L^p-IMPROVING PROPERTIES OF CERTAIN SINGULAR MEASURES ON THE HEISENBERG GROUP

Pablo Rocha, Bahía Blanca

Received January 20, 2020. Published online April 21, 2021.
Communicated by Jiří Spurný

Abstract. Let μ_A be the singular measure on the Heisenberg group $\mathbb{H}^n$ supported on the graph of the quadratic function $\varphi(y) = y^t Ay$, where A is a $2n \times 2n$ real symmetric matrix. If $\det(2A \pm J) \neq 0$, we prove that the operator of convolution by μ_A on the right is bounded from $L^{(2n+2)/(2n+1)}(\mathbb{H}^n)$ to $L^{2n+2}(\mathbb{H}^n)$. We also study the type set of the measures $\mathrm{d}\nu_\gamma(y,s) = \eta(y)|y|^{-\gamma}\mathrm{d}\mu_A(y,s)$, for $0 \leqslant \gamma < 2n$, where η is a cut-off function around the origin on $\mathbb{R}^{2n}$. Moreover, for $\gamma = 0$ we characterize the type set of ν_0.

Keywords: Heisenberg group; singular Borel measure; L^p-improving property

MSC 2020: 43A80, 42A38

1. Introduction

Let I_n be the $n \times n$ identity matrix and J be the $2n \times 2n$ skew-symmetric matrix given by

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{1}$$

The Heisenberg group is $\mathbb{H}^n = \mathbb{R}^{2n} \times \mathbb{R}$ endowed with the group law (non-commutative)

$$(x,t) \cdot (y,s) = (x + y, t + s + \langle x, y \rangle),$$

where $\langle x, y \rangle$ is the standard symplectic form on $\mathbb{R}^{2n}$, i.e. $\langle x, y \rangle = x^t Jy$ with neutral element $(0,0)$ and with inverse $(x,t)^{-1} = (-x,-t)$. The topology in $\mathbb{H}^n$ is induced by $\mathbb{R}^{2n+1}$, so the borelian sets of $\mathbb{H}^n$ are identified with those of $\mathbb{R}^{2n+1}$. The Haar measure in $\mathbb{H}^n$ is the Lebesgue measure of $\mathbb{R}^{2n+1}$, thus $L^p(\mathbb{H}^n) \equiv L^p(\mathbb{R}^{2n+1})$. Given

DOI: 10.21136/MB.2021.0014-20

a borelian function $f\colon \mathbb{H}^n \to \mathbb{C}$ and a Borel measure μ on $\mathbb{H}^n$, define the convolution by μ on the right by

$$(2) \qquad (f*\mu)(x,t) = \int_{\mathbb{H}^n} f((x,t)\cdot(y,s)^{-1})\,\mathrm{d}\mu(y,s),$$

provided the integral exists.

A Borel measure μ on the Heisenberg group $\mathbb{H}^n$ is said to be *L^p-improving* if the operator $T_\mu : f \mapsto f*\mu$ is bounded from $L^p(\mathbb{H}^n)$ into $L^q(\mathbb{H}^n)$ for some $1 \leqslant p < q < \infty$. A remarkable fact is that singular measures can be L^p-improving. If in (2) we replace the Heisenberg group $\mathbb{H}^n$ by $\mathbb{R}^n$ with the ordinary convolution in $\mathbb{R}^n$ and considering there $\mu = \eta\sigma_M$, where σ_M is the surface measure on a given manifold M (in $\mathbb{R}^n$) and η is a smooth cut-off function, then the L^p-improving properties of a measure of this type are closely related to the existence of a certain amount of curvature of the manifold M (see [5], [6], [7]). A similar result holds on general Lie groups (see Theorem 1.1, page 362 in [9]).

A more delicate problem consists in determining the exact range of pairs (p,q) for which $L^p * \mu \subseteq L^q$ embeds continuously. Given a manifold M (in $\mathbb{H}^n$), define the *type set* $E_{\eta\sigma_M}$ by

$$E_{\eta\sigma_M} = \Big\{\Big(\frac{1}{p},\frac{1}{q}\Big) \in [0,1]\times[0,1]\colon\ \|T_{\eta\sigma_M}\|_{p,q} < \infty\Big\}.$$

A very interesting survey of results concerning the type sets for convolution operators with singular measures in $\mathbb{R}^n$ can be found in [8].

In the $\mathbb{H}^n$ setting, Secco in [10] and [11] obtained L^p-improving properties of measures supported on curves in $\mathbb{H}^1$, under certain assumptions. In [9], Ricci and Stein showed that the type set of the measure given by (3) for the case $\varphi \equiv 0$, $\gamma = 0$ and $n = 1$ is the triangle with vertices $(0,0)$, $(1,1)$ and $(\frac{3}{4},\frac{1}{4})$. In [3] and [4], the author jointly with Godoy generalized the work of Ricci and Stein for the case $\varphi(w) = w^t A w = \sum\limits_{j=1}^{n} \alpha_j |w_j|^2$, where A is a $2n\times 2n$ real diagonal matrix such that $a_{ii} = a_{(i+1)(i+1)}$ for $i = 2j-1$ with $j = 1,2,\ldots,n$, $\alpha_j = a_{(2j-1)(2j-1)}$, $w_j \in \mathbb{R}^2$, $0 \leqslant \gamma < 2n$ and $n \in \mathbb{N}$. There we also gave some examples of surfaces with degenerate curvature at the origin.

Let $\varphi\colon \mathbb{R}^{2n} \to \mathbb{R}$ be the function defined by $\varphi(y) = y^t A y$, where A is a $2n\times 2n$ real symmetric matrix. It is well known that if A is an arbitrary matrix, then there exists a symmetric matrix $\tilde{A}$ such that $y^t A y = y^t \tilde{A} y$ for all y. We consider two borelian measures on $\mathbb{H}^n$ supported on the graph of φ, μ_A and ν_γ, $0 \leqslant \gamma < 2n$, given by

$$\mu_A(E) = \int_{\mathbb{R}^{2n}} \chi_E(y,\varphi(y))\,\mathrm{d}y$$

and

$$\nu_\gamma(E) = \int_{\mathbb{R}^{2n}} \chi_E(y, \varphi(y))\eta(y)|y|^{-\gamma}\,\mathrm{d}y, \tag{3}$$

where $\eta : \mathbb{R}^{2n} \to [0,1]$ is a smooth cut-off function such that $\eta(y) = 1$ if $|y| \leqslant 1$, $\eta(y) = 0$ if $|y| \geqslant 2$, and E is a borelian set of $\mathbb{H}^n$. Let $T_{\mu_A} f = f * \mu_A$ and $T_{\nu_\gamma} f = f * \nu_\gamma$ be the operators of convolution by μ_A and ν_γ on the right, respectively.

We are interested in studying the L^p-improving properties of the operator T_{μ_A} and in the characterization of the type set E_{ν_γ}. We point out that our measure μ_A is not the surface measure on the graph $\mathrm{gr}(\varphi)$ of φ, however the measures $\eta\mu_A$ and $\eta\sigma_{\mathrm{gr}(\varphi)}$ are equivalent, see Proposition 2 below, so $E_{\eta\mu_A} = E_{\eta\sigma_{\mathrm{gr}(\varphi)}}$.

The following restrictions for the type sets E_{ν_γ}, $0 \leqslant \gamma < 2n$, were proved in [3] and [4] for the case $\varphi(w_1, \ldots, w_n) = \sum\limits_{j=1}^{n} \alpha_j |w_j|^2$ with $w_j \in \mathbb{R}^2$. It is easy to see that such an argument works as well for our function $\varphi(y) = y^t A y$. Thus, if $(1/p, 1/q) \in E_{\nu_\gamma}$, $0 \leqslant \gamma < 2n$, then

$$p \leqslant q, \quad \frac{1}{q} \geqslant \frac{2n+1}{p} - 2n, \quad \frac{1}{q} \geqslant \frac{1}{(2n+1)p}. \tag{4}$$

Another necessary condition for the pair $(1/p, 1/q)$ to be in E_{ν_γ} is the following:

$$\frac{1}{q} \geqslant \frac{1}{p} - \frac{2n-\gamma}{2n+2}. \tag{5}$$

This last condition is relevant only for the case $0 < \gamma < 2n$. Let D be the point of intersection, in the $(1/p, 1/q)$ plane, of the lines $1/q = (2n+1)/p - 2n$, $1/q = 1/p - (2n-\gamma)/(2n+2)$, and let D' be its symmetric image with respect to the symmetry axis $1/q = 1 - 1/p$. So

$$D = \Bigl(\frac{4n^2 + 2n + \gamma}{2n(2n+2)}, \frac{2n + (2n+1)\gamma}{2n(2n+2)}\Bigr) = \Bigl(\frac{1}{p_D}, \frac{1}{q_D}\Bigr) \quad \text{and} \quad D' = \Bigl(1 - \frac{1}{q_D}, 1 - \frac{1}{p_D}\Bigr).$$

Since $0 \leqslant \gamma < 2n$, it is clear that $\|T_{\nu_\gamma} f\|_p \leqslant c\|f\|_p$ for all Borel functions $f \in L^p(\mathbb{H}^n)$ and all $1 \leqslant p \leqslant \infty$, so $(1/p, 1/p) \in E_{\mu_\gamma}$. Thus, for $0 < \gamma < 2n$ the set E_{ν_γ} is contained in the closed trapezoid with vertices $(0,0)$, $(1,1)$, D and D', and the set E_{ν_0} is contained in the closed triangle with vertices $(0,0)$, $(1,1)$ and $((2n+1)/(2n+2), 1/(2n+2))$.

In Section 3, our main result appears. There we prove that the operator T_{μ_A} is bounded from $L^{(2n+2)/(2n+1)}(\mathbb{H}^n)$ to $L^{2n+2}(\mathbb{H}^n)$, see Theorem 3 below. This result allows us to characterize the type set E_{ν_0} as well as the interior of E_{ν_γ} for $0 < \gamma < 2n$.

More precisely, we show that E_{ν_0} is the closed triangle with vertices $(0,0)$, $(1,1)$ and $((2n+1)/(2n+2), 1/(2n+2))$ and the interior of E_{ν_γ} coincides with the interior of the closed trapezoid with vertices $(0,0)$, $(1,1)$, D and D', see Theorem 4 and Theorem 6 below.

Throughout this paper, c will denote a positive real constant not necessarily the same at each occurrence. The symbol $A \lesssim B$ stands for the inequality $A \leqslant cB$ for a constant c. We use the following convention for the Fourier transform in $\mathbb{R}^n$ $\hat{f}(\xi) = \int f(x) \mathrm{e}^{-\mathrm{i}\xi\cdot x}\,\mathrm{d}x$. The Fourier transform $\widehat{u}$ of a distribution u on $\mathbb{R}^n$ is the distribution defined by $(\widehat{u}, \varphi) = (u, \widehat{\varphi})$ for all rapidly decreasing functions φ on $\mathbb{R}^n$.

2. Preliminaries

In the sequel J will denote the $2n \times 2n$ skew-symmetric matrix defined in (1). It is easy to check that

(a) $J^2 = -I$,

(b) $J^t = -J$,

(c) $x^t J x = 0$ for all $x \in \mathbb{R}^{2n}$,

(d) $x^t J y = -y^t J x$ for all $x, y \in \mathbb{R}^{2n}$.

Lemma 1. *Let A be a $2n \times 2n$ real diagonal matrix. Then*

$$\det(A \pm J) = (a_{11}a_{(n+1)(n+1)} + 1) \cdot (a_{22}a_{(n+2)(n+2)} + 1) \ldots (a_{nn}a_{(2n)(2n)} + 1),$$

where the a_{ii}'s are the diagonal entries of A.

Proof. Since $\det(A+J) = \det((A+J)^t) = \det(A-J)$, it is sufficient to prove the statement of the lemma for $\det(A+J)$. Applying induction on n, the lemma follows. □

Proposition 2. *Let A be a $2n\times 2n$ real symmetric matrix. Then the graph of the function $\varphi(y) = y^t A y$ generates all the group $\mathbb{H}^n$. Moreover, the measure $\nu_0 = \eta\mu_A$ is equivalent to the measure $\eta\sigma$, where η is a cut-off function and σ is the surface measure on the graph of φ.*

Proof. The first statement will follow if we prove that $(x,0)$ and $(0,t)$ belong to the set $G_{\mathrm{gr}(\varphi)}$ generated by the graph $\mathrm{gr}(\varphi)$ of φ, since $(x,t) = (x,0)\cdot(0,t)$. It is clear that $(x,\varphi(x)) \in G_{\mathrm{gr}(\varphi)}$, so $(-t^{1/2}x, \varphi(t^{1/2}x)) = (-t^{1/2}x, \varphi(-t^{1/2}x)) \in G_{\mathrm{gr}(\varphi)}$ for all $x \in \mathbb{R}^{2n}$ and all $t > 0$. From that it follows that $(0, t\varphi(x)) \in G_{\mathrm{gr}(\varphi)}$ for all $t > 0$ and all x. If A is a non-null matrix, then $(0,-t) = (0,t)^{-1} \in G_{\mathrm{gr}(\varphi)}$ and $(x,0) = (x,\varphi(x))\cdot(0,-\varphi(x)) \in G_{\mathrm{gr}(\varphi)}$. If A is the null matrix, it is sufficient to prove that $(0,t) \in G_{\mathrm{gr}(\varphi)}$ for all t. Indeed, for x and y such that $\langle x, y\rangle \neq 0$ we have $(0,t) = (x,0)\cdot(ty/\langle x,y\rangle, 0)\cdot(-x - ty/\langle x,y\rangle, 0) \in G_{\mathrm{gr}(\varphi)}$. So $G_{\mathrm{gr}(\varphi)} = \mathbb{H}^n$.

For the second part of the proposition, we have that the surface measure on the graph of φ is given by

$$\sigma(E) = \int_{\varphi^{-1}(E)} \sqrt{\det[(\partial_{x_i}\varphi, \partial_{x_j}\varphi)_x]}\,\mathrm{d}x,$$

where $\varphi(x) = (x, \varphi(x))$ and E is a borelian set of $\mathbb{R}^{2n+1}$ (see pages 43–45 in [1]). A computation gives

$$\det[(\partial_{x_i}\varphi, \partial_{x_j}\varphi)_x] = 1 + \sum_{j=1}^{2n} (\partial_{x_j}\varphi(x))^2 \quad \forall\, x.$$

So

$$\int_{\mathbb{R}^{2n}} \chi_E(\varphi(x))\eta(x)\,\mathrm{d}x \leqslant \int_{\varphi^{-1}(E)} \sqrt{\det[(\partial_{x_i}\varphi, \partial_{x_j}\varphi)_x]}\eta(x)\,\mathrm{d}x \lesssim \int_{\mathbb{R}^{2n}} \chi_E(\varphi(x))\eta(x)\,\mathrm{d}x.$$

Then ν_0 is equivalent to $\eta\sigma$. □

The λ-twisted convolution is defined by

$$(f \times_\lambda g)(x) = \int_{\mathbb{R}^{2n}} f(x-y)g(y)\mathrm{e}^{-\mathrm{i}\lambda x^t J y}\,\mathrm{d}y.$$

Given a $2n \times 2n$ real symmetric matrix A, we put

$$e_A(x) = \mathrm{e}^{\mathrm{i}x^t A x}.$$

It is easy to check, using the properties (b) and (c) of the matrix J, that

$$(f \times_\lambda e_{\lambda A})(x) = e_{\lambda A}(x)(e_{\lambda A}(\cdot)f(\cdot))\widehat{\ }(\lambda(2A+J)x),$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^{2n}} f(x)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x$ is the Fourier transform of f. Thus, for each $f \in L^1(\mathbb{R}^{2n}) \cap L^2(\mathbb{R}^{2n})$ we have

$$\|f \times_\lambda e_{\lambda A}\|_{L^2(\mathbb{R}^{2n})} = (2\pi)^n|\lambda|^{-n}|\det(2A \pm J)|^{-1/2}\|f\|_{L^2(\mathbb{R}^{2n})} \tag{6}$$

if $\det(2A \pm J) \neq 0$.

3. Main result

To prove the $L^{(2n+2)/(2n+1)}(\mathbb{H}^n) - L^{2n+2}(\mathbb{H}^n)$ boundedness of the operator T_{μ_A} we embed our operator in an analytic family $\{T_z\}$ of operators on the strip $-n \leqslant \Re(z) \leqslant 1$, and then we apply the complex interpolation theorem.

Theorem 3. *If* $\det(2A \pm J) \neq 0$*, then the operator* T_{μ_A} *is bounded from* $L^{(2n+2)/(2n+1)}(\mathbb{H}^n)$ *to* $L^{2n+2}(\mathbb{H}^n)$*.*

P r o o f. To prove the statement of the theorem we consider the family $\{|s|^{z-1}\}$ of functions initially defined when $\Re(z) > 0$ and $s \in \mathbb{R}\backslash\{0\}$. This family of functions can be extended in the z variable to an analytic family of distributions on $\mathbb{C}\setminus\{-2k\colon k \in \mathbb{N} \cup \{0\}\}$. By abuse of notation, we denote this extension by $|s|^{z-1}$. The family $\{|s|^{z-1}\}$ has simple poles in $z = -2k$ for $k \in \mathbb{N} \cup \{0\}$. Since the meromorphic continuation of the function $\Gamma(\frac{1}{2}z)$ (we keep the notation for his continuation) has simple poles at the same points (i.e. $z = -2k$), the family $\{I_z\}$ of distributions defined by

$$I_z(s) = \frac{2^{-z/2}}{\Gamma\left(\frac{1}{2}z\right)}|s|^{z-1} \tag{7}$$

results in an entire family of distributions (see pages 55–56 in [2]).

From this construction and by taking the ratios of the corresponding residues at $z = 0$, we have $I_0 = \delta$, where δ is the Dirac distribution at the origin on $\mathbb{R}$ (see equation (3), page 57 in [2]), also $\widehat{I_z} = cI_{1-z}$ for a real constant c independent of z (see equation (12′), page 173 in [2]).

For $z \in \mathbb{C}$, we also define U_z as the distribution on $\mathbb{H}^n$ given by the tensor product

$$U_z = \delta_{\mathbb{R}^{2n}} \otimes I_z,$$

where $\delta_{\mathbb{R}^{2n}}$ is the Dirac distribution at the origin on $\mathbb{R}^{2n}$ and I_z is given by (7). Let $\{T_z\}$ be the analytic family of operators on the strip $-n \leqslant \Re(z) \leqslant 1$, given by

$$T_z f = f * \mu_A * U_z.$$

It is clear that $T_0 = T_{\mu_A}$. For $\Re(z) = 1$ we have

$$\|T_z f\|_\infty = \|f * \mu_A * U_z\|_\infty \leqslant \|f\|_1 \|\mu_A * U_z\|_\infty.$$

Since $\mu_A * U_{1+ib}(x,t) = I_{1+ib}(t - \varphi(x)) = (2^{-(1+\mathrm{i}b)/2}/\Gamma(\frac{1}{2}(1+\mathrm{i}b)))|t - \varphi(x)|^{\mathrm{i}b}$, it follows that

$$\|T_{1+\mathrm{i}b}\|_{1,\infty} \leqslant \left|\frac{2^{-(1+\mathrm{i}b)/2}}{\Gamma(\frac{1}{2}(1+\mathrm{i}b))}\right| \quad \forall\, b \in \mathbb{R}.$$

For $\Re(z) = -n$ we will prove that the operator T_z is bounded on $L^2(\mathbb{H}^n)$. This is equivalent to showing that

$$\int_{\mathbb{R}^{2n}} |(T_z f)^\lambda(x)|^2 \,\mathrm{d}x \leqslant c \int_{\mathbb{R}^{2n}} |f^\lambda(x)|^2 \,\mathrm{d}x,$$

where $h^\lambda(x) := \int_{\mathbb{R}} h(x,t) \mathrm{e}^{-\mathrm{i}\lambda t} \,\mathrm{d}t$. A computation gives

$$\begin{aligned}(T_{-n+\mathrm{i}b} f)^\lambda(x) &= \widehat{I}_{-n+\mathrm{i}b}(\lambda) \int_{\mathbb{R}^{2n}} f^\lambda(x-y) e_{\lambda A}(y) \mathrm{e}^{-\mathrm{i}\lambda x^t J y} \,\mathrm{d}y \\ &= \widehat{I}_{-n+\mathrm{i}b}(\lambda)(f^\lambda \times_\lambda e_{\lambda A})(x).\end{aligned}$$

From the identity in (6) and since $\widehat{I}_z = cI_{1-z}$, we get

$$\|(T_{-n+\mathrm{i}b} f)^\lambda\|_{L^2(\mathbb{R}^{2n})} = \Big|\frac{c 2^{-(1+n-\mathrm{i}b)/2}}{\Gamma\big(\frac{1}{2}(1+n-\mathrm{i}b)\big)}\Big| (2\pi)^n |\det(2A \pm J)|^{-1/2} \|f^\lambda\|_{L^2(\mathbb{R}^{2n})}$$

for each $b \in \mathbb{R}$. So T_{-n+ib} is bounded on $L^2(\mathbb{H}^n)$ if $\det(2A \pm J) \neq 0$. Finally, it is easy to see, with the aid of the Stirling formula (see e.g. [12]), that the family $\{T_z\}$ satisfies, on the strip $-n \leqslant \Re(z) \leqslant 1$, the hypothesis of the complex interpolation theorem (see [13], page 205) and so $T_0 = T_{\mu_A}$ is bounded from $L^{(2n+2)/(2n+1)}(\mathbb{H}^n)$ into $L^{2n+2}(\mathbb{H}^n)$. □

Theorem 4. *Let* ν_0 *be the measure defined by* (3) *with* $\gamma = 0$. *If* $\det(2A \pm J) \neq 0$, *then the type set* E_{ν_0} *is the closed triangle with vertices* $(0,0)$, $(1,1)$ *and* $((2n+1)/(2n+2), 1/(2n+2))$.

P r o o f. Since the inequality $T_{\nu_0} f \leqslant T_{\mu_A} f$ holds for each borelian function $f \geqslant 0$, the theorem follows from the restrictions that appear in (4), Theorem 3 and the Riesz convexity theorem. □

Corollary 5. *If* $\det(2A \pm J) \neq 0$, *then the operator* T_{μ_A} *is bounded from* $L^p(\mathbb{H}^n)$ *into* $L^p(\mathbb{H}^n)$ *if and only if* $p = (2n+2)/(2n+1)$ *and* $q = 2n+2$.

P r o o f. The "*if*" part of the corollary is Theorem 3. To see the reciprocal we introduce the action of the dilation group $\mathbb{R}^{>0}$ on $\mathbb{H}^n$, i.e. $\delta \cdot (x,t) = (\delta x, \delta^2 t)$, $\delta > 0$. For a function f defined on $\mathbb{H}^n$ we put $f_\delta(x,t) = f(\delta \cdot (x,t))$. It is easy to check that

$$(T_{\mu_A} f)_\delta = \delta^{2n} T_{\mu_A}(f_\delta).$$

If $\|T_{\mu_A} f\|_q \leqslant c_{p,q} \|f\|_p$, then

$$\delta^{-(2n+2)/q} \|T_{\mu_A} f\|_q = \|(T_{\mu_A} f)_\delta\|_q = \delta^{2n} \|T_{\mu_A}(f_\delta)\|_q \leqslant \delta^{2n} c \|f_\delta\|_p = \delta^{2n-(2n+2)/p} c \|f\|_p$$

for all $\delta > 0$. So $1/q = 1/p - 2n/(2n+2)$. Since $T_{\nu_0} f \leqslant T_{\mu_A} f$ for $f \geqslant 0$, from Theorem 4 it follows that $p = (2n+2)/(2n+1)$ and $q = 2n+2$. □

Theorem 6. *Let ν_γ be the measure defined by equation* (3) *with* $0 < \gamma < 2n$. *If* $\det(2A \pm J) \neq 0$, *then the type set E_{ν_γ} is contained in the closed trapezoid with vertices* $(0,0)$, $(1,1)$, *D and D', where*

$$D = \Big(\frac{4n^2+2n+\gamma}{2n(2n+2)}, \frac{2n+(2n+1)\gamma}{2n(2n+2)}\Big) = \Big(\frac{1}{p_D}, \frac{1}{q_D}\Big) \quad \text{and} \quad D' = \Big(1-\frac{1}{q_D}, 1-\frac{1}{p_D}\Big)$$

and with the only possible exception of the closed segment joining the two points D and D'.

Proof. For each $k \in \mathbb{N} \cup \{0\}$ we define the sets $A_k \subset \mathbb{R}^{2n}$ by

$$A_k = \{y \in \mathbb{R}^{2n} \colon\ 2^{-k} < |y| \leqslant 2^{-k+1}\}.$$

Let $\nu_{\gamma,k}$ be the fractional Borel measure given by

$$\nu_{\gamma,k}(E) = \int_{A_k} \chi_E(y,\varphi(y))\eta(y)|y|^{-\gamma}\,\mathrm{d}y$$

and let $T_{\nu_{\gamma,k}}$ be its corresponding convolution operator, i.e. $T_{\nu_{\gamma,k}} f = f * \nu_{\gamma,k}$. Now, it is clear that $\nu_\gamma = \sum\limits_k \nu_{\gamma,k}$ and $\|T_{\nu_\gamma}\|_{p,q} \leqslant \sum\limits_k \|T_{\nu_{\gamma,k}}\|_{p,q}$. For $f \geqslant 0$ we have that

$$\int_{\mathbb{H}^n} f(y,s)\,\mathrm{d}\nu_{\gamma,k}(y,s) \leqslant 2^{k\gamma} \int_{\mathbb{R}^{2n}} f(y,\varphi(y))\eta(y)\,\mathrm{d}y.$$

Thus $\|T_{\nu_{\gamma,k}}\|_{p,q} \leqslant c2^{k\gamma}\|T_{\nu_0}\|_{p,q}$, from Theorem 4 it follows that

$$\|T_{\nu_{\gamma,k}}\|_{(2n+2)/(2n+1),2n+2} \leqslant c2^{k\gamma}.$$

It is easy to check that $\|T_{\nu_{\gamma,k}}\|_{1,1} \leqslant |\nu_{\gamma,k}(\mathbb{R}^{2n+1})| \sim \int_{A_k} |y|^{-\gamma}\,\mathrm{d}y = c2^{-k(2n-\gamma)}$. For $0 < \theta < 1$ we define

$$\Big(\frac{1}{p_\theta}, \frac{1}{q_\theta}\Big) = \Big(\frac{2n+1}{2n+2}, \frac{1}{2n+2}\Big)(1-\theta) + (1,1)\theta.$$

By the Riesz convexity theorem we have

$$\|T_{\nu_{\gamma,k}}\|_{p_\theta,q_\theta} \leqslant c2^{k\gamma(1-\theta)-k(2n-\gamma)\theta}.$$

Choosing θ such that $k\gamma(1-\theta) - k(2n-\gamma)\theta = 0$ yields $\sup\limits_{k\in\mathbb{N}}\|T_{\nu_{\gamma,k}}\|_{p_\theta,q_\theta} \leqslant c < \infty$. A simple computation gives $\theta = (2n-\gamma)/(2n)$, then $(1/p_\theta, 1/q_\theta) = (1/p_D, 1/q_D)$, so

$\|T_{\nu_{\gamma,k}}\|_{p_D,q_D} \leqslant c$, where c is independent of k. Interpolating once again, but now between the points $(1/p_D, 1/q_D)$ and $(1,1)$ we obtain for each $0<\tau<1$ fixed

$$\|T_{\nu_{\gamma,k}}\|_{p_\tau,q_\tau} \leqslant c2^{-k(2n-\gamma)\tau}.$$

Since $\|T_{\nu_\gamma}\|_{p,q} \leqslant \sum\limits_k \|T_{\nu_{\gamma,k}}\|_{p,q}$ and $0<\gamma<2n$, it follows that

$$\|T_{\nu_\gamma}\|_{p_\tau,q_\tau} \leqslant c\sum_{k\in\mathbb{N}} 2^{-k(2n-\gamma)\tau} < \infty.$$

By duality we also have

$$\|T_{\nu_\gamma}\|_{q_\tau/(q_\tau-1),p_\tau/(p_\tau-1)} \leqslant c_\tau < \infty.$$

Finally, the theorem follows from the Riesz convexity theorem, and the restrictions that appear in (4) and (5). □

We conclude this note with the following remarks.

R e m a r k 7. Let ν_0 be the measure of compact support defined by (3), but now with $\det(2A \pm J) = 0$. In this case, by Theorem 1.1 in [9] and Proposition 2, we can be sure that the type set E_{ν_0} has a nonempty interior.

R e m a r k 8. Lemma 1 provides us with examples of diagonal matrices A such that $\det(2A \pm J) = 0$. By the above remark we know that the interior of the type set of measure $\nu_0 = \eta\mu_A$ is nonempty. If $n \geqslant 2$ and A also satisfies that $\varphi(y) = y^tAy = \sum\limits_{j=1}^n \alpha_j|y_j|^2$ ($\alpha_j \in \mathbb{R}$ and $y_j \in \mathbb{R}^2$), then the type set of ν_0 is the closed triangle with vertices $(0,0)$, $(1,1)$ and $((2n+1)/(2n+2), 1/(2n+2))$. This result is independent of the value of $\det(2A \pm J)$ (see Theorem 1, page 102 in [3]).

These final comments illustrate the limits of the techniques used in this note as well as of those developed in the works [3] and [4].

A c k n o w l e d g e m e n t . I express my thanks to Prof. Fulvio Ricci for his numerous useful suggestions. My thanks also go to the referee for the useful suggestions and corrections which made the manuscript more readable.

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Author's address: *Pablo Rocha*, Departamento de Matemática, Universidad Nacional del Sur, Av. Alem 1253-Bahía Blanca 8000, Buenos Aires, Argentina, e-mail: `pablo.rocha@uns.edu.ar`.