OSCILLATION PROPERTIES OF SECOND-ORDER QUASILINEAR DIFFERENCE EQUATIONS WITH UNBOUNDED DELAY AND ADVANCED NEUTRAL TERMS

George E. Chatzarakis, Athens,
Ponnuraj Dinakar, Srinivasan Selvarangam,
Ethiraju Thandapani, Chennai

Received June 10, 2020. Published online July 12, 2021.
Communicated by Josef Diblík

Abstract. We obtain some new sufficient conditions for the oscillation of the solutions of the second-order quasilinear difference equations with delay and advanced neutral terms. The results established in this paper are applicable to equations whose neutral coefficients are unbounded. Thus, the results obtained here are new and complement some known results reported in the literature. Examples are also given to illustrate the applicability and strength of the obtained conditions over the known ones.

Keywords: oscillation; quasilinear difference equation; delay and advanced neutral terms

MSC 2020: 39A10

1. Introduction

We are dealing with the oscillatory properties of solutions of a second-order quasilinear difference equation with delay and advanced neutral terms of the form

$$\Delta(\zeta(i)(\Delta\chi(i))^{\alpha}) + \varrho(i)\psi^{\beta}(\sigma(i)) = 0, \quad i \geqslant i_0 > 0, \tag{1.1}$$

where $\chi(i) = \psi(i) + \varrho_1(i)\psi(i-\kappa) + \varrho_2(i)\psi(i+l)$, subject to the following conditions:

(C_1) $\{\zeta(i)\}$ and $\{\varrho(i)\}$ are real positive sequences with $\sum\limits_{i=i_0}^{\infty} \zeta^{-1/\alpha}(i) = \infty$;

(C_2) α, β are ratios of odd positive integers and l and κ are positive integers;

(C_3) $\{\sigma(i)\}$ is a sequence of integers and $\lim\limits_{i\to\infty} \sigma(i) = \infty$;

DOI: 10.21136/MB.2021.0109-20

(C$_4$) $\{\varrho_1(i)\}$ and $\{\varrho_2(i)\}$ are real sequences with $\varrho_1(i) \geqslant 0$, $\varrho_2(i) \geqslant 1$, and $\varrho_2(i) \not\equiv 1$ eventually;

(C$_5$) $\{\varrho_1(i)\}$ and $\{\varrho_2(i)\}$ are real sequences with $\varrho_2(i) \geqslant 0$, $\varrho_1(i) \geqslant 1$, and $\varrho_1(i) \not\equiv 1$ eventually.

We say a real sequence $\{\psi(i)\}$ is a solution of (1.1) if it is defined and satisfies (1.1) for all $i \geqslant i_0$. We consider only those solutions of $\{\psi(i)\}$ of (1.1) that satisfy $\sup\{|\psi(i)|\colon\ i \geqslant N\} > 0$ for all $N \geqslant i_0$; moreover, we assume tacitly that (1.1) possesses such solutions. Such a solution $\{\psi(i)\}$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise. Equation (1.1) is said to be oscillatory if all solutions of (1.1) are oscillatory.

The problem of oscillation and asymptotic behavior solutions to various classes of delay and advanced type neutral difference equations have been widely investigated in the literature, see for example [1], [2], [4], [5], [9], [10], [13], [14], [16], [17], [21]–[27] and the references cited therein. However, oscillation results for mixed type neutral difference equations are relatively scarce in the literature; some results can be found, for example, in [3], [6], [7], [8], [11], [12], [15], [19], [18], [20] and the references cited therein.

From the review of literature, we note that results obtained in [3], [6], [7], [8], [11], [12], [15], [18], [19], [20] require both of $\{\varrho_1(i)\}$ and $\{\varrho_2(i)\}$ to be constant or bounded sequences, and hence, the results established in these papers cannot be applied to the cases where $\lim\limits_{i\to\infty} \varrho_1(i) = \infty$ and/or $\lim\limits_{i\to\infty} \varrho_2(i) = \infty$. Motivated by this observation, we wish to develop new sufficient conditions which can be applied to the cases where $\lim\limits_{i\to\infty} \varrho_1(i) = \infty$ and/or $\lim\limits_{i\to\infty} \varrho_2(i) = \infty$. Therefore, the results obtained in the present paper are new and complement some existing results in the literature. Thus, we hope that the present paper will contribute significantly to the study of oscillation of the solutions of the second-order mixed type neutral difference equations.

2. Auxiliary lemmas

In this section, we present some lemmas that will play a significant role in establishing our main results. For the sake of convenience, we define the following notation:

$$F(i) = \sum_{s=N}^{i-1} \zeta^{-1/\alpha}(s), \quad \xi(i) = \frac{1}{\varrho_2(i-l)}\Big(1 - \frac{1}{\varrho_2(i-2l)} - \frac{\varrho_1(i-l)}{\varrho_2(i-\kappa-2l)}\Big) > 0,$$
$$\varphi(i) = \frac{1}{\varrho_1(i+\kappa)}\Big(1 - \frac{1}{\varrho_1(i+2\kappa)}\frac{F(i+2\kappa)}{F(i+\kappa)} - \frac{\varrho_2(i+\kappa)}{\varrho_1(i+2\kappa+l)}\frac{F(i+2\kappa+l)}{F(i+\kappa)}\Big) > 0,$$

for any $N \geqslant i_0$ and for all sufficiently large i.

Lemma 2.1 ([28]). *If $E > 0$, $D \geqslant 0$ and $\alpha > 0$, then*

$$Du - Eu^{1+1/\alpha} \leqslant \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{D^{\alpha+1}}{E^{\alpha}},$$

where equality holds if and only if $D = E$.

Lemma 2.2. *Assume that* (C_1)–(C_4) *(or* (C_1)–(C_3) *and* (C_5)*) hold, and let $\{\psi(i)\}$ be an eventually positive solution of* (1.1)*. Then there is an integer $i_1 \geqslant i_0$ such that, for $i \geqslant i_1$,*

$$\chi(i) > 0,\ \Delta\chi(i) > 0 \quad \text{and} \quad \Delta(\zeta(i)(\Delta\chi(i))^{\alpha}) < 0. \tag{2.1}$$

Proof. The proof is standard and so we omit the details. □

Lemma 2.3. *Assume that* (C_1)–(C_4) *(or* (C_1)–(C_3) *and* (C_5)*) hold, and $\{\psi(i)\}$ is a positive solution of* (1.1) *such that* (2.1) *holds. Then*

$$\chi(i) \geqslant F(i)\zeta^{1/\alpha}(i)\Delta\chi(i) \tag{2.2}$$

and

$$\left\{\frac{\chi(i)}{F(i)}\right\} \text{ is decreasing for all } i \geqslant N \geqslant i_1. \tag{2.3}$$

Proof. Since $\{\zeta(i)(\Delta\chi(i))^{\alpha}\}$ is decreasing for all $i \geqslant i_1$, we have

$$\chi(i) = \chi(i_1) + \sum_{s=i_1}^{i-1} \frac{(\zeta(s)(\Delta\chi(s))^{\alpha})^{1/\alpha}}{\zeta^{1/\alpha}(s)} \geqslant \zeta^{1/\alpha}(i)F(i)\Delta\chi(i).$$

Furthermore,

$$\Delta\Big(\frac{\chi(i)}{F(i)}\Big) = \frac{1}{\zeta^{1/\alpha}(i)} \frac{F(i)\zeta^{1/\alpha}(i)\Delta\chi(i) - \chi(i)}{F(i)F(i+1)} \leqslant 0, \quad \text{since } \Delta F(i) = \frac{1}{\zeta^{1/\alpha}(i)}.$$

The proof is now completed. □

Lemma 2.4. *Assume that* (C_1)–(C_4) *hold. If $\{\psi(i)\}$ is an eventually positive solution of* (1.1) *such that* (2.1) *holds, then $\{\chi(i)\}$ satisfies the inequality*

$$\Delta(\zeta(i)(\Delta\chi(i))^{\alpha}) + \varrho(i)\xi^{\beta}(\sigma(i))\chi^{\beta}(\sigma(i) - l) \leqslant 0 \tag{2.4}$$

for sufficiently large i.

P r o o f. Let $\{\psi(i)\}$ be an eventually positive solution of (1.1) such that $\psi(i) > 0$, $\psi(i-\kappa) > 0$, $\psi(i+l) > 0$, $\psi(\sigma(i)) > 0$ and $\chi(i)$ satisfies (2.1) for all $i \geqslant i_1$ for an integer $i_1 \geqslant i_0$. From the definition of $\chi(i)$ we obtain

$$\psi(i) = \frac{1}{\varrho_2(i-l)}(\chi(i-l) - \psi(i-l) - \varrho_1(i-l)\psi(i-\kappa-l)) \tag{2.5}$$

and

$$\psi(i) < \frac{1}{\varrho_2(i-l)}\chi(i-l). \tag{2.6}$$

Using (2.6) in (2.5), we have

$$\begin{aligned} \psi(i) &\geqslant \frac{1}{\varrho_2(i-l)}\Big(\chi(i-l) - \frac{1}{\varrho_2(i-2l)}\chi(i-2l) - \frac{\varrho_1(i-l)}{\varrho_2(i-\kappa-2l)}\chi(i-\kappa-2l)\Big) \\ &\geqslant \frac{1}{\varrho_2(i-l)}\Big(1 - \frac{1}{\varrho_2(i-2l)} - \frac{\varrho_1(i-l)}{\varrho_2(i-\kappa-2l)}\Big)\chi(i-l) \end{aligned} \tag{2.7}$$

for $i \geqslant i_1$, where we have used $\{\chi(i)\}$ is strictly increasing. Since $\lim\limits_{i\to\infty} \sigma(i) = \infty$, we can choose an integer $i_2 \geqslant i_1$ such that $\sigma(i) \geqslant i_2$ for all $i \geqslant i_2$. From (2.7) we have

$$\psi(\sigma(i)) \geqslant \xi(\sigma(i))\chi(\sigma(i) - l), \quad i \geqslant i_2. \tag{2.8}$$

Combining (1.1) with (2.8), we conclude that (2.4) is satisfied. The proof of the lemma is complete. □

Lemma 2.5. *Assume that* (C_1)–(C_3) *and* (C_5) *hold. If* $\{\psi(i)\}$ *is an eventually positive solution of* (1.1) *such that* (2.1) *holds, then* $\{\chi(i)\}$ *satisfies the inequality*

$$\Delta(\zeta(i)(\Delta\chi(i))^\alpha) + \varrho(i)\varphi^\beta(\sigma(i))\chi^\beta(\sigma(i)+\kappa) \leqslant 0 \tag{2.9}$$

for sufficiently large i.

P r o o f. Let $\{\psi(i)\}$ be an eventually positive solution of (1.1) such that $\psi(i) > 0$, $\psi(i-\kappa) > 0$, $\psi(i+l) > 0$, $\psi(\sigma(i)) > 0$ and $\chi(i)$ satisfies (2.1) for all $i \geqslant i_1$ for an integer $i_1 \geqslant i_0$. Following a similar argument as in the proof of Lemma 2.4 and taking into account that $\{\chi(i)/F(i)\}$ is decreasing for all $i \geqslant i_2$ for an integer $i_2 \geqslant i_1$, we obtain

$$\begin{aligned} \psi(i) &\geqslant \frac{1}{\varrho_1(i+\kappa)}\Big(\chi(i+\kappa) - \frac{\chi(i+2\kappa)}{\varrho_1(i+2\kappa)} - \frac{\varrho_2(i+\kappa)\chi(i+2\kappa+l)}{\varrho_1(i+2\kappa+l)}\Big) \\ &\geqslant \frac{1}{\varrho_1(i+\kappa)}\Big(1 - \frac{F(i+2\kappa)}{F(i+\kappa)\varrho_1(i+2\kappa)} - \frac{\varrho_2(i+\kappa)F(i+2\kappa+l)}{\varrho_1(i+2\kappa+l)F(i+\kappa)}\Big)\chi(i+\kappa). \end{aligned} \tag{2.10}$$

Since $\lim_{i\to\infty}\sigma(i)=\infty$, we choose an integer $i_3\geqslant i_2$ such that $\sigma(i)\geqslant i_3$ for all $i\geqslant i_3$. Thus, from (2.10) we obtain

$$\psi(\sigma(i))=\varphi(\sigma(i))\chi(\sigma(i)+\kappa),\quad i\geqslant i_3. \tag{2.11}$$

Combining (1.1) and (2.11), we conclude that (2.9) is satisfied. The proof of the lemma is complete. □

3. Main results

In this section, we present several sufficient conditions for the oscillation of all solutions of (1.1).

Theorem 3.1. *Assume that* (C_1)–(C_4) *hold and* $i+l\geqslant\sigma(i)$. *If* $\beta=\alpha$ *and there exists a positive nondecreasing sequence* $\{\eta(i)\}$ *such that for all sufficiently large integer* $N\geqslant i_1$,

$$\limsup_{i\to\infty}\sum_{s=N}^{i}\Big(\eta(s)\varrho(s)\xi^{\alpha}(\sigma(s))\frac{F^{\alpha}(\sigma(s)-l)}{F^{\alpha}(s)}-\frac{\Delta\eta(s)}{F^{\alpha}(s+1)}\Big)=\infty, \tag{3.1}$$

then every solution of (1.1) *is oscillatory.*

Proof. Let $\{\psi(i)\}$ be a nonoscillatory solution of (1.1). With no loss of generality, we may assume that there is an integer $i_1\geqslant i_0$ such that $\psi(i)>0$, $\psi(i-\kappa)>0$, $\psi(i+l)>0$, $\psi(\sigma(i))>0$ and $\chi(i)$ satisfies (2.1) for all $i\geqslant i_1$. Proceeding as in the proof of Lemmas 2.3 and 2.4, we see that (2.2), (2.3) and (2.4) hold for all $i\geqslant i_1$. Define

$$\omega(i)=\eta(i)\frac{\zeta(i)(\Delta\chi(i))^{\alpha}}{\chi^{\alpha}(i)},\quad i\geqslant i_1. \tag{3.2}$$

Clearly $\omega(i)>0$ for $i\geqslant i_1$, and from (2.4) we obtain

$$\begin{aligned}\Delta\omega(i)\leqslant&-\eta(i)\varrho(i)\frac{\xi^{\alpha}(\sigma(i))\chi^{\alpha}(\sigma(i-l))}{\chi^{\alpha}(i)}+\frac{\Delta\eta(i)\zeta(i+1)(\Delta\chi(i+1))^{\alpha}}{\chi^{\alpha}(i+1)}\\&-\frac{\eta(i)\zeta(i+1)(\Delta\chi(i+1))^{\alpha}}{\chi^{\alpha}(i)\chi^{\alpha}(i+1)}\Delta\chi^{\alpha}(i).\end{aligned} \tag{3.3}$$

From $i\geqslant\sigma(i)-l$ and by (2.3) we get

$$\frac{\chi(\sigma(i)-l)}{\chi(i)}\geqslant\frac{F(\sigma(i)-l)}{F(i)},\quad i\geqslant i_2\geqslant i_1. \tag{3.4}$$

Using (3.4) in (3.3) yields

$$\Delta\omega(i) \leqslant -\eta(i)\varrho(i)\xi^{\alpha}(\sigma(i))\frac{F^{\alpha}(\sigma(i)-l)}{F^{\alpha}(i)} + \Delta\eta(i)\frac{\zeta(i+1)(\Delta\chi(i+1))^{\alpha}}{\chi^{\alpha}(i+1)} - \frac{\eta(i)\zeta(i)(\Delta\chi(i))^{\alpha}}{\chi^{\alpha}(i)\chi^{\alpha}(i+1)}\Delta\chi^{\alpha}(i), \quad i \geqslant i_2. \tag{3.5}$$

From (2.2), it is easy to see that

$$\frac{1}{F^{\alpha}(i+1)} \geqslant \zeta(i+1)\frac{(\Delta\chi(i+1))^{\alpha}}{\chi^{\alpha}(i+1)}. \tag{3.6}$$

In view of (3.6), $\chi(i) > 0$ and $\Delta\chi(i) > 0$, (3.5) yields

$$\Delta\omega(i) \leqslant -\eta(i)\varrho(i)\xi^{\alpha}(\sigma(i))\frac{F^{\alpha}(\sigma(i)-l)}{F^{\alpha}(i)} + \frac{\Delta\eta(i)}{F^{\alpha}(i+1)} \tag{3.7}$$

for $i \geqslant i_2$. Summing up (3.7) from i_2 to i, we get

$$\sum_{s=i_2}^{i}\Big(\eta(s)\varrho(s)\xi^{\alpha}(\sigma(s))\frac{F^{\alpha}(\sigma(s)-l)}{F^{\alpha}(s)} - \frac{\Delta\eta(s)}{F^{\alpha}(s+1)}\Big) \leqslant \omega(i_2),$$

which contradicts (3.1). The proof of the theorem is complete. □

Theorem 3.2. *Assume that* (C$_1$)–(C$_4$) *hold and* $i + l \geqslant \sigma(i)$. *If there exists a positive nondecreasing sequence* $\{\eta(i)\}$ *such that for all sufficiently large integers* $N \geqslant i_1$,

$$\limsup_{i\to\infty}\sum_{s=N}^{i}\Big(E_1(s) - \Big(\frac{\alpha}{\beta}\Big)^{\alpha}\frac{(\Delta\eta(i))^{\alpha+1}f(s)}{(\alpha+1)^{\alpha+1}\eta^{\alpha}(s)\delta^{\alpha}(s)}\Big) = \infty, \tag{3.8}$$

where

$$E_1(i) = \eta(i)\varrho(i)\xi^{\beta}(\sigma(i))\frac{F^{\beta}(\sigma(i)-l)}{F^{\beta}(i)} \quad \text{and} \quad \delta(i) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ M_1 & \text{if } \alpha < \beta, \\ M_2F^{-1+\beta/\alpha}(i+1) & \text{if } \alpha > \beta \end{cases}$$

for all $M_1 > 0$, $M_2 > 0$, *then every solution of* (1.1) *is oscillatory.*

P r o o f. Let $\{\psi(i)\}$ be a nonoscillatory solution of (1.1). With no loss of generality, we may assume that there is an integer $i_1 \geqslant i_0$ such that $\psi(i) > 0$, $\psi(i-\kappa) > 0$, $\psi(i+l) > 0$, $\psi(\sigma(i)) > 0$ and $\chi(i)$ satisfies (2.1) for all $i \geqslant i_1$. Define

$$\omega(i) = \eta(i)\frac{\zeta(i)(\Delta\chi(i))^{\alpha}}{\chi^{\beta}(i)}, \quad i \geqslant i_1. \tag{3.9}$$

Clearly $\omega(i) > 0$ for $i \geqslant i_1$, and from (2.4) and (3.9) we get

$$\begin{aligned}\Delta\omega(i) \leqslant & -\eta(i)\varrho(i)\xi^{\beta}(\sigma(i))\frac{\chi^{\beta}(\sigma(i)-l)}{\chi^{\beta}(i)} + \frac{\Delta\eta(i)}{\eta(i+1)}\omega(i+1) \\ & - \frac{\eta(i)}{\eta(i+1)}\omega(i+1)\frac{\Delta\chi^{\beta}(i)}{\chi^{\beta}(i)}, \quad i \geqslant i_2 \geqslant i_1.\end{aligned} \tag{3.10}$$

By discrete mean value theorem (see [1]), we have

$$\Delta\chi^{\beta}(i) \geqslant \beta\frac{\chi^{\beta}(i)}{\chi(i+1)}\Delta\chi(i).$$

Using this and (3.4) in (3.10) gives

$$\Delta\omega(i) \leqslant -E_1(i) + \frac{\Delta\eta(i)}{\eta(i+1)}\omega(i+1) - \beta\frac{\eta(i)}{\eta(i+1)}\omega(i+1)\frac{\Delta\chi(i)}{\chi(i+1)}. \tag{3.11}$$

Since $f^{1/\alpha}(i)\Delta\chi(i)$ is decreasing, we have from (3.9) and (3.11)

$$\begin{aligned}\Delta\omega(i) \leqslant & -E_1(i) + \frac{\Delta\eta(i)}{\eta(i+1)}\omega(i+1) - \beta\frac{\eta(i)}{\eta^{1+1/\alpha}(i+1)}\omega^{1+1/\alpha}(i+1)\chi^{-1+\beta/\alpha}(i+1) \\ \leqslant & -E_1(i) + \frac{\Delta\eta(i)}{\eta(i+1)}\omega(i+1) - \beta\frac{\eta(i)\delta(i)}{\eta^{1+1/\alpha}(i+1)\zeta^{1/\alpha}(i)}\omega^{1+1/\alpha}(i+1),\end{aligned} \tag{3.12}$$

where we have used that $\chi(i)$ is increasing for $\beta > \alpha$ and $\chi(i)/F(i)$ is decreasing for $\beta < \alpha$. Applying Lemma 2.1 with

$$D = \frac{\Delta\eta(i)}{\eta(i+1)}, \quad E = \frac{\beta\eta(i)\delta(i)}{\eta^{1+1/\alpha}(i+1)\zeta^{1/\alpha}(i)},$$

we have from (3.12) that

$$\Delta\omega(i) \leqslant -E_1(i) + \frac{(\alpha/\beta)^{\alpha}(\Delta\eta(i))^{\alpha+1}\zeta(i)}{(\alpha+1)^{\alpha+1}\eta^{\alpha}(i)\delta^{\alpha}(i)}, \quad i \geqslant i_2.$$

Summing up the last inequality from i_2 to i, we get

$$\sum_{s=i_2}^{i}\Big(-E_1(s) + \frac{(\alpha/\beta)^{\alpha}(\Delta\eta(s))^{\alpha+1}\zeta(s)}{(\alpha+1)^{\alpha+1}\eta^{\alpha}(s)\delta^{\alpha}(s)}\Big) \leqslant \omega(i_2),$$

which contradicts (3.8). The proof of the theorem is complete. □

Theorem 3.3. *Assume that* (C_1)–(C_4) *hold,* $\alpha = \beta$ *and* $i + l \leqslant \sigma(i)$*. If there exists a positive nondecreasing real sequence* $\{\eta(i)\}$ *such that for all sufficiently large* $N \geqslant i_1 \geqslant i_0$,

$$\limsup_{i\to\infty} \sum_{s=N}^{i} \Big(\eta(s)\varrho(s)\xi^{\alpha}(\sigma(s)) - \frac{\Delta\eta(s)}{F^{\alpha}(s+1)}\Big) = \infty, \tag{3.13}$$

then every solution of (1.1) *is oscillatory.*

P r o o f. Let $\{\psi(i)\}$ be a nonoscillatory solution of (1.1). With no loss of generality, we may assume that there is an integer $i_1 \geqslant i_0$ such that $\psi(i) > 0$, $\psi(i-\kappa) > 0$, $\psi(i+l) > 0$, $\psi(\sigma(i)) > 0$ and $\chi(i)$ satisfies (2.1) for all $i \geqslant i_1$. Proceeding as in the proof of Theorem 3.1, we arrive at (3.3) for $i \geqslant i_2 \geqslant i_1$. From $i + \kappa \leqslant \sigma(i)$ we have $i \leqslant \sigma(i) - l$ and so

$$\frac{\chi(\sigma(i)-l)}{\chi(i)} \geqslant 1. \tag{3.14}$$

Using (3.14) in (3.3) yields

$$\begin{aligned} \Delta\omega(i) \leqslant\ & -\eta(i)\varrho(i)\xi^{\alpha}(\sigma(i)) + \Delta\eta(i)\frac{\zeta(i+1)(\Delta\chi(i+1))^{\alpha}}{\chi^{\alpha}(i+1)} \\ & - \frac{\eta(i)\zeta(i)(\Delta\chi(i))^{\alpha}\Delta\chi^{\alpha}(i)}{\chi^{\alpha}(i)\chi^{\alpha}(i+1)}, \quad i \geqslant i_2. \end{aligned} \tag{3.15}$$

Taking into account that (2.2) holds and using the fact that $\Delta\chi(i) > 0$, (3.15) takes the form

$$\Delta\omega(i) \leqslant -\eta(i)\varrho(i)\xi^{\alpha}(\sigma(i)) + \frac{\Delta\eta(i)}{F^{\alpha}(i+1)}, \quad i \geqslant i_2.$$

The remaining part of the proof is similar to that of Theorem 3.1 and the details are omitted. The proof of the theorem is complete. □

Theorem 3.4. *Assume that* (C_1)–(C_4) *hold and* $i+l \leqslant \sigma(i)$*. If there exists a positive nondecreasing real sequence* $\{\eta(i)\}$ *such that for all sufficiently large* $N \geqslant i_1 \geqslant i_0$,

$$\limsup_{i\to\infty} \sum_{s=N}^{i} \Big(\eta(s)\varrho(s)\xi^{\beta}(\sigma(s)) - \Big(\frac{\alpha}{\beta}\Big)^{\alpha} \frac{(\Delta\eta(s))^{\alpha+1}\zeta(s)}{(\alpha+1)^{\alpha+1}\eta^{\alpha}(s)\delta^{\alpha}(s)}\Big) = \infty, \tag{3.16}$$

where $\delta(i)$ *is defined as in Theorem* 3.3*, then every solution of* (1.1) *is oscillatory.*

P r o o f. The proof follows from Theorem 3.2 and (3.14) and so the details are omitted. □

Theorem 3.5. *Assume that* $\alpha=\beta$, (C_1)–(C_3) *and* (C_5) *hold, and* $i-\kappa\geqslant\sigma(i)$. *If there exists a positive nondecreasing real sequence* $\{\eta(i)\}$ *such that for all sufficiently large* $N\geqslant i_1\geqslant i_0$,

$$\limsup_{i\to\infty}\sum_{s=N}^{i}\Big(E_2(s)-\frac{\Delta\eta(s)}{F^{\alpha}(s+1)}\Big)=\infty, \tag{3.17}$$

where

$$E_2(i)=\eta(i)\varrho(i)\varphi^{\alpha}(\sigma(i))\frac{F^{\alpha}(\sigma(i)+\kappa)}{F^{\alpha}(i)},$$

then every solution of (1.1) *is oscillatory.*

Proof. Let $\{\psi(i)\}$ be a nonoscillatory solution of (1.1). With no loss of generality, we may assume that there is an integer $i_1\geqslant i_0$ such that $\psi(i)>0$, $\psi(i-\kappa)>0$, $\psi(i+l)>0$, $\psi(\sigma(i))>0$ and $\chi(i)$ satisfies (2.1) for all $i\geqslant i_1$. Proceeding as in the proof of Lemmas 2.3 and 2.5, we have (2.2), (2.3) and (2.9) hold for $i\geqslant i_2$ for an integer $i_2\geqslant i_1$. Define $\omega(i)$ by (3.2). Then it follows from (3.2) and (2.9) that

$$\begin{aligned}\Delta\omega(i)\leqslant&-\eta(i)\varrho(i)\varphi^{\alpha}(\sigma(i))\frac{\chi^{\alpha}(\sigma(i)+\kappa)}{\chi^{\alpha}(i)}+\Delta\eta(i)\frac{\zeta(i+1)(\Delta\chi(i+1))^{\alpha}}{\chi^{\alpha}(i+1)}\\&-\frac{\eta(i)\zeta(i+1)(\Delta\chi(i+1))^{\alpha}\Delta\chi^{\alpha}(i)}{\chi^{\alpha}(i)\chi^{\alpha}(i+1)},\quad i\geqslant i_2.\end{aligned} \tag{3.18}$$

Since $i-\kappa\geqslant\sigma(i)$, we have $i+1\geqslant i\geqslant\sigma(i)+\kappa$, and from (2.3) we get

$$\frac{\chi(\sigma(i)+\kappa)}{\chi(i)}\geqslant\frac{F(\sigma(i)+\kappa)}{F(i)}. \tag{3.19}$$

Substituting (3.19) into (3.18) yields

$$\Delta\omega(i)\leqslant-E_2(i)+\Delta\eta(i)\frac{\zeta(i+1)(\Delta\chi(i+1))^{\alpha}}{\chi^{\alpha}(i+1)}-\frac{\eta(i)\zeta(i+1)(\Delta\chi(i+1))^{\alpha}\Delta\chi^{\alpha}(i)}{\chi^{\alpha}(i)\chi^{\alpha}(i+1)},$$

where $i\geqslant i_2$. The rest of the proof is similar to that of Theorem 3.1 and hence the details are not repeated. The proof of the theorem is complete. □

Theorem 3.6. *Assume that* (C_1)–(C_3) *and* (C_5) *hold and* $i-\kappa\geqslant\sigma(i)$. *If there exists a positive nondecreasing real sequence* $\{\eta(i)\}$ *such that for all sufficiently large integers* $N\geqslant i_1$,

$$\limsup_{i\to\infty}\sum_{s=N}^{i}\Big(E_3(s)-\Big(\frac{\alpha}{\beta}\Big)^{\alpha}\frac{(\Delta\eta(s))^{\alpha+1}\zeta(s)}{(\alpha+1)^{\alpha+1}\eta^{\alpha}(s)\delta^{\alpha}(s)}\Big)=\infty, \tag{3.20}$$

where

$$E_3(i)=\eta(i)\varrho(i)\varphi^{\beta}(\sigma(i))\frac{F^{\beta}(\sigma(i)+\kappa)}{F^{\beta}(i+1)},$$

then every solution of (1.1) *is oscillatory.*

P r o o f. The proof follows from Theorem 3.2 by using (3.19) instead of (3.4), and so the details are not repeated. This completes the proof. $\square$

Theorem 3.7. *Assume that* (C_1)–(C_3) *and* (C_5) *hold* $\alpha = \beta$ *and* $i - \kappa \leqslant \sigma(i)$. *If there exists a positive nondecreasing real sequence* $\{\eta(i)\}$ *such that for all sufficiently large integers* $N \geqslant i_1 \geqslant i_0$,

$$\limsup_{i\to\infty} \sum_{s=N}^{i} \Big(E_4(s) - \frac{\Delta\eta(s)}{F^{\alpha}(s+1)} \Big) = \infty, \tag{3.21}$$

where

$$E_4(i) = \eta(i)\varrho(i)\varphi^{\alpha}(\sigma(i)),$$

then every solution of (1.1) *is oscillatory.*

P r o o f. Proceeding as in the proof of Theorem 3.5, we arrive at (3.18). For $i-\kappa \leqslant \sigma(i)$, we see that $i \leqslant \sigma(i)+\kappa$, and so since using $\{\chi(i)\}$ is increasing, we have

$$\frac{\chi(\sigma(i)+\kappa)}{\chi(i)} \geqslant 1. \tag{3.22}$$

Using (3.22) in (3.18), we obtain

$$\begin{aligned} \Delta\omega(i) \leqslant E_4(i) + \Delta\eta(i)\frac{\zeta(i+1)(\Delta\chi(i+1))^{\alpha}}{\chi^{\alpha}(i+1)} \\ - \frac{\eta(i)\zeta(i+1)(\Delta\chi(i+1))^{\alpha}\Delta\chi^{\alpha}(i)}{\chi^{\alpha}(i)\chi^{\alpha}(i+1)}, \quad i \geqslant i_2. \end{aligned}$$

The remaining part of the proof is similar to that of Theorem 3.5 and so the details are omitted. The proof of the theorem is complete. $\square$

Theorem 3.8. *Assume that* (C_1)–(C_3) *and* (C_5) *hold and* $i - \kappa \leqslant \sigma(i)$. *If there exists a positive nondecreasing real sequence* $\{\eta(i)\}$ *such that for all sufficiently large integers* $N \geqslant i_1 \geqslant i_0$,

$$\limsup_{i\to\infty} \sum_{s=N}^{i} \Big(E_5(s) - \Big(\frac{\alpha}{\beta}\Big)^{\alpha} \frac{(\Delta\eta(i))^{\alpha+1}\zeta(s)}{(\alpha+1)^{\alpha+1}\eta^{\alpha}(s)\delta^{\alpha}(s)} \Big) = \infty, \tag{3.23}$$

where $\delta(i)$ *is defined as in Theorem* 3.2 *and* $E_5(i) = \eta(i)\varrho(i)\varphi^{\beta}(\sigma(i))$, *then every solution of* (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 3.2 and using (2.9) instead of (2.4) we obtain

$$\Delta\omega(i) \leqslant -\eta(i)\varrho(i)\varphi^{\beta}(\sigma(i))\frac{\chi^{\beta}(\sigma(i)+\kappa)}{\chi^{\beta}(i)} + \frac{\Delta\eta(i)}{\eta(i+1)}\omega(i+1) - \frac{\eta(i)}{\eta(i+1)}\omega(i+1)\frac{\Delta\chi^{\beta}(i)}{\chi^{\beta}(i)}, \quad i \geqslant i_2. \tag{3.24}$$

Using (3.22) in (3.24) yields

$$\Delta\omega(i) \leqslant -E_5(i) + \frac{\Delta\eta(i)}{\eta(i+1)}\omega(i+1) - \frac{\eta(i)}{\eta(i+1)}\omega(i+1)\frac{\Delta\chi^{\beta}(i)}{\chi^{\beta}(i)}.$$

The rest part of the proof is similar to that of Theorem 3.2 and so the details are omitted. The proof of the theorem is complete. □

4. Examples

In this section, we present several examples to illustrate the importance of the main results.

Example 4.1. Consider the second-order neutral difference equation

$$\Delta((\Delta(\psi(i)+\psi(i-1)+i\psi(i+2)))^3) + (i^4+1)\psi^3(i+1) = 0, \quad i \geqslant 10. \tag{4.1}$$

Here, $\alpha = \beta = 3$, $\zeta(i) = \varrho_1(i) = 1$, $\varrho_2(i) = n$, $\varrho(i) = n^4+1$, $\kappa = 1$, $l = 2$ and $\sigma(i) = i+1$. It is clear that (C_1)–(C_4) hold, $i+\kappa \geqslant \sigma(i)$, and

$$\xi(i) = \frac{1}{i-2}\Big(1 - \frac{1}{i-4} - \frac{1}{i-5}\Big) = \frac{1}{i-2}\frac{i^2-11i+29}{(i-4)(i-5)} > 0,$$
$$F(i) = \sum_{s=N}^{i-1}\frac{1}{\zeta^{1/\alpha}(s)} = \sum_{s=10}^{i-1}\Delta s = i-10.$$

With $\eta(i) = 1$, we see that (3.1) becomes

$$\limsup_{i\to\infty}\sum_{s=10}^{i}\frac{(s^4+1)}{(s-2)^3}\frac{(s^2-11s+29)^3}{(s-4)^3(s-5)^3}\frac{(s-11)^3}{(s-10)^3} = \infty,$$

which, in view of Theorem 3.1, means that all solutions of (4.1) are oscillatory.

E x a m p l e 4.2. Consider the second-order neutral difference equation

$$\Delta\Big(\frac{1}{i^{1/3}}(\Delta(\psi(i)+2\psi(i-1)+8\psi(i+2)))^{1/3}\Big)+(i^2+1)\psi^{1/3}(i+3)=0, \quad i\geqslant 2. \tag{4.2}$$

Here, $\alpha=\beta=\frac{1}{3}$, $\zeta(i)=i^{-1/3}$, $\varrho_1(i)=2$, $\varrho_2(i)=8$, $\varrho(i)=i^2+1$, $\kappa=1$, $l=2$, and $\sigma(i)=i+3$. It is clear that (C_1)–(C_4) hold, $\sigma(i)\geqslant(i+\kappa)$ and

$$\xi(i)=\frac{1}{8}\Big(1-\frac{1}{8}-\frac{2}{8}\Big)=\frac{5}{64}>0, \quad F(i)=\sum_{s=2}^{i-1}s=\frac{i^2-i-2}{2}.$$

With $\eta(i)=1$, (3.13) becomes

$$\limsup_{i\to\infty}\sum_{s=2}^{i}(s^2+1)\Big(\frac{5}{64}\Big)^{1/3}=\infty,$$

which, in view of Theorem 3.3, means that all solutions of (4.2) are oscillatory.

E x a m p l e 4.3. Consider the second-order neutral difference equation

$$\Delta((\Delta(\psi(i)+3i\psi(i-1)+i\psi(i+2)))^3)+i^5\psi^3(i-2)=0, \quad i\geqslant 2. \tag{4.3}$$

Here, $\alpha=\beta=3$, $\zeta(i)=1$, $\varrho_1(i)=3i$, $\varrho_2(i)=i$, $\varrho(i)=i^5$, $\kappa=1$, $l=2$, and $\sigma(i)=i-2$. It is clear that (C_1)–(C_3) and (C_5) hold, $i-\kappa\geqslant\sigma(i)$ and

$$\begin{aligned}\varphi(i)&=\frac{1}{3(i+1)}\Big(1-\frac{1}{3(i+2)}\frac{i}{i-1}-\frac{i+1}{3(i+4)}\frac{(i+1)}{(i-1)}\Big)\\&=\frac{1}{3(i+1)}\frac{6i^4+24i^3-39i^2-69i+14}{9(i^4+4i^3-3i^2-10i+8)}>0.\end{aligned}$$

With $\eta(i)=1$, we see that (3.17) holds for $N>2$. Therefore, in view of Theorem 3.5, every solution of (4.3) is oscillatory.

E x a m p l e 4.4. Consider the second-order neutral difference equation

$$\Delta^2(\psi(i)+2^i\psi(i-2)+\psi(i+1))+9(2^i)\psi(i-1)=0, \quad i\geqslant 5. \tag{4.4}$$

Here, $\alpha=\beta=1$, $\zeta(i)=\varrho_2(i)=1$, $\varrho_1(i)=2^i$, $\varrho(i)=9(2^i)$, $\kappa=2$, $l=1$, and $\sigma(i)=i-1$. It is clear that (C_1)–(C_3) and (C_5) hold and $i-\kappa<\sigma(i)$. Also $F(i)=i-5$ and so

$$\varphi(i)=\frac{1}{4^{i+1}}\Big(2^i-\frac{9i-8}{32(i-3)}\Big)>0.$$

With $\eta(i)=1$, it is easy to see that (3.21) holds. Therefore, in view of Theorem 3.7, every solution of (4.4) is oscillatory. In fact, $\{\psi(i)\}=\{(-1)^i\}$ is such a solution.

5. Conclusion

In this paper, we have established several new oscillation theorems for equation (1.1) by using Ricatti transformation technique and summation averaging method. Furthermore, none of the results obtained in the literature can be used for the above examples to get any conclusion since the coefficients $\{\varrho_1(i)\}$ and $\{\varrho_2(i)\}$ are unbounded. Thus, the results established in this paper are new and complement the existing results.

References

[1] *R. P. Agarwal*: Difference Equations and Inequalities: Theory, Methods, and Applications. Pure and Applied Mathematics, Marcel Dekker 228. Marcel Dekker, New York, 2000. zbl MR doi

[2] *R. P. Agarwal, M. Bohner, S. R. Grace, D. O'Regan*: Discrete Oscillation Theory. Hindawi Publishing, New York, 2005. zbl MR doi

[3] *R. P. Agarwal, S. R. Grace, E. Akın-Bohner*: On the oscillation of higher order neutral difference equations of mixed type. Dyn. Syst. Appl. *11* (2002), 459–469. zbl MR

[4] *G. E. Chatzarakis, G. N. Miliaras*: Asymptotic behavior in neutral difference equations with several retarded arguments. Rocky Mt. J. Math. *45* (2015), 131–156. zbl MR doi

[5] *G. E. Chatzarakis, G. N. Miliaras, I. P. Stavroulakis, E. Thandapani*: Asymptotic behaviour of non-oscillatory solutions of first-order neutral difference equations. Panam. Math. J. *23* (2013), 111–129. zbl MR

[6] *S. R. Grace*: Oscillation of certain neutral difference equations of mixed type. J. Math. Anal. Appl. *224* (1998), 241–254. zbl MR doi

[7] *S. R. Grace, J. Alzabut*: Oscillation results for nonlinear second order difference equations with mixed neutral terms. Adv. Difference Equ. *2020* (2020), Article ID 8, 12 pages. MR doi

[8] *S. R. Grace, S. Dontha*: Oscillation of higher order neutral difference equations of mixed type. Dyn. Syst. Appl. *12* (2003), 521–532. zbl MR

[9] *J. Jiang*: Oscillation of second order nonlinear neutral delay difference equations. Appl. Math. Comput. *146* (2003), 791–801. zbl MR doi

[10] *S. H. Saker*: New oscillation criteria for second-order nonlinear neutral delay difference equations. Appl. Math. Comput. *142* (2003), 99–111. zbl MR doi

[11] *D. Seghar, E. Thandapani, S. Pinelas*: Some new oscillation theorems for second order difference equations with mixed neutral terms. Br. J. Math. Comput. Sci. *8* (2015), 121–133.

[12] *S. Selvarangam, S. Geetha, E. Thandapani, S. Pinelas*: Classification of solutions of second order nonlinear neutral difference equations of mixed type. Dyn. Contin. Discrete Impuls. Syst., Ser. B, Appl. Algorithms *23* (2016), 433–447. zbl

[13] *S. Selvarangam, E. Thandapani, S. Pinelas*: Oscillation theorems for second order nonlinear neutral difference equations. J. Inequal. Appl. *2014* (2014), Article ID 417, 15 pages. zbl MR doi

[14] *Y. G. Sun, S. H. Saker*: Oscillation for second-order nonlinear neutral delay difference equations. Appl. Math. Comput. *163* (2005), 909–918. zbl MR doi

[15] *E. Thandapani, V. Balasubramanian*: Oscillation and asymptotic behavior of second order nonlinear neutral difference equations of mixed arguments. Transylv. J. Math. Mech. *5* (2013), 149–156. MR

[16] *E. Thandapani, V. Balasubramanian*: Some oscillation results for second order neutral type difference equations. Differ. Equ. Appl. *3* (2013), 319–330. zbl MR doi

[17] *E. Thandapani, V. Balasubramanian*: Some oscillation theorems for second order nonlinear neutral type difference equations. Malaya J. Mat. *3* (2013), 34–43. zbl
[18] *E. Thandapani, N. Kavitha*: Oscillation theorems for second-order nonlinear neutral difference equation of mixed type. J. Math. Comput. Sci. *1* (2011), 89–102. MR
[19] *E. Thandapani, N. Kavitha, S. Pinelas*: Comparison and oscillation theorem for second-order nonlinear neutral difference equations of mixed type. Dyn. Syst. Appl. *21* (2012), 83–92. zbl MR
[20] *E. Thandapani, N. Kavitha, S. Pinelas*: Oscillation criteria for second-order nonlinear neutral difference equations of mixed type. Adv. Difference Equ. *2012* (2012), Article ID 4, 10 pages. zbl MR doi
[21] *E. Thandapani, K. Mahalingam*: Necessary and sufficient conditions for oscillation of second order neutral difference equations. Tamkang J. Math. *34* (2003), 137–145. zbl MR
[22] *E. Thandapani, K. Mahalingam*: Oscillation and nonoscillation of second order neutral delay difference equations. Czech. Math. J. *53* (2003), 935–947. zbl MR doi
[23] *E. Thandapani, D. Seghar, S. Selvarangam*: Oscillation of second order quasilinear difference equations with several neutral terms. Transylv. J. Math. Mech. *7* (2015), 67–74. zbl MR
[24] *E. Thandapani, S. Selvarangam*: Oscillation theorems for second order quasilinear neutral difference equations. J. Math. Comput. Sci. *2* (2012), 866–879. MR
[25] *E. Thandapani, S. Selvarangam*: Oscillation of solutions of second order neutral type difference equations. Nonlinear Funct. Anal. Appl. *20* (2015), 329–336. zbl
[26] *E. Thandapani, P. Sundaram, I. Gyori*: Oscillation of second order nonlinear neutral delay difference equations. J. Math. Phys. Sci. *31* (1997), 121–132. MR
[27] *D.-M. Wang, Z.-T. Xu*: Oscillation of second-order quasilinear neutral delay difference equations. Acta Math. Appl. Sin., Engl. Ser. *27* (2011), 93–104. zbl MR doi
[28] *S.-Y. Zhang, Q.-R. Wang*: Oscillation of second-order nonlinear neutral dynamic equations on time scales. Appl. Math. Comput. *216* (2010), 2837–2848. zbl MR doi

Authors' addresses: *George E. Chatzarakis*, Department of Electrical and Electronic Engineering Educators, School of Pedagogical and Technological Education, Marousi 15122, Athens, Greece, e-mail: `gea.xatz@aspete.gr`, `geaxatz@otenet.gr`; *Ponnuraj Dinakar*, *Srinivasan Selvarangam*, Department of Mathematics, Presidency College, University of Madras, Chennai 600 005, India, e-mail: `dinakarraj@gmail.com`, `selvarangam.9962@gmail.com`; *Ethiraju Thandapani*, Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai 600 005, India, e-mail: `ethandapani@yahoo.co.in`.