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# WEAKLY FUZZY TOPOLOGICAL ENTROPY

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*Abstract.* In 2005, İ. Tok fuzzified the notion of the topological entropy R. A. Adler et al. (1965) using the notion of fuzzy compactness of C. L. Chang (1968). In the present paper, we have proposed a new definition of the fuzzy topological entropy of fuzzy continuous mapping, namely weakly fuzzy topological entropy based on the notion of weak fuzzy compactness due to R. Lowen (1976) along with its several properties. We have shown that the topological entropy R. A. Adler et al. (1965) of continuous mapping $\psi\colon (X,\tau)\to(X,\tau)$, where $(X,\tau)$ is compact, is equal to the weakly fuzzy topological entropy of $\psi\colon (X,\omega(\tau))\to(X,\omega(\tau))$. We have also established an example that shows that the fuzzy topological entropy of İ. Tok (2005) cannot give such a bridge result to the topological entropy of Adler et al. (1965). Moreover, our definition of the weakly fuzzy topological entropy can be applied to find the topological entropy (namely weakly fuzzy topological entropy $h_w(\psi)$) of the mapping $\psi\colon X\to X$ (where $X$ is either compact or weakly fuzzy compact), whereas the topological entropy $h_a(\psi)$ of Adler does not exist for the mapping $\psi\colon X\to X$ (where $X$ is non-compact weakly fuzzy compact). Finally, a product theorem for the weakly fuzzy topological entropy has been established.

*Keywords*: weakly fuzzy compact; weakly fuzzy compact topological dynamical system; weakly fuzzy topological entropy

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## 1. Introduction

The topological entropy initiated by Adler et al. (see [1]) becomes a mature field in the ergodic theory, especially in the theory of dynamical systems. The topological entropy of Adler et al. (see [1]) was defined for a compact topological dynamical system. Then Bowen in [4] generalized the concept for non-compact spaces, but it is metric-dependent. In 2009, Liu et al. (see [17]) defined the notion of the topological entropy for arbitrary dynamical system. The contributions of Cánovas and Rodriguez

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(see [6]), Goodwyn (see [14], [15]), Kwietniak and Oprocha (see [16]), Bowen (see [3]), Cánovas and López (see [7]) and Thomas (see [27]) in the flow of the research works related to the topological entropy are worth to be mentioned.

In the fuzzy mathematics, the notions of the entropy of the fuzzy partitions and the entropy of the fuzzy process were initiated and studied by Dumitrescu (see [9], [10], [11]) and Dumitrescu and Barbu (see [13]). They also defined and investigated the concept of the fuzzy dynamical system in [9], [12], [13] in terms of the fuzzy measure space and the fuzzy measure preserving transformation and in this settings, the entropy was studied. The notion of the entropy of the fuzzy dynamical system was also studied by Markechová (see [20]–[23]) and Riečan and Markechová (see [26]). The notion of the topological entropy of the fuzzified dynamical system was introduced by Cánovas and Kupka in [5]. In Cánovas and Kupka's approach, they started with a compact metric space and a continuous self-mapping and so their definition of the entropy is metric-dependent. In 2005, Tok (see [28]) fuzzified the concept of the topological entropy for the fuzzy compact topological spaces and the concept was generalized for the arbitrary fuzzy topological spaces by Afsan and Basu, see [2]. The notions of the fuzzy topological entropy of Tok; Afsan and Basu were metric-independent.

Tok in [28] used the notion of the fuzzy compactness due to Chang (see [8]) to introduce the concept of the fuzzy topological entropy. But the fuzzy compactness fails to satisfy the important parallel properties like the compactness of the ordinary topology and thus the most intersecting properties of the topological entropy are not found in the investigation of Tok (see [28]). On the other hand, Lowen's definition of the weak fuzzy compactness (see [18]) satisfies all the desired properties which are possessed by the compactness in an ordinary topological space. Lowen in [18] showed that the compactness of a topological space $(X, \tau)$ implies the weakly fuzzy compactness of the fuzzy topological space $(X, \omega(\tau))$, but not its fuzzy compactness. So no bridge result between purely topological notions and fuzzy ones could be achieved using the fuzzy topological entropy of Tok, see [28]. In the present paper, in Section 3, we have established a new definition of the fuzzy topological entropy, namely weakly fuzzy topological entropy for weakly fuzzy compact topological spaces that can enable us to remove this serious limitation of the fuzzy topological entropy of Tok, see [28]. Our new definition of the fuzzy topological entropy is also metric-independent.

To compare our new definition of the weakly fuzzy topological entropy with the topological entropy, we note that the definition of the weakly fuzzy topological entropy can be applied to find the topological entropy (namely weakly fuzzy topological entropy $h_w(\psi)$) of the mapping $\psi\colon X \to X$ (where $X$ is either compact or weakly fuzzy compact), but the topological entropy $h_a(\psi)$ of Adler does not exist for the mapping $\psi\colon X \to X$ (where $X$ is non-compact weakly fuzzy compact). In fact, Adler's definition of the topological entropy is applicable only for a compact topological space $(X,\tau)$ and for a continuous mapping $\psi\colon (X,\tau) \to (X,\tau)$, i.e. for a compact topological dynamical system $(X,\psi)$. And our definition of the topological entropy (weakly fuzzy topological entropy) can be applied to a weakly fuzzy compact topological space $(X,\delta)$ and a fuzzy continuous mapping $\psi\colon (X,\delta) \to (X,\delta)$, i.e. to a weakly fuzzy compact topological dynamical system $(X,\psi)$. Thus, Adler's definition of the topological entropy cannot be applied to a non-compact weakly fuzzy compact topological dynamical system $(X,\psi)$ to calculate topological entropy $h_a(\psi)$ of the mapping $\psi$, but using the definition of the weakly fuzzy topological entropy of $(X,\psi)$, we can calculate the weakly fuzzy topological entropy $h_w(\psi)$ of $\psi$. Again, if $(X,\tau)$ is a compact topological space, Lowen in [18] showed that $(X,\omega(\tau))$ is a weakly fuzzy compact topological space. And if the mapping $\psi\colon (X,\tau) \to (X,\tau)$ is a continuous mapping, then the mapping $\psi\colon (X,\omega(\tau)) \to (X,\omega(\tau))$ is fuzzy continuous. Thus, if $((X,\tau),\psi)$ is a compact topological dynamical system, then $((X,\omega(\tau)),\psi)$ is a weakly fuzzy compact topological dynamical system. Thus, using the definition of the weakly fuzzy topological entropy, we can calculate the weakly fuzzy topological entropy $h_w(\psi)$ of $\psi$ with $h_w(\psi) = h_a(\psi)$ (Theorem 3.3). Thus, the definition of the weakly fuzzy topological entropy can be applied to a compact dynamical topological system $(X,\psi)$ and a weakly compact dynamical system $(X,\psi)$ both to find the weakly fuzzy topological entropy $h_w(\psi)$, but Adler's definition of the topological entropy $h_a(\psi)$ is applicable only to a compact topological dynamical system, but not to a non-compact weakly fuzzy compact topological dynamical system.

We have achieved several important properties of the weakly fuzzy topological entropy in Section 3. We have also established a product theorem (Theorem 3.6) for the weakly fuzzy topological entropy.

## 2. Preliminaries

Throughout this paper, the symbol $I$ is used for the unit closed interval $[0,1]$ and the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^+$ and $\mathbb{Z}^-$ stand for the set of all natural numbers, the set of all integers, the set of all positive integers and the set of all negative integers, respectively.

**Definition 2.1** ([30])**.** Let $X$ be a nonempty set. Then any function with domain $X$ and codomain $I$ is called a *fuzzy subset* of $X$ and the set of all fuzzy subsets of $X$ is denoted by $I^X$.

The support of a fuzzy set $A$ is the set $\{x \in X\colon A(x) > 0\}$ and is denoted by $\mathrm{supp}(A)$. A fuzzy set with only nonzero value $p \in (0,1]$ at only one element $x \in X$ is called a *fuzzy point* and is denoted by $x_p$ and the set of all fuzzy points of a set $X$ is denoted by $Pt(X)$. For any two fuzzy sets $A$, $B$ of $X$, $A \leqslant B$ if and only if $A(x) \leqslant B(x)$ for all $x \in X$. A fuzzy point $x_\lambda$ is said to be "in a fuzzy set $A$" (denoted by $x_\lambda \in A$) if $x_\lambda \leqslant A$, i.e. if $\lambda \leqslant A(x)$. The constant fuzzy set of $X$ with value $\varepsilon \in [0,1]$ is denoted by $\underline{\varepsilon}$. A fuzzy set $A$ is said to be quasi-coincident with $B$ (written as $A\hat{q}B$, see [24]) if $A(x) + B(x) > 1$ for some $x \in X$. A fuzzy set $A$ is said not to be quasi-coincident with $B$ (written as $A\bar{q}B$, see [24]) if $A(x) + B(x) \leqslant 1$ for all $x \in X$.

Let $\mu, \nu \in I^X$. Then their join $\mu \vee \nu \in I^X$ and meet $\mu \wedge \nu \in I^X$ are defined by $(\mu \vee \nu)(x) = \max\{\mu(x), \nu(x)\}$ and $(\mu \vee \nu)(x) = \min\{\mu(x), \nu(x)\}$ for all $x \in X$, respectively. The general definitions of the notions of the join and the meet were introduced by Chang in [8] to initiate the notion of the fuzzy topology.

Let $\{\mu_\alpha \in I^X\colon \alpha \in \Delta\}$ be an arbitrary family of fuzzy sets of $X$. Then their join $\bigvee\limits_{\alpha\in\Delta} \mu_\alpha (= \vee\{\mu_\alpha\colon \alpha \in \Delta\}) \in I^X$ and meet $\bigwedge\limits_{\alpha\in\Delta} \mu_\alpha (= \wedge\{\mu_\alpha\colon \alpha \in \Delta\}) \in I^X$ are defined by $\Big(\bigvee\limits_{\alpha\in\Delta} \mu_\alpha\Big)(x) = \sup\{\mu_\alpha(x)\colon \alpha \in \Delta\}$ and $\Big(\bigwedge\limits_{\alpha\in\Delta} \mu_\alpha\Big)(x) = \inf\{\mu_\alpha(x)\colon \alpha \in \Delta\}$ for all $x \in X$, respectively. The complement of a fuzzy set $\mu \in I^X$ is denoted by $\mu' \in I^X$ and is defined by $\mu'(x) = 1 - \mu(x)$ for all $x \in X$.

**Definition 2.2** ([8]). A family $\delta$ of fuzzy subsets of $X$ is called a *fuzzy topology* on $X$ if

(i) $\underline{0}, \underline{1} \in \delta$,

(ii) $\mu \wedge \nu \in \delta$ for all $\mu, \nu \in \delta$,

(iii) $\vee\{\mu \in \delta_0\} \in \delta$ for any subfamily $\delta_0$ of $\delta$.

The pair $(X, \delta)$ is called a *fuzzy topological space*. The members of $\delta$ are called *fuzzy open sets* and their complements are called *fuzzy closed sets* of $(X, \delta)$.

Throughout the paper, spaces $(X, \delta)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) represent nonempty fuzzy topological spaces due to Chang, see [8]. A fuzzy open set $A$ of $X$ is called *fuzzy quasi-neighborhood* (see [24]) (or *fuzzy neighbourhood*, see [8]) of a fuzzy point $x_\lambda$ if $x_\lambda \hat{q} A$ (or $x_\lambda \in A$). The collection of all fuzzy quasi-neighborhoods (or fuzzy quasi-neighborhoods) of a fuzzy point $x_\lambda$ is denoted by $\mathcal{Q}(X, x_\lambda)$ (or $\mathcal{N}(X, x_\lambda)$).

A collection $\Sigma$ of fuzzy subsets of $X$ is called a *fuzzy cover* of a fuzzy subset $\eta$ of $X$ if $\vee\{A\colon A \in \Sigma\} \geqslant \eta$. A fuzzy cover $\Sigma$ of $\eta = \underline{1}$ is known as a fuzzy cover of the fuzzy topological space $X$. If the members of a fuzzy cover $\Sigma$ of a fuzzy subset $\eta$ of $X$ are fuzzy open (or fuzzy closed), then the cover $\Sigma$ is called a *fuzzy open cover* (or *fuzzy closed cover*) of $\eta$.

**Definition 2.3** ([8]). A fuzzy subset $\eta$ of a fuzzy topological space $X$ is said to be fuzzy compact (= fuzzy quasi-compact, see [18]) if every fuzzy cover of $\eta$ by fuzzy open sets of $X$ has a finite subcover of $\eta$. If $\underline{1}$ is fuzzy compact subset of $X$, $X$ is called a *fuzzy compact space.*

Originally, Lowen in [18] introduced another notion of fuzzy compactness of a fuzzy subset in fuzzy topological spaces and the notion of weakly fuzzy compactness of the whole space. In the present paper, we have renamed a fuzzy compact subset of Lowen as weakly fuzzy compact subset to differentiate it from fuzzy compact subset of Chang.

**Definition 2.4** ([18]). A fuzzy subset $\eta$ of a fuzzy topological space $X$ is said to be weakly fuzzy compact (originally, fuzzy compact, see [18]) if for every fuzzy cover $\Sigma$ of $\eta$ by fuzzy open sets of $X$ and for every $\varepsilon > 0$, there exists a finite subfamily $\{A_i\colon\ i = 1, 2, \ldots, n\}$ of $\Sigma$ such that $\bigvee\limits_{i=1}^{n} A_i \geqslant \eta - \underline{\varepsilon}$. If $\underline{1}$ is a weakly fuzzy compact subset of $X$, $X$ is called a *weakly fuzzy compact space.*

**Definition 2.5** ([25]). Let $X$ and $Y$ be two ordinary sets and $\psi : X \to Y$ be an ordinary mapping. Let $A$ be a fuzzy set of $X$. Then $\psi(A)$ is defined by

$$\psi(A)(y) = \begin{cases} \sup\{A(x) : \psi(x) = y\} & \text{if } \psi^{-1}(y) \neq \emptyset, \\ 0 & \text{if } \psi^{-1}(y) = \emptyset. \end{cases}$$

Clearly, $\psi(A)$ is a fuzzy set of $Y$. And for a fuzzy set $B$ of $Y$, $\psi^{-1}(B)$ is defined by $\psi^{-1}(B)(x) = B(\psi(x))$. Clearly, $\psi^{-1}(B)$ is a fuzzy set of $X$.

**Definition 2.6** ([8]). A mapping $\psi\colon\ X \to Y$ is fuzzy continuous if $\psi^{-1}(U)$ is fuzzy open in $Y$ for each fuzzy open set $U$ in $X$. A bijective fuzzy continuous mapping $\psi\colon\ X \to Y$ is called *fuzzy homeomorphism* if $\psi^{-1}$ is also fuzzy continuous.

Following the theorem of Ming and Ming (see [25]) gives some characterizations of fuzzy continuous functions.

**Theorem 2.1** ([25]). *For a mapping $\psi\colon\ X \to Y$, following conditions are equivalent:*

(i) *$\psi$ is fuzzy continuous,*

(ii) *$\psi^{-1}(V)$ is fuzzy closed in $Y$ for each fuzzy open set $V$ in $X$,*

(iii) *for every fuzzy point $x_\lambda \in Pt(X)$ and every $V \in \mathcal{Q}(Y, \psi(x_\lambda))$ there exists $U \in \mathcal{Q}(X, x_\lambda)$ such that $\psi(U) \leqslant V$,*

(iv) *for every fuzzy point $x_\lambda \in Pt(X)$ and every $V \in \mathcal{N}(Y, \psi(x_\lambda))$ there exists $U \in \mathcal{N}(X, x_\lambda)$ such that $\psi(U) \leqslant V$.*

Let $\Sigma$ and $\Omega$ be two fuzzy open covers of a fuzzy topological space $X$. Define their join by $\Sigma \vee \Omega = \{U \wedge V\colon U \in \Sigma, V \in \Omega\}$. Clearly, the join $\Sigma \vee \Omega$ is a fuzzy open cover of $X$. It is well-known that $\Omega$ is called a *refinement* of $\Sigma$ (denoted by $\Sigma \prec \Omega$) if for each $V \in \Omega$ there exists $U \in \Sigma$ such that $V \leqslant U$. In this paper, the cardinality of a family $\Omega$ of fuzzy subsets or ordinary subsets of $X$ is denoted by $|\Omega|$. The join of covers of topological space $X$ is defined in an analogous way.

In a compact topological space $X$, the concept of the topological entropy of a continuous self-mapping $\psi\colon X \to X$ was defined by Adler et al. (see [1]). The pair $(X, \psi)$ is called a *compact topological dynamical system*.

**Definition 2.7** ([1])**.** Let $(X, \psi)$ be a compact topological dynamical system and $\Sigma$ be an open cover of $X$. Let $N_a(\Sigma) = \min\{|\Omega|\colon \Omega \subset \Sigma \text{ and } \Sigma \text{ is a cover of } X\}$. Let $H_a(\Sigma) = \log N_a(\Sigma)$. Then the limit

$$h_a(\psi, \Sigma) = \lim_{n\to\infty} \frac{1}{n} H_a(\vee\{\psi^{-i}(\Sigma)\colon i = 0, 1, 2, \ldots, n-1\})$$

exists. The quantity $h_a(\psi) = \sup\{h_a(\psi, \Sigma)\colon \Sigma \text{ is an open cover of } X\}$ is called the *topological entropy of the continuous mapping* $\psi$.

If $X$ is a fuzzy compact space and $\psi\colon X \to Y$ is a fuzzy continuous mapping, then the pair $(X, \psi)$ is called a *fuzzy compact topological dynamical system*. For a fuzzy compact dynamical system, Tok in [28] defined the notion of the fuzzy topological entropy.

**Definition 2.8** ([28])**.** Let $(X, \psi)$ be a fuzzy compact topological dynamical system and $\Sigma$ be a fuzzy open cover of $X$. Let $N_t(\Sigma) = \min\{|\Omega|\colon \Omega \subset \Sigma \text{ and } \Sigma \text{ is a cover of } X\}$. Since $X$ is fuzzy compact, $N_t(\Sigma)$ is a positive integer. Let $H_t(\Sigma) = \log N_t(\Sigma)$. Then Tok (see [28]) showed that

$$h_t(\psi, \Sigma) = \lim_{n\to\infty} \frac{1}{n} H_t(\vee\{\psi^{-i}(\Sigma)\colon i = 0, 1, 2, \ldots, n-1\})$$

exists. The quantity $h_t(\psi) = \sup\{h_t(\psi, \Sigma)\colon \Sigma \text{ is a fuzzy open cover of } X\}$ is called the *fuzzy topological entropy of the fuzzy continuous function* $\psi$.

Afsan and Basu in [2] introduced the notion of the fuzzy topological entropy for a fuzzy continuous self-mapping $\psi\colon X \to X$, where $X$ is an arbitrary fuzzy topological space, in terms of the fuzzy compact subsets of $X$.

**Definition 2.9** ([2])**.** Let $X$ be a fuzzy topological space, $\Sigma$ be a fuzzy open cover of $X$ and $C(X, \psi) = \{K \text{ is a nonempty fuzzy compact subset of } X \text{ such that } \psi(K) \leqslant K\}$. For each $K \in C(X, \psi)$, let $N^\star(\Sigma, K) = \min\{|\Omega|\colon \Omega \subset \Sigma, K \leqslant \vee\Omega\}$. Let $H^\star(\Sigma, K) = \log N^\star(\Sigma, K)$. Then the quantity

$$h^\star(\psi, \Sigma, K) = \lim_{n\to\infty} \frac{1}{n} H^\star(\vee\{\psi^{-i}(\Sigma)\colon i = 0, 1, 2, \ldots, n-1\}, K)$$

is called the *fuzzy topological entropy of* $\psi$ *on* $K$ *relative to* $\Sigma$,

$$h^\star(\psi, K) = \sup\{h^\star(\psi, \Sigma, K)\colon\ \Sigma \text{ is a fuzzy open cover of } X\}$$

is called the *fuzzy topological entropy of* $\psi$ *on* $K$ and

$$h^\star(\psi) = \sup\{h^\star(\psi, K)\colon\ K \in C(X, \psi)\}$$

is called the *fuzzy topological entropy of* $\psi$.

Afsan and Basu in [2] also achieved the following result.

**Theorem 2.2** ([2]). *Let* $(X, \psi)$ *be a fuzzy compact topological dynamical system. Then* $h^\star(\psi) = h_t(\psi)$.

## 3. Weakly fuzzy topological entropy

In this section, we have established a new definition of the fuzzy topological entropy, namely weakly fuzzy topological entropy for the weakly fuzzy compact dynamical system $(X, \psi)$ along with its several properties. The new definition of the fuzzy topological entropy is metric-independent. In Theorem 3.3, we have achieved a bridge result between the weakly fuzzy topological entropy and the topological entropy of Adler el al. (see [1]). Then with the help of Example 3.1, we have shown that such a bridge result between the fuzzy topological entropy of Tok (see [28]) and the topological entropy of Adler el al. (see [1]) is not possible. We have also established a product theorem of the weakly fuzzy topological entropies in Theorem 3.6.

**Definition 3.1.** Let $(X, \delta)$ be a fuzzy topological space and $\psi\colon\ (X, \delta) \to (X, \delta)$ be a fuzzy continuous mapping. Then the pair $(X, \psi)$ is called a *fuzzy topological dynamical system.* If $X$ is weakly fuzzy compact, $(X, \psi)$ is called a *weakly fuzzy compact topological dynamical system.*

Since every fuzzy compact space (see [8]) is weakly fuzzy compact (see [18]), fuzzy compact topological dynamical systems (see [2]) are weakly fuzzy compact topological dynamical systems.

**Definition 3.2.** Let $(X, \psi)$ be a weakly fuzzy compact topological dynamical system and $\Sigma$ be a fuzzy open cover of $X$ and $\varepsilon > 0$. Let $N_w(\Sigma, \varepsilon) = \min\Big\{|\Omega|\colon\ \Omega$ is a finite subfamily of $\Sigma$ such that $\bigvee\limits_{\mu\in\Omega} \mu \geqslant \underline{1-\varepsilon}\Big\}$. Then $H_w(\Sigma, \varepsilon) = \log N_w(\Sigma, \varepsilon)$ is called the *weakly fuzzy topological* $\varepsilon$*-entropy of* $\Sigma$.

**Lemma 3.1.** *Let $(X,\psi)$ be a weakly fuzzy compact topological dynamical system, $\varepsilon>0$ and $\Sigma$ and $\Omega$ be fuzzy open covers of $X$. Then the following statements hold:*

(a) $H_w(\Sigma,\varepsilon)\geqslant 0$;

(b) $\Omega\prec\Sigma$ *implies* $H_w(\Omega,\varepsilon)\leqslant H_w(\Sigma,\varepsilon)$;

(c) $H_w(\Sigma\vee\Omega,,\varepsilon)\leqslant H_w(\Sigma,\varepsilon)+H_w(\Omega,\varepsilon)$;

(d) $H_w(\psi^{-1}(\Sigma),\varepsilon)\leqslant H_w(\Sigma,\varepsilon)$. *If $\psi$ is bijective, the equality holds.*

(e) *If $\psi$ is a homeomorphism,* $H_w(\psi(\Sigma),\varepsilon)=H_w(\Sigma,\varepsilon)$.

Proof. (a) Obvious.

(b) It is sufficient to show $N_w(\Omega,\varepsilon)\leqslant N_w(\Sigma,\varepsilon)$ to achieve this result. Let $N_w(\Sigma,\varepsilon)=n$. Let $\Sigma_n=\{U_i\colon i=1,2,\dots,n\}$ be a finite subfamily of $\Sigma$ such that $\bigvee\limits_{i=1}^{n}U_i\geqslant\underline{1-\varepsilon}$. Since $\Omega\prec\Sigma$, for each $i=1,2,\dots,n$, there exists $V_i\in\Omega$ such that $U_i\leqslant V_i$. Then $\Omega_n=\{V_i\colon i=1,2,\dots,n\}$ is a subfamily of $\Omega$ such that $\bigvee\limits_{i=1}^{n}V_i\geqslant\underline{1-\varepsilon}$. So $N_w(\Omega,\varepsilon)\leqslant n$. Hence $N_w(\Omega,\varepsilon)\leqslant N_w(\Sigma,\varepsilon)$.

(c) It is sufficient to show $N_w(\Sigma\vee\Omega,\varepsilon)\leqslant N_w(\Sigma,\varepsilon)\cdot N_w(\Omega,\varepsilon)$ to achieve this result. Let $N_w(\Sigma,\varepsilon)=m$ and $N_w(\Omega,\varepsilon)=n$. Let $\Sigma_m=\{U_i\colon i=1,2,\dots,m\}$ and $\Omega_n=\{V_i\colon i=1,2,\dots,n\}$ be respective subfamilies of $\Sigma$ and $\Omega$ such that $\bigvee\limits_{i=1}^{m}U_i\geqslant\underline{1-\varepsilon}$ and $\bigvee\limits_{j=1}^{n}V_j\geqslant\underline{1-\varepsilon}$. Consider the subfamily $\bigotimes\limits_{mn}=\{W_{ij}=U_i\wedge V_j\colon i=1,2,\dots,m;\ j=1,2,\dots,n\}$ of $\Sigma\vee\Omega$. Then $\bigvee\limits_{i=1}^{m}\bigvee\limits_{j=1}^{n}W_{ij}=\Bigl(\bigvee\limits_{i=1}^{m}U_i\Bigr)\wedge\Bigl(\bigvee\limits_{j=1}^{n}V_j\Bigr)\geqslant\underline{1-\varepsilon}$. So $N_w(\Sigma\vee\Omega),\varepsilon)\leqslant mn$. Hence $N_w(\Sigma\vee\Omega,\varepsilon))\leqslant N_w(\Sigma,\varepsilon)\cdot N_w(\Omega,\varepsilon)$.

(d) Since $\Sigma$ covers $X$, $\vee\{U\colon U\in\Sigma\}=\underline{1}$. Then $\vee\{\psi^{-1}(U)(x)\colon U\in\Sigma\}=\vee\{U(\psi(x))\colon U\in\Sigma\}=1$ for each $x\in X$ and so $\psi^{-1}(\Sigma)$ covers $X$. Let $N_w(\Sigma,\varepsilon)=n$. Let $\Sigma_n=\{U_i\colon i=1,2,\dots,n\}$ be a subfamily of $\Sigma$ such that $\bigvee\limits_{i=1}^{n}U_i\geqslant\underline{1-\varepsilon}$. Now since $\psi$ is fuzzy continuous, $\{\psi^{-1}(U_i)\colon i=1,2,\dots,n\}$ is a subfamily of the cover $\psi^{-1}(\Sigma)$ of $X$. We claim that $\bigvee\limits_{i=1}^{n}\psi^{-1}(U_i)\geqslant\underline{1-\varepsilon}$. Let $x\in X$. Then $U_i(\psi(x))\geqslant 1-\varepsilon$ for some $i\in\{1,2,\dots,n\}$ and so $\psi^{-1}(U_i)(x)=U_i(\psi(x))\geqslant 1-\varepsilon$. Thus $\bigvee\limits_{i=1}^{n}\psi^{-1}(U_i)\geqslant\underline{1-\varepsilon}$. Hence $N_w(\psi^{-1}(\Sigma),\varepsilon)\leqslant n=N_w(\Sigma,\varepsilon)$ and so $H_w(\psi^{-1}(\Sigma),\varepsilon)\leqslant H_w(\Sigma,\varepsilon)$.

The second part can be obtained applying the result of the first part on the mapping $\psi^{-1}$.

(e) Applying the second part of (d) on the mapping $\psi^{-1}$, we get the result. □

**Theorem 3.1.** *Let $(X,\psi)$ be a weakly fuzzy compact topological dynamical system, $\varepsilon>0$ and $\Sigma$ be a fuzzy open cover of $X$. Then*

$$\lim_{n\to\infty}\frac{1}{n}H_w(\vee\{\psi^{-i}(\Sigma)\colon i=0,1,2,\dots,n-1\},\varepsilon)$$

*exists.*

Proof. Consider the sequence $\{x_n \colon n \in \mathbb{N}\}$, $x_n = H_w(\vee\{\psi^{-i}(\Sigma) \colon i = 0, 1, 2, \ldots, n-1\}, \varepsilon)$ of positive real numbers. Then the sequence $\{x_n/n \colon n \in \mathbb{N}\}$ is convergent if $x_{m+n} \leqslant x_m + x_n$ for all $m, n \in \mathbb{N}$, see [29].

By Lemma 3.1,

$$\begin{aligned} x_{m+n} &= H_w( \vee \{\psi^{-i}(\Sigma) \colon i = 0, 1, \ldots, m+n-1\}, \varepsilon) \\ &= H_w( \vee \{\psi^{-i}(\Sigma) \colon i = 0, 1, \ldots, m-1\} \\ &\qquad\qquad \vee (\vee\{\psi^{-j}(\Sigma) \colon i = m, m+1, \ldots, m+n-1\}), \varepsilon) \\ &\leqslant H_w( \vee \{\psi^{-i}(\Sigma) \colon i = 0, 1, \ldots, m-1\}, \varepsilon) \\ &\quad + H_w(\vee\{\psi^{-j}(\Sigma) \colon i = m, m+1, \ldots, m+n-1\}, \varepsilon) \\ &\leqslant H_w( \vee \{\psi^{-i}(\Sigma) \colon i = 0, 1, \ldots, m-1\}, \varepsilon) \\ &\quad + H_w(\psi^{-m}(\vee\{\psi^{-j}(\Sigma) \colon j = 0, 1, \ldots, n-1\}), \varepsilon) \\ &\leqslant H_w( \vee \{\psi^{-i}(\Sigma) \colon i = 0, 1, \ldots, m-1\}, \varepsilon) \\ &\quad + H_w(\vee\{\psi^{-j}(\Sigma) \colon j = 0, 1, \ldots, n-1\}, \varepsilon). \end{aligned}$$

So $x_{m+n} \leqslant x_m + x_n$ for all $m, n \in \mathbb{N}$. □

**Definition 3.3.** Let $(X, \psi)$ be a weakly fuzzy compact topological dynamical system, $\varepsilon > 0$ and $\Sigma$ be a fuzzy open cover of $X$. Then

$$h_w(\psi, \Sigma, \varepsilon) = \lim_{n\to\infty} \frac{1}{n} H_w(\vee\{\psi^{-i}(\Sigma) \colon i = 0, 1, 2, \ldots, n-1\}, \varepsilon)$$

is called the *weakly fuzzy topological $\varepsilon$-entropy of $\psi$ relative to $\Sigma$* and

$$h_w(\psi, \varepsilon) = \sup\{h_w(\psi, \Sigma) \colon \Sigma \text{ is a fuzzy open cover of} X\}$$

is called the *weakly fuzzy topological $\varepsilon$-entropy of $\psi$*. And

$$h_w(\psi) = \sup\{h_w(\psi, \varepsilon) \colon \varepsilon > 0\}$$

is called the *weakly fuzzy topological entropy of $\psi$*.

The following theorem shows that for a fuzzy compact dynamical system, the weakly fuzzy topological entropy is equal to the fuzzy topological entropy of both Tok (see [28]) and Afsan and Basu (see [2]).

**Theorem 3.2.** *If $(X, \psi)$ is a fuzzy compact topological dynamical system, then $h_w(\psi) = h_t(\psi) = h^\star(\psi)$.*

P r o o f. The equality of $h_t(\psi)$ and $h^\star(\psi)$ was established in [2]. So we shall only show $h_w(\psi) = h_t(\psi)$. Since $X$ is fuzzy compact, it is also weakly fuzzy compact. Let $\Sigma$ be a fuzzy open cover of $X$ such that $N_t(\Sigma) = k$. Let $\varepsilon > 0$. Then there exists a minimal subfamily $\{U_i\colon\ i = 1, 2, \ldots, k\}$ of $\Sigma$ such that $\bigvee_{i=1}^{k} U_i = \underline{1} \geqslant \underline{1-\varepsilon}$. So $N_w(\Sigma, \varepsilon) \leqslant k = N_t(\Sigma)$. Then $h_w(\psi, \Sigma, \varepsilon) \leqslant h_t(\psi, \Sigma) \leqslant h_t(\psi)$ for any fuzzy open cover $\Sigma$ of $X$ and for any $\varepsilon > 0$ and so $h_w(\psi) \leqslant h_t(\psi)$.

Now suppose $\Omega$ be a fuzzy open cover of $X$. Since $X$ is fuzzy compact, there exists a minimal subfamily $\{V_j\colon\ j = 1, 2, \ldots, m\}$ of $\Omega$ such that $\bigvee_{j=1}^{m} V_j = \underline{1}$. Then for each $i \in \{1, 2, \ldots, m\}$ there exists $x_i \in X$ and $\varepsilon_i > 0$ such that $(\vee\{V_j\colon\ j = 1, 2, \ldots, m,\ j \neq i\})(x_i) < 1 - \varepsilon_i < \underline{1}$. Thus, $N_w(\Sigma, \varepsilon_i) = m = N_t(\Sigma)$ and so $h_t(\psi, \Omega) = h_w(\psi, \Omega, \varepsilon_i) \leqslant h_w(\psi, \varepsilon_i) \leqslant h_w(\psi)$ for all fuzzy open covers $\Omega$ of $X$. Therefore $h_t(\psi) \leqslant h_w(\psi)$. $\square$

Let $(X, \tau)$ be a given topological space and $\tau_{\mathbb{R}} = \{]r, \infty[\colon\ r \in \mathbb{R}\} \cup \{\emptyset\}$. Consider the space $I = [0, 1]$ with the subtopology $\tau_{\mathbb{R}}|_I$ and $\omega(\tau) = \{\mu \in X^I\colon\ \mu \text{ is continuous}\}$. Lowen in [19] showed that $\omega(\tau)$ is a fuzzy topology on $X$.

Using Theorem 2.1, we shall show the following lemma.

**Lemma 3.2.** *Let $(X, \tau)$ be a compact topological space and $\psi\colon\ (X, \tau) \to (X, \tau)$ be a continuous mapping. Then $\psi\colon\ (X, \omega(\tau)) \to (X, \omega(\tau))$, the fuzzy mapping induced by the ordinary mapping $\psi\colon\ (X, \tau) \to (X, \tau)$, is fuzzy continuous.*

P r o o f. Let $x_\lambda \in Pt(X)$ and $\mu \in \mathcal{N}(X, \psi(x_\lambda))$. Then the continuity of $\mu\colon (X, \tau) \to I$ ensures that $V = \{y \in X\colon\ \mu(y) > \lambda\}$ is open in $(X, \tau)$ containing $\psi(x)$. Since $\psi\colon\ (X, \tau) \to (X, \tau)$ is continuous, there exists a set $U \in \tau$ containing $x$ such that $\psi(U) \subset V$.

We define $\varrho\colon\ \tau \to [0, 1]$ by $\varrho(U) = \{\lambda\}$ and $\varrho(X - U) = \{0\}$. Then $\varrho \in \mathcal{N}(X, x_\lambda)$. We claim that $\psi(\varrho) \leqslant \mu$. Let $y \in X$. First suppose $\psi^{-1}(y) = \emptyset$, then $\psi(\varrho)(y) = 0$. So $\psi(\varrho)(y) \leqslant \mu(y)$ for all $y \in X$ with $\psi^{-1}(y) = \emptyset$. Now let $\psi^{-1}(y) \neq \emptyset$. Then either $\psi^{-1}(y) \cap U = \emptyset$ or $\psi^{-1}(y) \cap U \neq \emptyset$.

For the former case, clearly, $\psi(\varrho)(y) \leqslant \mu(y)$. So, let $z \in U$ such that $\psi(z) = y$. Then $\psi(\varrho)(y) = \sup\{\varrho(z)\colon\ \psi(z) = y\} = \lambda$. Again $y = \psi(x) \in \psi(U) \subset V$. Then $\mu(y) > \lambda$ and so $\psi(\varrho)(y) \leqslant \mu(y)$. Thus, $\psi(\varrho)(y) \leqslant \mu(y)$ for all $y \in Y$ and so $\psi(\varrho) \leqslant \mu$.

Hence, by Theorem 2.1, $\psi\colon\ (X, \omega(\tau)) \to (X, \omega(\tau))$ is fuzzy continuous. $\square$

Now we shall establish a bridge result between the weakly fuzzy topological entropy and the topological entropy of Adler, see [1].

**Theorem 3.3.** *Let $(X, \tau)$ be a compact topological space and $\psi\colon\ (X, \tau) \to (X, \tau)$ be a continuous mapping. Then $h_w(\psi) = h_a(\psi)$, where $h_a(\psi)$ is the topological* *entropy of the continuous mapping* $\psi\colon (X,\tau) \to (X,\tau)$ *due to Adler* (*see* [1]) *and* $h_w(\psi)$ *is the weakly fuzzy topological entropy of the fuzzy continuous mapping* $\psi\colon (X,\omega(\tau)) \to (X,\omega(\tau))$.

P r o o f. Lowen in [18] proved that fuzzy topological space $(X,\omega(\tau))$ is weakly fuzzy compact if and only if topological space $(X,\tau)$ is compact.

First, let $\Sigma_\omega$ be any fuzzy open cover for $X$. Then $\vee\{\mu\colon \mu \in \Sigma_\omega\} = \underline{1}$. Let $\varepsilon \in (0,1)$. Define $U_\mu = \{(x,r) \in X \times [0,1]\colon \mu(x) > r - \varepsilon\}$. Then $U_\mu$ is an open set of $X \times [0,1]$ with $\vee\{U_\mu : \mu \in \Sigma_\omega\} \supset X \times [0,1]$. The compactness of $X \times [0,1]$ ensures the existence of a finite (minimum) number of fuzzy open sets $\mu_1, \mu_2, \ldots, \mu_n \in \Sigma_\omega$ such that $\bigcup\{U_{\mu_i}\colon i = 1,2,\ldots,n\} \supset X \times [0,1]$.

Now we define the projection mapping $\pi_X\colon X \times [0,1] \to X$ defined by $\pi_X(x,t) = x$ for all $(x,t) \in X \times [0,1]$. Then $\{\pi_X(U_{\mu_i})\colon i = 1,2,\ldots,n\}$ is a minimal subcover of $\Sigma = \{\pi_X(U_\mu)\colon \mu \in \Sigma_\omega\}$ of $X$, i.e. $N_a(\Sigma) = n$.

Again, $\{\mu_1, \mu_2, \ldots, \mu_n\}$ is a subfamily (which may not be minimal) of $\Sigma_\omega$ with $\bigvee_{i=1}^{n} \mu_i \geqslant \underline{1-\varepsilon}$. Therefore $N_w(\Sigma_\omega,\varepsilon) \leqslant n = N_a(\Sigma)$. So $H_w(\Sigma_\omega,\varepsilon) \leqslant H_a(\Sigma)$. Hence, $h_w(\psi,\Sigma_\omega,\varepsilon) \leqslant h_a(\psi,\Sigma) \leqslant h_a(\psi)$. Therefore $\{h_w(\psi,\Sigma_\omega,\varepsilon)\colon \Sigma_\omega$ is a fuzzy open cover of $X\}$ is bounded above by $h_a(\psi)$. Therefore $h_w(\psi,\varepsilon) \leqslant h_a(\psi)$ and so $h_w(\psi) \leqslant h_a(\psi)$.

Now, let $\Sigma$ be any open cover for $X$. Then $\Sigma_\omega = \{\chi_U\colon U \in \Sigma\}$ is a fuzzy open cover for $X$. Since $(X,\omega(\tau))$ is weakly fuzzy compact, this cover has a finite minimal subfamily $\{\chi_{U_i}\colon i = 1,2,\ldots,n\}$ such that $\vee\{\chi_{U_i}\colon i = 1,2,\ldots,n_\varepsilon\} = \underline{1} > \underline{1-\varepsilon}$ for each given $\varepsilon > 0$, i.e. $N_w(\Sigma_\omega,\varepsilon) = n_\varepsilon$. Again, $\{U_i\colon i = 1,2,\ldots,n_\varepsilon\}$ is a subfamily (which may not be minimal) of $\Sigma_\omega$ with $\bigcup_{i=1}^{n_\varepsilon} \mu_i = X$. Therefore $N_a(\Sigma) \leqslant N_w(\Sigma_\omega,\varepsilon)$ and so $H_a(\Sigma) \leqslant H_w(\Sigma_\omega,\varepsilon)$. Hence $h_a(\psi,\Sigma) \leqslant h_w(\psi,\Sigma_\omega,\varepsilon) \leqslant h_w(\psi,\varepsilon) \leqslant h_w(\psi)$. Thus $h_a(\psi) \leqslant h_w(\psi)$. □

Theorem 3.3 shows that the weakly fuzzy topological entropy is a "good extension" of the topological entropy of Adler (see [1]) in the sense that if $\psi\colon (X,\tau) \to (X,\tau)$ is a continuous mapping, then its topological entropy due to Adler (see [1]) is equal to the weakly fuzzy topological entropy of $\psi\colon (X,\omega(\tau)) \to (X,\omega(\tau))$. Now we shall give an example that shows that the fuzzy topological entropy of Tok (see [28]) is not a "good extension" of the topological entropy of Adler (see [1]) in the above sense.

E x a m p l e 3.1. Let $X = [0,1]$ be the topological space with usual topology $\tau$. Suppose $\psi\colon (X,\tau) \to (X,\tau)$ be the identity mapping. Then evidently, $(X,\psi)$ is a compact topological dynamical system and $h_a(\psi) = 0$. Lowen in [18] showed that $(X,\omega(\tau))$ is not fuzzy compact (= fuzzy quasi-compact) due to Chang and so the definition of Tok (see [28]) fails to calculate the fuzzy topological entropy of the fuzzy continuous mapping $\psi\colon (X,\omega(\tau)) \to (X,\omega(\tau))$. Thus, the fuzzy topological entropy of Tok (see [28]) is not a "good extension" of the topological entropy of Adler, see [1].

Again, since Theorem 3.3 implies that $h_w(\psi) = h_a(\psi) = 0$, this example also shows that the result $h_w(\psi) = h_t(\psi)$ in not true in general for a fuzzy non-compact topological dynamical system.

Now we shall set an example which shows that the concepts of the weakly fuzzy topological entropy and the fuzzy topological entropy of Afsan and Basu (see [2]) are not equivalent.

Example 3.2. Let $X = [0,1]$ with the usual topology $\tau$. Then $(X,\omega(\tau))$ is weakly fuzzy compact. We define $\mu_n \in I^X$ for each $n \in \mathbb{N}$ by $\mu_n(x) = 1 - n^{-1}$ for all $x \in X$. Since $\mu_n \in I^X$ is continuous when $I$ is given, the subtopology $\tau_{\mathbb{R}}|_I$, $\mu_n \in \omega(\tau)$ for each $n \in \mathbb{N}$. Then clearly, $\Omega_0 = \{\mu_n\colon n \in \mathbb{N}\}$ is a cover of $\underline{1}$ by the fuzzy open sets of the fuzzy topological space $(X,\omega(\tau))$ without having any finite subcover. So $(X,\omega(\tau))$ is not fuzzy compact.

Consider the continuous mapping $\psi\colon X \to X$ defined by

$$\psi(x) = \begin{cases} 2x & \text{if } x \in [0,\frac{1}{2}],\\ 2(1-x) & \text{if } x \in [\frac{1}{2},1]. \end{cases}$$

Then evidently, $(X,\psi)$ is a compact topological dynamical system with $h_a(\psi) = \log 2$ and hence by Theorem 3.3, $h_w(\psi) = \log 2$.

We claim that $h^\star(\psi) = 0$. Let $K$ be a fuzzy compact set of $(X,\omega(\tau))$ with $\psi(K) \leqslant K$, i.e. $K \in C(X,\psi)$. Since $\Omega_0$ is a fuzzy open cover of $K$, there exist finite number of natural numbers $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that $\bigvee_{i=1}^{k} \mu_{n_i} \geqslant K$. Let $n_0 = \max\{n_1, n_2, \ldots, n_k\}$. Since $\mu_n \leqslant \mu_m$ whenever $n, m \in \mathbb{N}$ and $n \leqslant m$, $\mu_{n_0} \geqslant K$. Thus, the fuzzy compact set $K$ is covered by the subcover $\{\mu_{n_0}\}$ of $\Omega_0$ consisting of single member and so $N^\star(\Omega_0, K) = 1$. Hence $H^\star(\Omega_0, K) = 0$.

Now let $\Sigma$ be any fuzzy open cover of $(X,\omega(\tau))$. We claim that $\Sigma \prec \Omega_0$. Suppose $n \in \mathbb{N}$ and $\mu_n \in \Omega_0$. Since $\vee\{U\colon U \in \Sigma\} = \underline{1}$, $\sup\{U(x)\colon U \in \Sigma\} = 1$ for all $x \in X$. So there exists $U \in \Sigma$ such that $\mu_n(x) = 1 - n^{-1} \leqslant U(x)$ for all $x \in X$ and so $\Sigma \prec \Omega_0$.

Therefore by Lemma 3.1, $H^\star(\Sigma, K) = 0$ for any fuzzy open cover $\Sigma$ of $X$. So by Definition 2.9, $h^\star(\psi, \Sigma, K) = 0$ for any fuzzy open cover $\Sigma$ of $X$ and for any $K \in C(X,\psi)$. Hence by Definition 2.9, $h^\star(\psi) = 0$.

This example also shows that the fuzzy topological entropy of Afsan and Basu (see [2]) is not a "good extension" of the topological entropy of Adler, see [1].

**Theorem 3.4.** *Let $X$ and $Y$ be weakly fuzzy compact topological spaces. Further, $\psi\colon X\to X$ is fuzzy continuous and $\varphi\colon X\to Y$ is fuzzy homeomorphism. Then $h_w(\psi)=h_w(\varphi\psi\varphi^{-1})$.*

Proof. Since $\varphi$ is a fuzzy homeomorphism, $\Sigma$ is a fuzzy open cover of $X$ if and only if $\varphi(\Sigma)$ is a fuzzy open cover of $Y$. By Lemma 3.1,

$$\begin{aligned} h_w(\varphi\psi\varphi^{-1},\varphi(\Sigma),\varepsilon) &= \lim_{n\to\infty}\frac{1}{n}H_w(\vee\{(\varphi\psi\varphi^{-1})^{-i}(\varphi(\Sigma))\colon\ i=0,1,2,\dots,n-1\},\varepsilon)\\ &= \lim_{n\to\infty}\frac{1}{n}H_w(\vee\{((\varphi\psi^{-i}\varphi)(\Sigma))\colon\ i=0,1,2,\dots,n-1\},\varepsilon)\\ &= \lim_{n\to\infty}\frac{1}{n}H_w(\vee\{\psi^{-i}(\Sigma): i=0,1,2,\dots,n-1\},\varepsilon)\\ &= h_w(\psi,\Sigma,\varepsilon). \end{aligned}$$

Thus $h_w(\psi)=h_w(\varphi\psi\varphi^{-1})$. □

**Theorem 3.5.** *Let $(X,\psi)$ be a weakly fuzzy compact topological dynamical system. Then*
(a) $h_w(\psi^k)=k\cdot h_w(\psi)$ *if* $k\in\mathbb{Z}^+$;
(b) $h_w(\psi^k)=|k|\cdot h_w(\psi)$ *if $\psi$ is a fuzzy homeomorphism and* $k\in\mathbb{Z}$.

Proof. (a) Let $\Sigma$ be a fuzzy open cover of $X$ and $\varepsilon>0$. Since for any two fuzzy covers $\Lambda_1$ and $\Lambda_2$ of $X$, $\psi^{-1}(\Lambda_1\vee\Lambda_2)=\psi^{-1}(\Lambda_1)\vee\psi^{-1}(\Lambda_2)$, we have

$$\begin{aligned} &H_w(\vee\{(\psi^k)^{-i}(\vee\{\psi^{-j}(\Sigma))\colon\ j=0,1,2,\dots,k-1\},i=0,1,2,\dots,n-1\},\varepsilon)\\ &\qquad=H_w(\vee\{\psi^{-i}(\Sigma)\colon\ i=0,1,2,\dots,nk-1\},\varepsilon). \end{aligned}$$

Let $\Omega=\vee\{\psi^{-j}(\Sigma)\colon\ j=0,1,2,\dots,k-1\}$. Then

$$\begin{aligned} &h_w(\psi^k,\Omega,\varepsilon)\\ &\ =\lim_{n\to\infty}\frac{1}{n}H_w(\vee\{(\psi^k)^{-i}(\Omega)\colon\ i=0,1,2,\dots,n-1\},\varepsilon)\\ &\ =\lim_{n\to\infty}\frac{1}{n}H_w(\vee\{(\psi^k)^{-i}(\vee\{\psi^{-j}(\Sigma): j=0,1,2,\dots,k-1\})\colon\ i=0,1,2,\dots,n-1\},\varepsilon)\\ &\ =\lim_{n\to\infty}\frac{1}{n}H_w(\vee\{\psi^{-i}(\Sigma)\colon\ i=0,1,2,\dots,nk-1\},\varepsilon)\\ &\ =k\cdot\lim_{n\to\infty}\frac{1}{nk}H_w(\vee\{\psi^{-i}(\Sigma)\colon\ i=0,1,2,\dots,nk-1\},\varepsilon)\\ &\ =k\cdot h_w(\psi,\Sigma,\varepsilon) \end{aligned}$$

for all fuzzy open covers $\Sigma$ of $X$ and for all $\varepsilon>0$. Thus, $h_w(\psi^k,\varepsilon)\geqslant h_w(\psi^k,\Omega,\varepsilon)=k\cdot h_w(\psi,\Sigma,\varepsilon)$ for every fuzzy open cover $\Sigma$ of $X$ and so $h_w(\psi^k)\geqslant k\cdot h_w(\psi)$.

Now suppose $\Sigma$ be a fuzzy open cover of $X$ and $\varepsilon > 0$. Since $\vee\{\psi^{-i}(\Sigma)\colon i = 0,1,2,\ldots,nk-1\})$ is a refinement of $\vee\{(\psi^k)^{-i}(\Sigma)\colon i = 0,1,2,\ldots,n-1\})$, by Lemma 3.1,

$$\begin{aligned}
&h_w(\psi^k,\Sigma,\varepsilon)\\
&= \lim_{n\to\infty}\frac{1}{n}H_w(\vee\{(\psi^k)^{-i}(\vee\{\psi^{-j}(\Sigma)\colon j=0,1,2,\ldots,k-1\})\colon i=0,1,2,\ldots,n-1\},\varepsilon)\\
&\leqslant k\cdot\lim_{n\to\infty}\frac{1}{nk}H_w(\vee\{\psi^{-i}(\Sigma)\colon j=0,1,2,\ldots,nk-1\},\varepsilon)\\
&= k\cdot h_w(\psi,\Sigma,\varepsilon).
\end{aligned}$$

So $h_w(\psi^k,\Sigma,\varepsilon) \leqslant k\cdot h_w(\psi,\Sigma,\varepsilon)$ for every fuzzy open cover $\Sigma$ of $X$ and for all $\varepsilon > 0$. Hence $h_w(\psi^k,\Sigma,\varepsilon) \leqslant k\cdot h_w(\psi,\varepsilon)$. So $h_w(\psi) \geqslant h_w(\psi,\varepsilon) \geqslant h_w(\psi,\Sigma,\varepsilon) \geqslant h_w(\psi^k,\Sigma,\varepsilon)k^{-1}$ for every fuzzy open cover $\Sigma$ of $X$ and for all $\varepsilon > 0$. Thus $\{h_w(\psi^k,\Sigma,\varepsilon)\colon \Sigma$ is a fuzzy open cover of $X$ and $\varepsilon > 0\}$ is bounded above by $k\cdot h_w(\psi)$ and so $h_w(\psi^k) \leqslant k\cdot h_w(\psi)$. Therefore, by the first part of the theorem, $h_w(\psi^k) = k\cdot h_w(\psi)$.

(b) Let $k\in Z^-$. Then $-k\in Z^+$ and so by (a), $h_w(\psi^{-k}) = -k\cdot h_w(\psi)$. Again by Lemma 3.1(d), $h_w(\psi^{-k}) = h_w(\psi^k)$. Then $h_w(\psi^k) = -k\cdot h_w(\psi)$ and thus $h_w(\psi^k) = |k|\cdot h_w(\psi)$. □

**Theorem 3.6.** *Let $(X,\psi)$ and $(Y,\varphi)$ be weakly fuzzy compact topological dynamical systems and $\psi\times\varphi\colon X\times Y\to X\times Y$ be the mapping that sends $(x,y)\in X\times Y$ to $(\psi(x),\varphi(y))$. Then $h_w(\psi\times\varphi) = h_w(\psi)+h_w(\varphi)$.*

P r o o f. Lowen in [18] proved that $X\times Y$ is weakly fuzzy compact. If $\sigma$ and $\varrho$ are fuzzy topologies on $X$ and $Y$, respectively, then the fuzzy topology on $X\times Y$ is defined by $\sigma\times\varrho = \{\mu\times\nu\colon \mu\in\sigma,\ \nu\in\varrho\}$, where $\mu\times\nu$ is a fuzzy set on $X\times Y$ defined by $(\mu\times\nu)(x,y) = \min\{\mu(x),\nu(y)\}$ for all $(x,y)\in X\times Y$.

First, we shall establish the fuzzy continuity of $\psi\times\varphi$. Let $\mu\times\nu\in\sigma\times\varrho$. Then $\mu\in\sigma$ and $\nu\in\varrho$. The continuities of $\psi$ and $\varphi$ ensure that $\psi^{-1}(\mu)\in\sigma$ and $\varphi^{-1}(\nu)\in\varrho$ and so $\psi^{-1}(\mu)\times\varphi^{-1}(\nu)\in\sigma\times\varrho$. We claim that $(\psi\times\varphi)^{-1}(\mu\times\nu) = \psi^{-1}(\mu)\times\varphi^{-1}(\nu)$. In fact, $((\psi\times\varphi)^{-1}(\mu\times\nu))(x,y) = (\mu\times\nu)((\psi\times\varphi)(x,y)) = (\mu\times\nu)((\psi(x),\varphi(y)) = \min\{\mu(\psi(x)),\nu(\varphi(y))\} = \min\{(\psi^{-1}(\mu))(x),(\nu^{-1}(\varphi))(y)\} = (\psi^{-1}(\mu)\times\varphi^{-1}(\nu))(x,y)$ for each $(x,y)\in X\times Y$. Therefore $\psi\times\varphi$ is fuzzy continuous.

Let $\Sigma$ be a fuzzy open cover of $X\times Y$ and $\varepsilon > 0$. Then the fact

$$\begin{aligned}
\vee\{U\times V\colon U\times V\in\Sigma\} = (&\vee\{U\in\sigma\colon U\times V\in\Sigma \text{ for some } V\in\varrho\})\\
&\times(\vee\{V\in\varrho\colon U\times V\in\Sigma \text{ for some } U\in\sigma\})
\end{aligned}$$

ensures that $\Sigma = \{U \times V \colon U \in \sigma, V \in \varrho\}$ is a fuzzy open cover of $X \times Y$ if and only if $\Sigma_X = \{U \in \sigma \colon U \times V \in \Sigma$ for some $V \in \varrho\}$ is a fuzzy open cover of $X$ and $\Sigma_Y = \{V \in \varrho \colon U \times V \in \Sigma$ for some $U \in \sigma\}$ is a fuzzy open cover of $Y$.

Let $N_w(\Sigma, \varepsilon) = mn$. Let $\Sigma_{mn} = \{U_i \times V_j \colon i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n\}$ be a finite minimal subfamily of $\Sigma$ such that $\bigvee_{i=1}^{m} \bigvee_{j=1}^{n} U_i \times V_j = \Big(\bigvee_{i=1}^{m} U_i\Big) \times \Big(\bigvee_{j=1}^{n} V_i\Big) \geqslant \underline{1_{X\times Y} - \varepsilon}$. So $\bigvee_{i=1}^{m} U_i \geqslant \underline{1_X - \varepsilon}$ and $\bigvee_{j=1}^{n} V_j \geqslant \underline{1_Y - \varepsilon}$. Thus $N_w(\Sigma_X, \varepsilon) \leqslant m$ and $N_w(\Sigma_Y, \varepsilon) \leqslant n$.

Again, if $N_w(\Sigma_X, \varepsilon) = m$ and $N_w(\Sigma_Y, \varepsilon) = n$, then there exist finite minimal subfamilies $\{U_i \colon i = 1, 2, \ldots, m\} \subset \Sigma_X$ and $\{V_j \colon j = 1, 2, \ldots, n\} \subset \Sigma_Y$ such that $\bigvee_{i=1}^{m} U_i \geqslant \underline{1_X - \varepsilon}$ and $\bigvee_{j=1}^{n} V_j \geqslant \underline{1_Y - \varepsilon}$. Then $\Sigma_{mn} = \{U_i \times V_j \colon i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n\}$ is a finite subfamily of $\Sigma = \Sigma_X \times \Sigma_Y$ such that $\bigvee_{i=1}^{m} \bigvee_{j=1}^{n} U_i \times V_j \geqslant \underline{1_{X\times Y} - \varepsilon}$. So $N_w(\Sigma, \varepsilon) \leqslant mn$.

Therefore $N_w(\Sigma, \varepsilon) = N_w(\Sigma_X, \varepsilon) \cdot N_w(\Sigma_Y, \varepsilon)$. Thus $H_w(\Sigma, \varepsilon) = H_w(\Sigma_X, \varepsilon) + H_w(\Sigma_Y, \varepsilon)$ which ensures that $h_w(\psi \times \varphi) = h_w(\psi) + h_w(\varphi)$. □

Remark 3.1. Since the product of two fuzzy compact spaces (= quasi-compact) is not fuzzy compact, the result of Theorem 3.6 cannot be obtained for the entropies of Tok (see [28]) or Afsan and Basu (see [2]).

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