla u-\Theta(u)|\nabla u\,\mathrm{d}x+\frac{\lambda_2^pC^p}{p}\int_{\Omega_k^h}\omega|\nabla u|^p\,\mathrm{d}x.
\end{aligned}$$

We apply the previous results (1) and (2) and we get that

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}}\omega|\nabla u-\Theta(u)|^p\,\mathrm{d}x=0.$$

□

Now, let u and v be two weak solutions of degenerate elliptic problem (1.1) and let h, k be two positive real numbers such that $1<k<h$. For the solution u, we take $\varphi=T_k(u-T_h(v))$ in equality (3.1), and for the solution v, we take $\varphi=T_k(v-T_h(u))$ as test function. We have

$$\begin{aligned}
\int_\Omega \alpha(u)T_k(u-T_h(v))\,\mathrm{d}x+\int_\Omega \omega(|\nabla u-\Theta(u)|^{p-2}(\nabla u-\Theta(u)))\nabla T_k(u-T_h(v))\,\mathrm{d}x\\
=\int_\Omega fT_k(u-T_h(v))\,\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega \alpha(v)T_k(v-T_h(u))\,\mathrm{d}x+\int_\Omega \omega(|\nabla v-\Theta(v)|^{p-2}(\nabla v-\Theta(v)))\nabla T_k(v-T_h(u))\,\mathrm{d}x\\
=\int_\Omega fT_k(v-T_h(u))\,\mathrm{d}x.
\end{aligned}$$

We divide the two equalities above by k and we pass to the limit where $k\to 0$ and $h\to\infty$. We find by applying Dominated Convergence Theorem that

$$\|\alpha(u)-\alpha(v)\|_1+\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}(k;h)=0, \tag{3.5}$$

where

$$\begin{aligned}\mathcal{I}(k;h) = &\int_\Omega \omega(|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u)))\nabla T_k(u - T_h(v))\,\mathrm{d}x\\ &+ \int_\Omega \omega(|\nabla v - \Theta(v)|^{p-2}(\nabla v - \Theta(v)))\nabla T_k(v - T_h(u))\,\mathrm{d}x.\end{aligned}$$

We will prove that

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}(k;h) \geqslant 0.$$

We consider the following decomposition:

$$\begin{aligned}\Omega_1(h) &= \{|u| \leqslant h; |v| \leqslant h\}; \quad \Omega_2(h) = \{|u| \leqslant h; |v| > h\},\\ \Omega_3(h) &= \{|u| > h; |v| \leqslant h\}; \quad \Omega_4(h) = \{|u| > h; |v| > h\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_i(k;h) = &\int_{\Omega_i(h)} \omega(|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u)))\nabla T_k(u - T_h(v))\,\mathrm{d}x\\ &+ \int_{\Omega_i(h)} \omega(|\nabla v - \Theta(v)|^{p-2}(\nabla v - \Theta(v)))\nabla T_k(v - T_h(u))\,\mathrm{d}x\end{aligned}$$

for $i = 1, \dots, 4$. Firstly, we have

$$\begin{aligned}\mathcal{I}_1(k;h) = &\int_{\Omega_h^k(1)} \omega(|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))\\ &\qquad - |\nabla v - \Theta(v)|^{p-2}(\nabla v - \Theta(v)))\nabla T_k(u - v)\,\mathrm{d}x\\ := &\ \mathcal{I}_1^1(k;h) + \mathcal{I}_1^2(k;h),\end{aligned}$$

where

$$\begin{aligned}\Omega_h^k(1) &= \{|u - v| \leqslant k;\ |u| \leqslant h;\ |v| \leqslant h\},\\ \mathcal{I}_1^1(k;h) &= \int_{\Omega_h^k(1)} \omega(|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))\\ &\qquad - |\nabla v - \Theta(v)|^{p-2}(\nabla v - \Theta(v)))\Phi_\theta(u;v)\,\mathrm{d}x,\\ \mathcal{I}_1^2(k;h) &= \int_{\Omega_h^k(1)} \omega(|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))\\ &\qquad - |\nabla v - \Theta(v)|^{p-2}(\nabla v - \Theta(v)))\Psi_\theta(u;v)\,\mathrm{d}x\end{aligned}$$

and

$$\Phi_\theta(u;v) = (\nabla u - \Theta(u)) - (\nabla v - \Theta(v)); \quad \Psi_\theta(u;v) = \Theta(u) - \Theta(v).$$

We prove that

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1(k;h) = 0.$$

For this, we divide the proof into two cases according to the value of p.

Case ($1 < p \leqslant 2$): Let $\varepsilon > 0$. We apply Young's inequality and we find

$$\begin{aligned}\mathcal{I}_1^2(k;h) \leqslant \frac{\varepsilon}{p'} \int_{\Omega_h^k(1)} \omega|(|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u)) \\ - (|\nabla v - \Theta(v)|^{p-2}(\nabla v - \Theta(v)))|^{p'}\,\mathrm{d}x \\ + \frac{1}{\varepsilon p} \int_{\Omega_h^k(1)} \omega|\Theta(u) - \Theta(v)|^p\,\mathrm{d}x.\end{aligned}$$

We apply Lemma 2.9 and hypothesis (H_4) and get

$$|\mathcal{I}_1^2(k;h)| \leqslant \varepsilon C_6 \mathcal{I}_1^1(k;h) + \frac{C_7}{\varepsilon} k^p,$$

which implies that

$$\lim_{k\to 0} \frac{1}{k} |\mathcal{I}_1^2(k;h)| \leqslant \varepsilon C_6 \lim_{k\to 0} \frac{1}{k} \mathcal{I}_1^1(k;h). \tag{3.6}$$

If $\lim\limits_{k\to 0} k^{-1}\mathcal{I}_1^1(k,h) = 0$, the above inequality (3.6) becomes

$$\lim_{k\to 0} \frac{1}{k} \mathcal{I}_1^2(k,h) = 0, \quad \text{i.e.} \ \lim_{h\to\infty} \lim_{k\to 0} \frac{1}{k} \mathcal{I}_1(k,h) = 0.$$

If $0 < \lim\limits_{k\to 0} k^{-1}\mathcal{I}_1^1(k,h) < \infty$, we take

$$\varepsilon = \frac{1}{h \lim_{k\to 0} k^{-1} \mathcal{I}_1^1(k,h)}$$

in (3.6), we deduce that

$$\lim_{h\to\infty} \lim_{k\to 0} \frac{1}{k} \mathcal{I}_1^2(k,h) = 0.$$

It follows that

$$\lim_{h\to\infty} \lim_{k\to 0} \frac{1}{k} \mathcal{I}_1(k,h) \geqslant 0.$$

If $\lim\limits_{k\to 0} k^{-1}\mathcal{I}_1^1(k,h) = \infty$, we have by using hypothesis (H_4) that

$$\begin{aligned}|\mathcal{I}_1^2(k;h)| &\leqslant k\lambda_2 \int_{\Omega_h^k(1)} \omega||\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u)) - |\nabla v - \Theta(v)|^{p-2}(\nabla v - \Theta(v))| \\ &\leqslant k\lambda_2 \int_{\Omega_h^k(1)} \omega(|\nabla u - \Theta(u)|^{p-1} + |\nabla v - \Theta(v)|^{p-1})\,\mathrm{d}x.\end{aligned}$$

Consequently,

$$\frac{1}{k}|\mathcal{I}_1^2(k;h)| \leqslant \lambda_2 \int_{\Omega_h^k(1)} \omega(|\nabla u - \Theta(u)|^{p-1} + |\nabla v - \Theta(v)|^{p-1})\,\mathrm{d}x.$$

On the other hand, for the solution u we take $\varphi = T_k(u)$ in equality (3.1) and we find

$$\int_{\{|u|\leqslant k\}} \omega(|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u)))\nabla u \,\mathrm{d}x \leqslant kC_8.$$

This implies that

$$\int_{\{|u|\leqslant k\}} \omega|\nabla u - \Theta(u)|^p \,\mathrm{d}x \leqslant kC_8 + C_9 \int_{\{|u|\leqslant k\}} \omega|\Theta(u)|^p \,\mathrm{d}x \leqslant kC_8 + C_{10}k^p \leqslant C_{11}k^p.$$

Similarly, we prove that

$$\int_{\{|u|\leqslant k\}} \omega|\nabla v - \Theta(v)|^p \,\mathrm{d}x \leqslant C_{12}k^p.$$

Therefore

$$\frac{1}{k}|\mathcal{I}_1^2(k;h)| \leqslant \lambda_2 C_{13}(h+k)^p, \quad \text{i.e. } \lim_{k\to 0}\frac{1}{k}|\mathcal{I}_1^2(k;h)| \leqslant \lambda_2 C_{14}h^p.$$

Thus, it follows that

$$\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^1(k,h) + \lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^2(k,h) = \infty.$$

Then

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1(k,h) = \infty.$$

Case ($p > 2$): We use Young's inequality to deduce

$$\frac{1}{k}|\mathcal{I}_1^2(k,h)| \leqslant \frac{C_{15}\varepsilon(k+h)}{p'k} + \frac{C_{16}}{p\varepsilon}k^{p-1} \quad \forall\, \varepsilon > 0.$$

Then, we take $\varepsilon = k/h^2$ and obtain

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^2(k,h) = 0.$$

Consequently,

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1(k,h) \geqslant 0.$$

Secondly, we have

$$\begin{aligned}
\mathcal{I}_2(k;h) = &\int_{\Omega_2(h)} \omega|\nabla v - \Theta(v)|^{p-2}(\nabla v - \Theta(v))\nabla T_k(v-u)\,\mathrm{d}x \\
&+ \int_{\Omega_2(h)} \omega|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))\nabla T_k(u - h\,\mathrm{sign}(v))\,\mathrm{d}x \\
:=& \mathcal{I}_2^1(k;h) + \mathcal{I}_2^2(k;h),
\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_2^1(k;h) &= \int_{\Omega_2(h)} \omega|\nabla v - \Theta(v)|^{p-2}(\nabla v - \Theta(v))\nabla T_k(v-u)\,\mathrm{d}x \\ &= \int_{\Omega_{h,k}^{2,1}} \omega|\nabla v - \Theta(v)|^{p-2}(\nabla v - \Theta(v))\nabla v\,\mathrm{d}x \\ &\quad - \int_{\Omega_{h,k}^{2,1}} \omega|\nabla v - \Theta(v)|^{p-2}(\nabla v - \Theta(v))\nabla u\,\mathrm{d}x, \\ \mathcal{I}_2^2(k;h) &= \int_{\Omega_2(h)} \omega|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))\nabla T_k(u - h\,\mathrm{sign}(v))\,\mathrm{d}x \\ &= \int_{\Omega_{h,k}^{2,2}} \omega|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))\nabla u\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\Omega_{h,k}^{2,1} &= \{|u| \leqslant h; |v| > h; |v-u| \leqslant k\}; \\ \Omega_{h,k}^{2,2} &= \{|u| \leqslant h; |v| > h; |u - h\,\mathrm{sign}(v)| \leqslant k\}.\end{aligned}$$

On one hand, since ω is a positive function, then, applying Lemmas 2.7 and 2.8, we get

$$\mathcal{I}_2^2(k;h) \geqslant 0.$$

In the same manner, we prove that

$$\int_{\Omega_{h,k}^{2,1}} \omega|\nabla v - \Theta(v)|^{p-2}(\nabla v - \Theta(u))\nabla v\,\mathrm{d}x \geqslant 0.$$

On the other hand, by Hölder's inequality, we have

$$\begin{aligned}&\left|\int_{\Omega_{h,k}^{2,1}} \omega|\nabla v - \Theta(v)|^{p-2}(\nabla v - \Theta(u))\nabla u\,\mathrm{d}x\right| \\ &\qquad \leqslant \left(\int_{\Omega_{h,k}^{2,1}} \omega|\nabla v - \Theta(v)|^p\,\mathrm{d}x\right)^{(p-1)/p}\left(\int_{\Omega_{h,k}^{2,1}} \omega|\nabla u|^p\,\mathrm{d}x\right)^{1/p}.\end{aligned}$$

Hence, by application of Lemma 3.4, we get

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\Omega_{h,k}^{2,1}} \omega|\nabla v - \Theta(v)|^{p-2}(\nabla v - \Theta(u))\nabla u\,\mathrm{d}x = 0.$$

Then

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_2^1(k;h) \geqslant 0.$$

Therefore

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_2(k;h) \geqslant 0.$$

Finally, in the same manner, we show that

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}(\mathcal{I}_3(k;h)+\mathcal{I}_4(k;h))\geqslant 0.$$

Hence

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}(k;h)\geqslant 0.$$

Therefore, inequality (3.5) becomes

$$\|\alpha(u)-\alpha(v)\|_1\leqslant 0.$$

This implies that

$$u=v\quad\text{a.e. in }\Omega.$$

This completes the proof of Theorem 3.2. □

References

[1] *A. Abassi, A. El Hachimi, A. Jamea*: Entropy solutions to nonlinear Neumann problems with L^1-data. Int. J. Math. Stat. *2* (2008), 4–17. zbl MR

[2] *A. C. Cavalheiro*: Weighted Sobolev spaces and degenerate elliptic equations. Bol. Soc. Parana. Mat. (3) *26* (2008), 117–132. zbl MR doi

[3] *A. C. Cavalheiro*: Existence and uniqueness of solutions for some degenerate nonlinear Dirichlet problems. J. Appl. Anal. *19* (2013), 41–54. zbl MR doi

[4] *A. C. Cavalheiro*: Existence results for Dirichlet problems with degenerated p-Laplacian and p-biharmonic operators. Appl. Math. E-Notes *13* (2013), 234–242. zbl MR

[5] *A. C. Cavalheiro*: Existence and uniqueness of solutions for Dirichlet problems with degenerate nonlinear elliptic operators. Differ. Equ. Dyn. Syst. *24* (2016), 305–317. zbl MR doi

[6] *Y. Chen, S. Levine, M. Rao*: Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. *66* (2006), 1383–1406. zbl MR doi

[7] *J. I. Diaz, F. De Thelin*: On a nonlinear parabolic problem arising in some models related to turbulent flows. SIAM J. Math. Anal. *25* (1994), 1085–1111. zbl MR doi

[8] *P. Drábek*: The least eigenvalues of nonhomogeneous degenerated quasilinear eigenvalue problems. Math. Bohem. *120* (1995), 169–195. zbl MR doi

[9] *P. Drábek, A. Kufner, V. Mustonen*: Pseudo-monotonicity and degenerated or singular elliptic operators. Bull. Aust. Math. Soc. *58* (1998), 213–221. zbl MR doi

[10] *P. A. Hästö*: The $p(x)$-Laplacian and applications. J. Anal. *15* (2007), 53–62. zbl MR

[11] *J. Heinonen, T. Kilpeläinen, O. Martio*: Nonlinear Potential Theory of Degenerate Elliptic Equations. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1993. zbl MR

[12] *J. L. Lions*: Quelques méthodes de résolution des problèmes aux limites non linéaires. Etudes mathematiques. Dunod, Paris, 1969. (In French.) zbl MR

[13] *M. Růžička*: Electrorheological Fluids: Modeling and Mathematical Theory. Lectures Notes in Mathematics 1748. Springer, Berlin, 2000. zbl MR doi

[14] *B. O. Turesson*: Nonlinear Potential Theory and Weighted Sobolev Spaces. Lecture Notes in Mathematics 1736. Springer, Berlin, 2000. zbl MR doi

Authors' address: *Abdelali Sabri*, *Ahmed Jamea* (corresponding author), *Hamad Talibi Alaoui*, Equipe de Mathématique Appliquée à la Physique et à l'Industrie (EMAPI), Faculté des Sciences, Université Chouaib Doukkali, El Jadida, 24000, Morocco, e-mail: abdelali.sabri21@gmail.com, a.jamea77@gmail.com, talibi_1@hotmail.fr.