GENERALIZED ATOMIC SUBSPACES FOR OPERATORS IN HILBERT SPACES

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Abstract. We introduce the notion of a g-atomic subspace for a bounded linear operator and construct several useful resolutions of the identity operator on a Hilbert space using the theory of g-fusion frames. Also, we shall describe the concept of frame operator for a pair of g-fusion Bessel sequences and some of their properties.

Keywords: frame; atomic subspace; g-fusion frame; K-g-fusion frame

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1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer in 1952 to study some fundamental problems in non-harmonic Fourier series (see [7]). Later on, after some decades, frame theory was popularized by Daubechies, Grossman, Meyer (see [5]). At present, frame theory has been widely used in signal and image processing, filter bank theory, coding and communications, system modeling and so on. Several generalizations of frames, namely K-frames, g-frames, fusion frames etc. have been introduced in recent times.

K-frames were introduced by Gavruta (see [8]) to study the atomic system with respect to a bounded linear operator. Using frame theory techiques, the author also studied the atomic decompositions for operators on reproducing kernel Hilbert spaces, see [9]. Sun in [15] introduced a g-frame and a g-Riesz basis in complex Hilbert spaces and discussed several properties of them. Huang in [12] began to study K-g-frame by combining K-frame and g-frame. Casazza (see [3]) was first to introduce the notion of fusion frames or frames of subspaces and gave various ways to obtain a resolution of the identity operator from a fuison frame. The concept of

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an atomic subspace with respect to a bounded linear operator were introduced by Bhandari and Mukherjee in [2]. Construction of K-g-fusion frames and their dual were presented by Sadri and Rahimi (see [1]) to generalize the theory of K-frame, fusion frame and g-frame. Ghosh and Samanta in [11] studied the stability of dual g-fusion frames in Hilbert spaces.

In this paper, we present some useful results about resolution of the identity operator on a Hilbert space using the theory of g-fusion frames. We give the notion of g-atomic subspace with respect to a bounded linear operator. The frame operator for a pair of g-fusion Bessel sequences are discussed and some properties are going to be established.

The paper is organized as follows: in Section 2, we briefly recall the basic definitions and results. Various ways of obtaining resolution of the identity operator on a Hilbert space in g-fusion frame are studied in Section 3. g-atomic subspaces are introduced and discussed in Section 4. In Section 5, frame operators for a pair of g-fusion Bessel sequences are given and various properties are established.

Throughout this paper, H is considered to be a separable Hilbert space with associated inner product $\langle \cdot, \cdot \rangle$ and $\{H_j\}_{j\in J}$ are the collection of Hilbert spaces, where J is a subset of integers $\mathbb{Z}$. I_H is the identity operator on H. $\mathcal{B}(H_1, H_2)$ is a collection of all bounded linear operators from H_1 to H_2. In particular, $\mathcal{B}(H)$ denotes the space of all bounded linear operators on H. For $T \in \mathcal{B}(H)$, we denote $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for null space and range of T, respectively. Also, $P_V \in \mathcal{B}(H)$ is the orthonormal projection onto a closed subspace $V \subset H$. Define the space

$$l^2(\{H_j\}_{j\in J}) = \left\{ \{f_j\}_{j\in J} \colon\ f_j \in H_j,\ \sum_{j\in J} \|f_j\|^2 < \infty \right\}$$

with inner product given by

$$\langle \{f_j\}_{j\in J}, \{g_j\}_{j\in J} \rangle = \sum_{j\in J} \langle f_j, g_j \rangle_{H_j}.$$

Clearly $l^2(\{H_j\}_{j\in J})$ is a Hilbert space with the pointwise operations (see [1]).

2. Preliminaries

Theorem 2.1 ([6], Douglas' factorization theorem). *Let $U, V \in \mathcal{B}(H)$. Then the following conditions are equivalent:*

(1) $\mathcal{R}(U) \subseteq \mathcal{R}(V)$.
(2) $UU^* \leqslant \lambda^2 VV^*$ *for some* $\lambda > 0$.
(3) $U = VW$ *for some bounded linear operator* W *on* H.

Theorem 2.2 ([13]). *The set $\mathcal{S}(H)$ of all self-adjoint operators on H is a partially ordered set with respect to the partial order $\leqslant$ which is defined as for $T, S \in \mathcal{S}(H)$*

$$T \leqslant S \Leftrightarrow \langle Tf, f\rangle \leqslant \langle Sf, f\rangle \quad \forall\, f \in H.$$

Theorem 2.3 ([10]). *Let $V \subset H$ be a closed subspace and $T \in \mathcal{B}(H)$. Then $P_V T^* = P_V T^* P_{\overline{TV}}$. If T is a unitary operator (i.e. $T^*T = I_H$), then $P_{\overline{TV}} T = T P_V$.*

Definition 2.4 ([4]). A sequence $\{f_j\}_{j\in J}$ of elements in H is a frame for H if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{j\in J} |\langle f, f_j\rangle|^2 \leqslant B\|f\|^2 \quad \forall\, f \in H.$$

The constants A and B are called frame bounds.

Definition 2.5 ([3]). Let $\{W_j\}_{j\in J}$ be a collection of closed subspaces of H and $\{v_j\}_{j\in J}$ be a collection of positive weights. A family of weighted closed subspaces $\{(W_j, v_j)\colon\ j \in J\}$ is called a *fusion frame* for H if there exist constants $0 < A \leqslant B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{j\in J} v_j^2 \|P_{W_j}(f)\|^2 \leqslant B\|f\|^2 \quad \forall\, f \in H.$$

The constants A, B are called *fusion frame bounds*. If $A = B$, then the fusion frame is called a *tight fusion frame*, if $A = B = 1$, then it is called a *Parseval fusion frame*.

Definition 2.6 ([2]). Let $\{W_j\}_{j\in J}$ be a family of closed subspaces of H and $\{v_j\}_{j\in J}$ be a family of positive weights and $K \in \mathcal{B}(H)$. Then $\{(W_j, v_j)\colon\ j \in J\}$ is said to be an atomic subspace of H with respect to K if the following conditions hold:

(I) $\sum\limits_{j\in J} v_j f_j$ is convergent for all $\{f_j\}_{j\in J} \in \Big(\sum\limits_{j\in J} \oplus W_j\Big)_{l^2}$.

(II) For every $f \in H$ there exists $\{f_j\}_{j\in J} \in \Big(\sum\limits_{j\in J} \oplus W_j\Big)_{l^2}$ such that

$$K(f) = \sum_{j\in J} v_j f_j \quad \text{and} \quad \|\{f_j\}\|_{\left(\sum_{j\in J} \oplus W_j\right)_{l^2}} \leqslant C\|f\|_H$$

for some $C > 0$, where

$$\Big(\sum_{j\in J} \oplus W_j\Big)_{l^2} = \Big\{\{f_j\}_{j\in J}\colon\ f_j \in W_j,\ \sum_{j\in J} \|f_j\|^2 < \infty\Big\}$$

with inner product given by $\langle \{f_j\}_{j\in J}, \{g_j\}_{j\in J}\rangle = \sum\limits_{j\in J} \langle f_j, g_j\rangle_H$.

Definition 2.7 ([15]). A sequence $\{\Lambda_j \in \mathcal{B}(H, H_j)\colon\ j \in J\}$ is called a *generalized frame* or *g-frame* for H with respect to $\{H_j\}_{j\in J}$ if there are two positive constants A and B such that

$$A\|f\|^2 \leqslant \sum_{j\in J} \|\Lambda_j f\|^2 \leqslant B\|f\|^2 \quad \forall\, f \in H.$$

The constants A and B are called the *lower* and *upper frame bounds*, respectively.

Definition 2.8 ([14], [1]). Let $\{W_j\}_{j\in J}$ be a collection of closed subspaces of H and $\{v_j\}_{j\in J}$ be a collection of positive weights and let $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in J$. Then the family $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ is called a *generalized fusion frame* or a *g-fusion frame* for H with respect to $\{H_j\}_{j\in J}$ if there exist constants $0 < A \leqslant B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \leqslant B\|f\|^2 \quad \forall\, f \in H. \tag{2.1}$$

The constants A and B are called the *lower* and *upper bounds* of g-fusion frame, respectively. If $A = B$, then Λ is called *tight g-fusion frame* and if $A = B = 1$, then we say Λ is a *Parseval g-fusion frame.* If Λ satisfies only the condition

$$\sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \leqslant B\|f\|^2 \quad \forall\, f \in H,$$

then it is called a *g-fusion Bessel sequence* with bound B in H.

Definition 2.9 ([1]). Let $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ be a g-fusion Bessel sequence in H with a bound B. The synthesis operator T_Λ of Λ is defined as

$$T_\Lambda\colon\ l^2(\{H_j\}_{j\in J}) \to H, \quad T_\Lambda(\{f_j\}_{j\in J}) = \sum_{j\in J} v_j P_{W_j} \Lambda_j^* f_j \quad \forall\, \{f_j\}_{j\in J} \in l^2(\{H_j\}_{j\in J})$$

and the analysis operator is given by

$$T_\Lambda^*\colon\ H \to l^2(\{H_j\}_{j\in J}), \quad T_\Lambda^*(f) = \{v_j \Lambda_j P_{W_j}(f)\}_{j\in J} \quad \forall\, f \in H.$$

The g-fusion frame operator $S_\Lambda\colon\ H \to H$ is defined as

$$S_\Lambda(f) = T_\Lambda T_\Lambda^*(f) = \sum_{j\in J} v_j^2 P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}(f)$$

and it can be easily verified that

$$\langle S_\Lambda(f), f\rangle = \sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \quad \forall\, f \in H.$$

Furthermore, if Λ is a g-fusion frame with bounds A and B, then from (2.1),

$$\langle Af, f\rangle \leqslant \langle S_\Lambda(f), f\rangle \leqslant \langle Bf, f\rangle \quad \forall\, f \in H.$$

The operator S_Λ is bounded, self-adjoint, positive and invertible. Now, according to Theorem 2.2, we can write $AI_H \leqslant S_\Lambda \leqslant BI_H$ and this gives

$$B^{-1}I_H \leqslant S_\Lambda^{-1} \leqslant A^{-1}I_H.$$

Definition 2.10 ([1])**.** Let $\{W_j\}_{j\in J}$ be a collection of closed subspaces of H and $\{v_j\}_{j\in J}$ be a collection of positive weights and let $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in J$ and $K \in \mathcal{B}(H)$. Then the family $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ is called a *K-g-fusion frame* for H if there exist constants $0 < A \leqslant B < \infty$ such that

$$A\|K^*f\|^2 \leqslant \sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \leqslant B\|f\|^2 \quad \forall\, f \in H. \tag{2.2}$$

Theorem 2.11 ([1])**.** *Let Λ be a g-fusion Bessel sequence in H. Then Λ is a K-g-fusion frame for H if and only if there exists $A > 0$ such that $S_\Lambda \geqslant AKK^*$.*

Definition 2.12 ([3])**.** A family of bounded operators $\{T_j\}_{j\in J}$ on H is called a *resolution of identity operator* on H if for all $f \in H$ we have $f = \sum_{j\in J} T_j(f)$, provided the series converges unconditionally for all $f \in H$.

3. Resolution of the identity operator in g-fusion frame

In this section, we present several useful results of resolution of the identity operator on a Hilbert space using the theory of g-fusion frames.

Theorem 3.1. *Let $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ be a g-fusion frame for H with frame bounds C, D and S_Λ be its associated g-fusion frame operator. Then the family $\{v_j^2 P_{W_j}\Lambda_j^* T_j\}_{j\in J}$ is the resolution of the identity operator on H, where $T_j = \Lambda_j P_{W_j} S_\Lambda^{-1}$, $j \in J$. Furthermore, for all $f \in H$ we have*

$$\frac{C}{D^2}\|f\|^2 \leqslant \sum_{j\in J} v_j^2 \|T_j(f)\|^2 \leqslant \frac{D}{C^2}\|f\|^2.$$

P r o o f. For any $f \in H$ we have the reconstruction formula for g-fusion frame:

$$f = S_\Lambda S_\Lambda^{-1}(f) = \sum_{j\in J} v_j^2 P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} S_\Lambda^{-1}(f) = \sum_{j\in J} v_j^2 P_{W_j} \Lambda_j^* T_j(f).$$

Thus, $\{v_j^2 P_{W_j} \Lambda_j^* T_j\}_{j\in J}$ is a resolution of the identity operator on H. Since Λ is a g-fusion frame with bounds C and D, for each $f \in H$ we have

$$\begin{aligned}\sum_{j\in J} v_j^2 \|T_j(f)\|^2 &= \sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j} S_\Lambda^{-1}(f)\|^2 \leqslant D\|S_\Lambda^{-1}(f)\|^2 \leqslant D\|S_\Lambda^{-1}\|^2 \|f\|^2 \\ &\leqslant \frac{D}{C^2}\|f\|^2 \quad (\text{since } D^{-1} I_H \leqslant S_\Lambda^{-1} \leqslant C^{-1} I_H).\end{aligned}$$

On the other hand,

$$\sum_{j\in J} v_j^2 \|T_j(f)\|^2 = \sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j} S_\Lambda^{-1}(f)\|^2 \geqslant C\|S_\Lambda^{-1}(f)\|^2 \geqslant \frac{C}{D^2}\|f\|^2.$$

Therefore

$$\frac{C}{D^2}\|f\|^2 \leqslant \sum_{j\in J} v_j^2 \|T_j(f)\|^2 \leqslant \frac{D}{C^2}\|f\|^2 \quad \forall\, f \in H.$$

□

Theorem 3.2. *Let $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ be a g-fusion frame for H with frame bounds C, D and let $T_j\colon H \to H_j$ be a bounded operator such that $\{v_j^2 P_{W_j} \Lambda_j^* T_j\}_{j\in J}$ is a resolution of the identity operator on H. Then*

$$\frac{1}{D}\Big\|\sum_{j\in J} v_j^2 P_{W_j} \Lambda_j^* T_j(f)\Big\|^2 \leqslant \sum_{j\in J} v_j^2 \|T_j(f)\|^2 \quad \forall\, f \in H.$$

P r o o f. Assume $I \subset J$ with $|I| < \infty$. If our inequality holds for all finite subsets, then it would hold for all subsets. Let $f \in H$ and set $g = \sum_{j\in I} v_j^2 P_{W_j} \Lambda_j^* T_j(f)$. Then

$$\begin{aligned}\|g\|^4 &= \langle g, g\rangle^2 = \Big\langle g, \sum_{j\in I} v_j^2 P_{W_j} \Lambda_j^* T_j(f)\Big\rangle^2 = \Big(\sum_{j\in I} v_j \langle \Lambda_j P_{W_j}(g), v_j T_j(f)\rangle\Big)^2 \\ &\leqslant \Big(\sum_{j\in I} v_j \|\Lambda_j P_{W_j}(g)\| \|v_j T_j(f)\|\Big)^2 \leqslant \sum_{j\in I} v_j^2 \|\Lambda_j P_{W_j}(g)\|^2 \sum_{j\in I} \|v_j T_j(f)\|^2 \\ &\leqslant D\|g\|^2 \sum_{j\in I} \|v_j T_j(f)\|^2 \quad (\text{since } \Lambda \text{ is a } g\text{-fusion frame}) \\ &\Rightarrow \frac{1}{D}\|g\|^2 \leqslant \sum_{j\in I} \|v_j T_j(f)\|^2 \\ &\Rightarrow \frac{1}{D}\Big\|\sum_{j\in I} v_j^2 P_{W_j} \Lambda_j^* T_j(f)\Big\|^2 \leqslant \sum_{j\in I} v_j^2 \|T_j(f)\|^2 \quad \forall\, f \in H.\end{aligned}$$

Since the inequality holds for any finite subset $I \subset J$, we have

$$\frac{1}{D}\Big\|\sum_{j\in J} v_j^2 P_{W_j}\Lambda_j^* T_j(f)\Big\|^2 \leqslant \sum_{j\in J} v_j^2 \|T_j(f)\|^2 \quad \forall\, f \in H.$$

This completes the proof. □

Theorem 3.3. *Let $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ be a g-fusion frame for H with frame bounds C, D and let $T_j\colon\ H \to H_j$ be a bounded operator such that $\{v_j^2 P_{W_j}\Lambda_j^* T_j\}_{j\in J}$ is a resolution of the identity operator on H. If $T_j^*\Lambda_j P_{W_j} = T_j$, then*

$$\frac{1}{D}\|f\|^2 \leqslant \sum_{j\in J} v_j^2\|T_j(f)\|^2 \leqslant DE\|f\|^2 \quad \forall\, f \in H,$$

where $E = \sup_j \|T_j\|^2 < \infty$.

Proof. Since $\{v_j^2 P_{W_j}\Lambda_j^* T_j\}_{j\in J}$ is a resolution of the identity on H,

$$f = \sum_{j\in J} v_j^2 P_{W_j}\Lambda_j^* T_j(f), \quad f \in H.$$

Now, for each $f \in H$, using Theorem 3.2, we get

$$\begin{aligned}
\frac{1}{D}\|f\|^2 &= \frac{1}{D}\Big\|\sum_{j\in J} v_j^2 P_{W_j}\Lambda_j^* T_j(f)\Big\|^2 \leqslant \sum_{j\in J} v_j^2\|T_j(f)\|^2 \\
&= \sum_{j\in J} v_j^2\|T_j^*\Lambda_j P_{W_j}(f)\|^2 \quad (\text{since } T_j^*\Lambda_j P_{W_j} = T_j) \\
&\leqslant \sum_{j\in J} v_j^2\|T_j\|^2\|\Lambda_j P_{W_j}(f)\|^2 \\
&\leqslant E\sum_{j\in J} v_j^2\|\Lambda_j P_{W_j}(f)\|^2 \quad (\text{using } E = \sup_j \|T_j\|^2) \\
&\leqslant DE\|f\|^2 \quad (\text{since } \Lambda \text{is a } g\text{-fusion frame}).
\end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let $\{W_j\}_{j\in J}$ be a family of closed subspaces of H and $\{v_j\}_{j\in J}$ be a family of bounded weights and let $\Lambda_j \in \mathcal{B}(H, H_j)$, $j \in J$. Then $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ is a g-fusion frame for H if the following conditions hold:*

(I) *For all $f \in H$ there exists $A > 0$ such that*

$$\sum_{j\in J}\|\Lambda_j P_{W_j}(f)\|^2 \leqslant \frac{1}{A}\|f\|^2.$$

(II) *$\{v_j P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}\}_{j\in J}$ is a resolution of the identity operator on H.*

P r o o f. Since $\{v_j P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}\}_{j\in J}$ is a resolution of the identity operator on H, for $f \in H$ we have

$$f = \sum_{j\in J} v_j P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}(f).$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|f\|^4 = \langle f, f\rangle^2 &= \Big\langle \sum_{j\in J} v_j P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}(f), f \Big\rangle^2 \\
&= \Big(\sum_{j\in J} v_j \langle \Lambda_j P_{W_j}(f), \Lambda_j P_{W_j}(f)\rangle\Big)^2 = \Big(\sum_{j\in J} v_j \|\Lambda_j P_{W_j}(f)\|^2\Big)^2 \\
&\leqslant \sum_{j\in J} \|\Lambda_j P_{W_j}(f)\|^2 \sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \\
&\leqslant \frac{1}{A}\|f\|^2 \sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \quad \text{(using given condition (I))} \\
&\Rightarrow A\|f\|^2 \leqslant \sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 &\leqslant B \sum_{j\in J} \|\Lambda_j P_{W_j}(f)\|^2 \quad \Big(\text{where } B = \sup_{j\in J}\{v_j^2\}\Big) \\
&\leqslant \frac{B}{A}\|f\|^2 \quad \text{(using given condition (I))}
\end{aligned}$$

and hence, Λ is a g-fusion frame. □

4. g-ATOMIC SUBSPACE

In this section, we define a generalized atomic subspace or a g-atomic subspace of a Hilbert space with respect to a bounded linear operator.

Definition 4.1. Let $K \in \mathcal{B}(H)$ and $\{W_j\}_{j\in J}$ be a collection of closed subspaces of H, let $\{v_j\}_{j\in J}$ be a collection of positive weights and $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in J$. Then the family $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ is said to be a generalized atomic subspace or g-atomic subspace of H with respect to K if the following statements hold:

(I) Λ is a g-fusion Bessel sequence in H.

(II) For every $f \in H$ there exists $\{f_j\}_{j\in J} \in l^2(\{H_j\}_{j\in J})$ such that

$$K(f) = \sum_{j\in J} v_j P_{W_j} \Lambda_j^* f_j \quad \text{and} \quad \|\{f_j\}_{j\in J}\|_{l^2(\{H_j\}_{j\in J})} \leqslant C\|f\|_H$$

for some $C > 0$.

Theorem 4.2. *Let $K \in \mathcal{B}(H)$ and $\{W_j\}_{j\in J}$ be a collection of closed subspaces of H, let $\{v_j\}_{j\in J}$ be a collection of positive weights and $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in J$. Then the following statements are equivalent:*

(I) $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ *is a g-atomic subspace of H with respect to K.*

(II) Λ *is a K-g-fusion frame for H.*

P r o o f. (I) $\Rightarrow$ (II): Suppose Λ is a g-atomic subspace of H with respect to K. Then Λ is a g-fusion Bessel sequence, so there exists $B > 0$ such that

$$\sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \leqslant B\|f\|^2 \quad \forall\, f \in H.$$

Now, for any $f \in H$ we have

$$\|K^* f\| = \sup_{\|g\|=1} |\langle K^* f, g\rangle| = \sup_{\|g\|=1} |\langle f, Kg\rangle|,$$

by Definition 4.1, for $g \in H$ there exists $\{f_j\}_{j\in J} \in l^2(\{H_j\}_{j\in J})$ such that

$$K(g) = \sum_{j\in J} v_j P_{W_j} \Lambda_j^* f_j \quad \text{and} \quad \|\{f_j\}_{j\in J}\|_{l^2(\{H_j\}_{j\in J})} \leqslant C\|g\|_H$$

for some $C > 0$. Thus

$$\begin{aligned}
\|K^* f\| &= \sup_{\|g\|=1} \left|\left\langle f, \sum_{j\in J} v_j P_{W_j} \Lambda_j^* f_j \right\rangle\right| = \sup_{\|g\|=1} \left|\sum_{j\in J} v_j \langle \Lambda_j P_{W_j}(f), f_j\rangle\right| \\
&\leqslant \sup_{\|g\|=1} \Big(\sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2\Big)^{1/2} \Big(\sum_{j\in J} \|f_j\|^2\Big)^{1/2} \\
&\leqslant C \sup_{\|g\|=1} \Big(\sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2\Big)^{1/2} \|g\| \\
&\Rightarrow \frac{1}{C^2} \|K^* f\|^2 \leqslant \sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2.
\end{aligned}$$

Therefore Λ is a K-g-fusion frame for H with bounds $1/C^2$ and B.

(II) $\Rightarrow$ (I): Suppose that Λ is a K-g-fusion frame with the corresponding synthesis operator T_Λ. Then obviously Λ is a g-fusion Bessel sequence in H. Now, for each $f \in H$,

$$A\|K^* f\|^2 \leqslant \sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 = \|T_\Lambda^* f\|^2$$

gives $AKK^* \leqslant T_\Lambda T_\Lambda^*$ and by Theorem 2.1, exists $L \in \mathcal{B}(H, l^2(\{H_j\}_{j\in J}))$ such that $K = T_\Lambda L$. Define $L(f) = \{f_j\}_{j\in J}$ for every $f \in H$. Then for each $f \in H$ we have

$$K(f) = T_\Lambda L(f) = T_\Lambda(\{f_j\}_{j\in J}) = \sum_{j\in J} v_j P_{W_j} \Lambda_j^* f_j$$

and

$$\|\{f_j\}_{j\in J}\|_{l^2(\{H_j\}_{j\in J})} = \|L(f)\|_{l^2(\{H_j\}_{j\in J})} \leqslant C\|f\|,$$

where $C = \|L\|$. Hence, Λ is a g-atomic subspace of H with respect to K. □

Theorem 4.3. *Let $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ be a g-fusion frame for H. Then Λ is a g-atomic subspace of H with respect to its g-fusion frame operator S_Λ.*

Proof. Since Λ is a g-fusion frame in H, there exist $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \leqslant B\|f\|^2 \quad \forall\, f \in H.$$

Since $\mathcal{R}(T_\Lambda) = H = \mathcal{R}(S_\Lambda)$, by Theorem 2.1, there exists $\alpha > 0$ such that $\alpha S_\Lambda S_\Lambda^* \leqslant T_\Lambda T_\Lambda^*$ and therefore for each $f \in H$ we have

$$\alpha\|S_\Lambda^* f\|^2 \leqslant \|T_\Lambda^* f\|^2 = \sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \leqslant B\|f\|^2.$$

Thus, Λ is a S_Λ-g-fusion frame and hence by Theorem 4.2, Λ is a g-atomic subspace of H with respect to S_Λ. □

Theorem 4.4. *Let $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ and $\Gamma = \{(W_j, \Gamma_j, v_j)\}_{j\in J}$ be two g-atomic subspaces of H with respect to $K \in \mathcal{B}(H)$ with the corresponding synthesis operators T_Λ and T_Γ, respectively. If $T_\Lambda T_\Gamma^* = \theta_H$ (θ_H is a null operator on H) and $U, V \in \mathcal{B}(H)$ such that $U+V$ is invertible operator on H with $K(U+V) = (U+V)K$, then*

$$\{((U+V)W_j, (\Lambda_j + \Gamma_j)P_{W_j}(U+V)^*, v_j)\}_{j\in J}$$

is a g-atomic subspace of H with respect to K.

Proof. Since Λ and Γ are g-atomic subspaces with respect to K, by Theorem 4.2, they are K-g-fusion frames for H. So, for each $f \in H$ there exist positive constants (A_1, B_1) and (A_2, B_2) such that

$$A_1\|K^* f\|^2 \leqslant \sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \leqslant B_1\|f\|^2$$

and

$$A_2\|K^* f\|^2 \leqslant \sum_{j\in J} v_j^2 \|\Gamma_j P_{W_j}(f)\|^2 \leqslant B_2\|f\|^2.$$

Since $T_\Lambda T_\Gamma^* = \theta_H$, for any $f \in H$ we have

$$T_\Lambda\{v_j\Gamma_j P_{W_j}(f)\}_{j\in J} = \sum_{j\in J} v_j^2 P_{W_j}\Lambda_j^*\Gamma_j P_{W_j}(f) = 0. \tag{4.1}$$

Also, $U+V$ is invertible, so

$$\|K^*f\|^2 = \|((U+V)^{-1})^*(U+V)^*K^*f\|^2 \leqslant \|(U+V)^{-1}\|^2\|(U+V)^*K^*f\|^2. \tag{4.2}$$

Now, for any $f \in H$ we have

$$\begin{aligned}
&\sum_{j\in J} v_j^2\|(\Lambda_j+\Gamma_j)P_{W_j}(U+V)^*P_{(U+V)W_j}(f)\|^2\\
&\qquad= \sum_{j\in J} v_j^2\|(\Lambda_j+\Gamma_j)P_{W_j}(U+V)^*(f)\|^2 \quad \text{(using Theorem 2.3)}\\
&\qquad= \sum_{j\in J} v_j^2\langle(\Lambda_j+\Gamma_j)P_{W_j}(T^*f),(\Lambda_j+\Gamma_j)P_{W_j}(T^*f)\rangle \quad \text{(taking } T=U+V)\\
&\qquad= \sum_{j\in J} v_j^2(\|\Lambda_jP_{W_j}(T^*f)\|^2+\|\Gamma_jP_{W_j}(T^*f)\|^2+2\operatorname{Re}\langle TP_{W_j}\Lambda_j^*\Gamma_jP_{W_j}(T^*f),f\rangle)\\
&\qquad= \sum_{j\in J} v_j^2\|\Lambda_jP_{W_j}(T^*f)\|^2+\sum_{j\in J} v_j^2\|\Gamma_jP_{W_j}(T^*f)\|^2 \quad \text{(using (4.1))}\\
&\qquad\leqslant B_1\|T^*f\|^2+B_2\|T^*f\|^2 \quad \text{(since } \Lambda,\Gamma \text{ are } K\text{-}g\text{-fusion frames)}\\
&\qquad= (B_1+B_2)\|(U+V)^*f\|^2 \quad \text{(since} T=U+V)\\
&\qquad\leqslant (B_1+B_2)\|U+V\|^2\|f\|^2 \quad \text{(as } U+V \text{ is bounded)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{j\in J} v_j^2\|(\Lambda_j+\Gamma_j)P_{W_j}(U+V)^*P_{(U+V)W_j}(f)\|^2\\
&\qquad= \sum_{j\in J} v_j^2\|\Lambda_jP_{W_j}(U+V)^*f\|^2+\sum_{j\in J} v_j^2\|\Gamma_jP_{W_j}(U+V)^*f\|^2\\
&\qquad\geqslant \sum_{j\in J} v_j^2\|\Lambda_jP_{W_j}(U+V)^*f\|^2\\
&\qquad\geqslant A_1\|K^*(U+V)^*f\|^2 \quad \text{(since } \Lambda \text{ is } K\text{-}g\text{-fusion frame)}\\
&\qquad= A_1\|(U+V)^*K^*f\|^2 \quad \text{(using } K(U+V)=(U+V)K)\\
&\qquad\geqslant A_1\|(U+V)^{-1}\|^{-2}\|K^*f\|^2 \quad \text{(using (4.2))}.
\end{aligned}$$

Therefore $\{((U+V)W_j,(\Lambda_j+\Gamma_j)P_{W_j}(U+V)^*,v_j)\}_{j\in J}$ is a K-g-fusion frame and by Theorem 4.2, it is a g-atomic subspace of H with respect to K. □

Corollary 4.5. *Let* $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ *and* $\Gamma = \{(W_j, \Gamma_j, v_j)\}_{j\in J}$ *be two g-atomic subspaces of* H *with respect to* $K \in \mathcal{B}(H)$ *with the corresponding synthesis operators* T_Λ *and* T_Γ. *If* $T_\Lambda T_\Gamma^* = \theta_H$ *and* $U \in \mathcal{B}(H)$ *is an invertible operator with* $KU = UK$, *then* $\{(UW_j, (\Lambda_j + \Gamma_j)P_{W_j}U^*, v_j)\}_{j\in J}$ *is a g-atomic subspace of* H *with respect to* K.

Proof. The proof of this Corollary directly follows from Theorem 4.4 by putting $V = \theta_H$. □

Theorem 4.6. *Let* $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ *is a g-atomic subspace for* $K \in \mathcal{B}(H)$ *and* S_Λ *be the frame operator of* Λ. *If* $U \in \mathcal{B}(H)$ *is a positive and invertible operator on* H, *then* $\Lambda' = \{((I_H + U)W_j, \Lambda_j P_{W_j}(I_H + U)^*, v_j)\}_{j\in J}$ *is a g-atomic subspace of* H *with respect to* K. *Moreover, for any natural number* n, $\Lambda'' = \{((I_H + U^n)W_j, \Lambda_j P_{W_j}(I_H + U^n)^*, v_j)\}_{j\in J}$ *is a g-atomic subspace of* H *with respect to* K.

Proof. Since Λ is a g-atomic subspace with respect to K, by Theorem 4.2, it is a K-g-fusion frame for H. Then according to Theorem 2.11, there exists $A > 0$ such that $S_\Lambda \geqslant AKK^*$. Now, for each $f \in H$ we have

$$\begin{aligned}
&\sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(I_H + U)^* P_{(I_H+U)W_j}(f)\|^2 \\
&\qquad = \sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(I_H + U)^*(f)\|^2 \quad \text{(using Theorem 2.3)} \\
&\qquad \leqslant B\|(I_H + U)^*(f)\|^2 \quad \text{(since Λ is a K-g-fusion frame)} \\
&\qquad \leqslant B\|I_H + U\|^2\|f\|^2 \quad \text{(since $(I_H + U) \in \mathcal{B}(H)$)}.
\end{aligned}$$

Thus, Λ' is a g-fusion Bessel sequence in H. Also, for each $f \in H$ we have

$$\begin{aligned}
&\sum_{j\in J} v_j^2 P_{(I_H+U)W_j}(\Lambda_j P_{W_j}(I_H + U)^*)^* \Lambda_j P_{W_j}(I_H + U)^* P_{(I_H+U)W_j}(f) \\
&\quad = \sum_{j\in J} v_j^2 P_{(I_H+U)W_j}(I_H + U)P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}(I_H + U)^* P_{(I_H+U)W_j}(f) \\
&\quad = \sum_{j\in J} v_j^2 (P_{W_j}(I_H + U)^* P_{(I_H+U)W_j})^* \Lambda_j^*\Lambda_j (P_{W_j}(I_H + U)^* P_{(I_H+U)W_j}(f)) \\
&\quad = \sum_{j\in J} v_j^2 (P_{W_j}(I_H + U)^*)^* \Lambda_j^*\Lambda_j P_{W_j}(I_H + U)^*(f) \quad \text{(using Theorem 2.3)} \\
&\quad = \sum_{j\in J} v_j^2 (I_H + U)P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}(I_H + U)^*(f) \\
&\quad = (I_H + U)\sum_{j\in J} v_j^2 P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}(I_H + U)^*(f) = (I_H + U)S_\Lambda(I_H + U)^*(f).
\end{aligned}$$

This shows that the frame operator of Λ' is $(I_H+U)S_\Lambda(I_H+U)^*$. Now,

$$(I_H+U)S_\Lambda(I_H+U)^* \geqslant S_\Lambda \geqslant AKK^* \quad \text{(since } U,\ S_\Lambda \text{ are positive).}$$

Then by Theorem 2.11, we can conclude that Λ' is a K-g-fusion frame and therefore by Theorem 4.2, Λ' is a g-atomic subspace of H with respect to K. According to the preceding procedure, for any natural number n, the frame operator of Λ'' is $(I_H+U^n)S_\Lambda(I_H+U^n)^*$ and similarly, it can be shown that Λ'' is a g-atomic subspace of H with respect to K. □

5. Frame operator for a pair of g-fusion Bessel sequences

In this section, we shall discuss the frame operator for a pair of g-fusion Bessel sequences and establish some properties relative to frame operator. At the end of this section, we shall construct a new g-fusion frame for the Hilbert space $H \oplus X$, using the g-fusion frames of the Hilbert spaces H and X.

Definition 5.1. Let $\Lambda = \{(W_j, \Lambda_j, w_j)\}_{j\in J}$ and $\Gamma = \{(V_j, \Gamma_j, v_j)\}_{j\in J}$ be two g-fusion Bessel sequences in H with bounds D_1 and D_2. Then the operator $S_{\Gamma\Lambda}\colon H \to H$, defined by

$$S_{\Gamma\Lambda}(f) = \sum_{j\in J} v_j w_j P_{V_j} \Gamma_j^* \Lambda_j P_{W_j}(f) \quad \forall\, f \in H,$$

is called the frame operator for the pair of g-fusion Bessel sequences Λ and Γ.

Theorem 5.2. *The frame operator $S_{\Gamma\Lambda}$ for the pair of g-fusion Bessel sequences Λ and Γ is bounded and $S_{\Gamma\Lambda}^* = S_{\Lambda\Gamma}$.*

Proof. For each $f, g \in H$ we have

$$(5.1)\quad \langle S_{\Gamma\Lambda}(f), g\rangle = \Big\langle \sum_{j\in J} v_j w_j P_{V_j}\Gamma_j^*\Lambda_j P_{W_j}(f), g \Big\rangle = \sum_{j\in J} v_j w_j \langle \Lambda_j P_{W_j}(f), \Gamma_j P_{V_j}(g)\rangle.$$

By the Cauchy-Schwarz inequality, we obtain

$$(5.2)\qquad \begin{aligned} |\langle S_{\Gamma\Lambda}(f), g\rangle| &\leqslant \Big(\sum_{j\in J} v_j^2 \|\Gamma_j P_{V_j}(g)\|^2\Big)^{1/2} \Big(\sum_{j\in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2\Big)^{1/2} \\ &\leqslant \sqrt{D_2}\|g\|\sqrt{D_1}\|f\|. \end{aligned}$$

This shows that $S_{\Gamma\Lambda}$ is a bounded operator with $\|S_{\Gamma\Lambda}\| \leqslant \sqrt{D_1 D_2}$. Now,

$$
\begin{aligned}
(5.3)\qquad \|S_{\Gamma\Lambda} f\| &= \sup_{\|g\|=1} |\langle S_{\Gamma\Lambda}(f), g\rangle| \\
&\leqslant \sup_{\|g\|=1} \sqrt{D_2}\|g\| \Big(\sum_{j\in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2\Big)^{1/2} \quad (\text{using } (5.2)) \\
&\leqslant \sqrt{D_2} \Big(\sum_{j\in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2\Big)^{1/2}
\end{aligned}
$$

and similarly, it can be shown that

$$
(5.4)\qquad \|S^*_{\Gamma\Lambda} g\| \leqslant \sqrt{D_1} \Big(\sum_{j\in J} v_j^2 \|\Gamma_j P_{V_j}(g)\|^2\Big)^{1/2}.
$$

Also, for each $f, g \in H$ we have

$$
\begin{aligned}
\langle S_{\Gamma\Lambda}(f), g\rangle &= \Big\langle \sum_{j\in J} v_j w_j P_{V_j} \Gamma_j^* \Lambda_j P_{W_j}(f), g \Big\rangle = \sum_{j\in J} v_j w_j \langle f, P_{W_j} \Lambda_j^* \Gamma_j P_{V_j}(g)\rangle \\
&= \Big\langle f, \sum_{j\in J} w_j v_j P_{W_j} \Lambda_j^* \Gamma_j P_{V_j}(g) \Big\rangle = \langle f, S_{\Lambda\Gamma}(g)\rangle
\end{aligned}
$$

and hence $S^*_{\Gamma\Lambda} = S_{\Lambda\Gamma}$. □

Theorem 5.3. *Let $S_{\Gamma\Lambda}$ be the frame operator for a pair of g-fusion Bessel sequences Λ and Γ with bounds D_1 and D_2, respectively. Then the following statements are equivalent:*

(I) *$S_{\Gamma\Lambda}$ is bounded below.*

(II) *There exists $K \in \mathcal{B}(H)$ such that $\{T_j\}_{j\in J}$ is a resolution of the identity operator on H, where $T_j = v_j w_j K P_{V_j} \Gamma_j^* \Lambda_j P_{W_j}$, $j \in J$.*

If one of the given conditions holds, then Λ is a g-fusion frame.

Proof. (I) ⇒ (II): Suppose that $S_{\Gamma\Lambda}$ is bounded below. Then for each $f \in H$ there exists $A > 0$ such that

$$
\|f\|^2 \leqslant A\|S_{\Gamma\Lambda} f\|^2 \Rightarrow \langle I_H f, f\rangle \leqslant A\langle S^*_{\Gamma\Lambda} S_{\Gamma\Lambda} f, f\rangle \Rightarrow I_H^* I_H \leqslant A S^*_{\Gamma\Lambda} S_{\Gamma\Lambda}.
$$

So, by Theorem 2.1, there exists $K \in \mathcal{B}(H)$ such that $K S_{\Gamma\Lambda} = I_H$. Therefore for each $f \in H$ we have

$$
f = K S_{\Gamma\Lambda}(f) = K \sum_{j\in J} v_j w_j P_{V_j} \Gamma_j^* \Lambda_j P_{W_j}(f) = \sum_{j\in J} v_j w_j K P_{V_j} \Gamma_j^* \Lambda_j P_{W_j}(f) = \sum_{j\in J} T_j(f)
$$

and hence $\{T_j\}_{j\in J}$ is a resolution of the identity operator on H, where $T_j = v_j w_j K P_{V_j} \Gamma_j^* \Lambda_j P_{W_j}$.

(II) $\Rightarrow$ (I): Since $\{T_j\}_{j\in J}$ is a resolution of the identity operator on H, for any $f \in H$ we have

$$f = \sum_{j\in J} T_j(f) = \sum_{j\in J} v_j w_j K P_{V_j} \Gamma_j^* \Lambda_j P_{W_j}(f) = K \sum_{j\in J} v_j w_j P_{V_j} \Gamma_j^* \Lambda_j P_{W_j}(f) = K S_{\Gamma\Lambda}(f).$$

Thus, $I_H = K S_{\Gamma\Lambda}$. So, by Theorem 2.1, there exists $\alpha > 0$ such that $I_H I_H^* \leqslant \alpha S_{\Gamma\Lambda} S_{\Gamma\Lambda}^*$ and hence $S_{\Gamma\Lambda}$ is bounded below.

Last part: First we suppose that $S_{\Gamma\Lambda}$ is bounded below. Then for all $f \in H$ there exists $M > 0$ such that $\|S_{\Gamma\Lambda} f\| \geqslant M\|f\|$ and this implies that

$$M^2\|f\|^2 \leqslant \|S_{\Gamma\Lambda} f\|^2 \leqslant D_2 \sum_{j\in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \quad \text{(using (5.3))}$$
$$\Rightarrow \frac{M^2}{D_2}\|f\|^2 \leqslant \sum_{j\in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2.$$

Hence, Λ is a g-fusion frame for H with bounds M^2/D_2 and D_1.

Next, we suppose that the given condition (II) holds. Then for any $f \in H$ we have

$$f = \sum_{j\in J} v_j w_j K P_{V_j} \Gamma_j^* \Lambda_j P_{W_j}(f), \quad K \in \mathcal{B}(H).$$

By Cauchy-Schwarz inequality, for each $f \in H$ we have

$$\begin{aligned}
\|f\|^2 = \langle f, f\rangle &= \left\langle \sum_{j\in J} v_j w_j K P_{V_j} \Gamma_j^* \Lambda_j P_{W_j}(f), f \right\rangle = \sum_{j\in J} v_j w_j \langle \Lambda_j P_{W_j}(f), \Gamma_j P_{V_j}(K^* f)\rangle \\
&\leqslant \left(\sum_{j\in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2\right)^{1/2} \left(\sum_{j\in J} v_j^2 \|\Gamma_j P_{V_j}(K^* f)\|^2\right)^{1/2} \\
&\leqslant \sqrt{D_2}\|K^* f\| \left(\sum_{j\in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2\right)^{1/2} \\
&\leqslant \sqrt{D_2}\|K\|\|f\| \left(\sum_{j\in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2\right)^{1/2} \\
&\Rightarrow \frac{1}{D_2\|K\|^2}\|f\|^2 \leqslant \sum_{j\in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2.
\end{aligned}$$

Therefore, in this case Λ is also a g-fusion frame for H. □

Theorem 5.4. *Let $S_{\Gamma\Lambda}$ be the frame operator for a pair of g-fusion Bessel sequences Λ and Γ with bounds D_1 and D_2, respectively. Suppose $\lambda_1 < 1$, $\lambda_2 > -1$ such that for each $f \in H$, $\|f - S_{\Gamma\Lambda} f\| \leqslant \lambda_1\|f\| + \lambda_2\|S_{\Gamma\Lambda} f\|$. Then Λ is a g-fusion frame for H.*

P r o o f. For each $f \in H$ we have

$$\begin{aligned}
\|f\| - \|S_{\Gamma\Lambda} f\| &\leqslant \|f - S_{\Gamma\Lambda} f\| \leqslant \lambda_1 \|f\| + \lambda_2 \|S_{\Gamma\Lambda} f\| \\
&\Rightarrow (1-\lambda_1)\|f\| \leqslant (1+\lambda_2)\|S_{\Gamma\Lambda} f \\
&\Rightarrow \Big(\frac{1-\lambda_1}{1+\lambda_2}\Big)\|f\| \leqslant \sqrt{D_2}\Big(\sum_{j\in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2\Big)^{1/2} \quad \text{(using (5.3))}
\end{aligned}$$

$$\Rightarrow \frac{1}{D_2}\Big(\frac{1-\lambda_1}{1+\lambda_2}\Big)^2 \|f\|^2 \leqslant \sum_{j\in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2. \tag{5.5}$$

Thus, Λ is a g-fusion frame for H with bounds $(1-\lambda_1)^2(1+\lambda_2)^{-2}D_2^{-1}$ and D_1. □

Theorem 5.5. *Let $S_{\Gamma\Lambda}$ be the frame operator for a pair of g-fusion Bessel sequences Λ and Γ of bounds D_1 and D_2, respectively. Assume $\lambda \in [0,1)$ such that*

$$\|f - S_{\Gamma\Lambda} f\| \leqslant \lambda \|f\| \quad \forall\, f \in H.$$

Then Λ and Γ are g-fusion frames for H.

P r o o f. By putting $\lambda_1 = \lambda$ and $\lambda_2 = 0$ in (5.5), we get

$$\frac{(1-\lambda)^2}{D_2}\|f\|^2 \leqslant \sum_{j\in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2$$

and therefore Λ is a g-fusion frame. Now, for each $f \in H$ we have

$$\begin{aligned}
\|f - S_{\Gamma\Lambda}^* f\| &= \|(I_H - S_{\Gamma\Lambda})^* f\| \leqslant \|(I_H - S_{\Gamma\Lambda})\|\|f\| \leqslant \lambda\|f\| \\
&\Rightarrow (1-\lambda)\|f\| \leqslant \|S_{\Gamma\Lambda}^* f\| \leqslant \sqrt{D_1}\Big(\sum_{j\in J} v_j^2 \|\Gamma_j P_{V_j}(f)\|^2\Big)^{1/2} \quad \text{(using (5.4))} \\
&\Rightarrow \sum_{j\in J} v_j^2 \|\Gamma_j P_{V_j}(f)\|^2 \geqslant \frac{(1-\lambda)^2}{D_1}\|f\|^2 \quad \forall\, f \in H.
\end{aligned}$$

Hence, Γ is a g-fusion frame with bounds $(1-\lambda)^2/D_1$ and D_2. □

Definition 5.6. Let H and X be two Hilbert spaces. Define

$$H \oplus X = \{(f,g)\colon\ f \in H, g \in X\}.$$

Then $H \oplus X$ forms a Hilbert space with respect to point-wise operations and inner product defined by

$$\langle (f,g),(f',g')\rangle = \langle f,f'\rangle_H + \langle g,g'\rangle_X \quad \forall\, f,f' \in H \text{ and } \forall\, g,g' \in X.$$

Now, if $U \in \mathcal{B}(H,Z), V \in \mathcal{B}(X,Y)$, then for all $f \in H$, $g \in X$ we define

$$U \oplus V \in \mathcal{B}(H \oplus X, Z \oplus Y) \quad \text{by } (U\oplus V)(f,g) = (Uf, Vg),$$

and $(U \oplus V)^* = U^* \oplus V^*$, where Z, Y are Hilbert spaces and also we define $P_{M\oplus N}(f,g) = (P_M f, P_N g)$, where P_M, P_N and $P_{M\oplus N}$ are orthonormal projections onto the closed subspaces $M \subset H$, $N \subset X$ and $M \oplus N \subset H \oplus X$, respectively.

From here we assume that for each $j \in J$, $W_j \oplus V_j$ are the closed subspaces of $H \oplus X$ and $\Gamma_j \in \mathcal{B}(X, X_j)$, where $\{X_j\}_{j\in J}$ is the collection of Hilbert spaces and $\Lambda_j \oplus \Gamma_j \in \mathcal{B}(H \oplus X, H_j \oplus X_j)$.

Theorem 5.7. *Let $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ be a g-fusion frame for H with bounds A, B and $\Gamma = \{(V_j, \Gamma_j, v_j)\}_{j\in J}$ be a g-fusion frame for X with bounds C, D. Then $\Lambda \oplus \Gamma = \{(W_j \oplus V_j, \Lambda_j \oplus \Gamma_j, v_j)\}_{j\in J}$ is a g-fusion frame for $H \oplus X$ with bounds $\min\{A, C\}$, $\max\{B, D\}$. Furthermore, if S_Λ, S_Γ and $S_{\Lambda\oplus\Gamma}$ are g-fusion frame operators for Λ, Γ and $\Lambda \oplus \Gamma$, respectively, then we have $S_{\Lambda\oplus\Gamma} = S_\Lambda \oplus S_\Gamma$.*

Proof. Let $(f,g) \in H \oplus X$ be an arbitrary element. Then

$$\begin{aligned}
&\sum_{j\in J} v_j^2 \|(\Lambda_j \oplus \Gamma_j) P_{W_j\oplus V_j}(f,g)\|^2 \\
&\quad = \sum_{j\in J} v_j^2 \langle (\Lambda_j \oplus \Gamma_j) P_{W_j\oplus V_j}(f,g), (\Lambda_j \oplus \Gamma_j) P_{W_j\oplus V_j}(f,g)\rangle \\
&\quad = \sum_{j\in J} v_j^2 \langle \Lambda_j \oplus \Gamma_j (P_{W_j}(f), P_{V_j}(g)), \Lambda_j \oplus \Gamma_j (P_{W_j}(f), P_{V_j}(g))\rangle \\
&\quad = \sum_{j\in J} v_j^2 \langle (\Lambda_j P_{W_j}(f), \Gamma_j P_{V_j}(g)), (\Lambda_j P_{W_j}(f), \Gamma_j P_{V_j}(g))\rangle \\
&\quad = \sum_{j\in J} v_j^2 (\langle \Lambda_j P_{W_j}(f), \Lambda_j P_{W_j}(f)\rangle_H + \langle \Gamma_j P_{V_j}(g), \Gamma_j P_{V_j}(g)\rangle_X) \\
&\quad = \sum_{j\in J} v_j^2 (\|\Lambda_j P_{W_j}(f)\|_H^2 + \|\Gamma_j P_{V_j}(g)\|_X^2) \\
&\quad = \sum_{j\in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|_H^2 + \sum_{j\in J} v_j^2 \|\Gamma_j P_{V_j}(g)\|_X^2 \\
&\quad \leqslant B\|f\|_H^2 + D\|g\|_X^2 \quad \text{(since } \Lambda, \Gamma \text{ are } g\text{-fusion frames)} \\
&\quad \leqslant \max\{B, D\}(\|f\|_H^2 + \|g\|_X^2) = \max\{B, D\}\|(f,g)\|^2.
\end{aligned}$$

Similarly, it can be shown that

$$\min\{A, C\}\|(f,g)\|^2 \leqslant \sum_{j\in J} v_j^2 \|(\Lambda_j \oplus \Gamma_j) P_{W_j\oplus V_j}(f,g)\|^2.$$

Therefore, for all $(f,g) \in H \oplus X$ we have

$$A_1\|(f,g)\|^2 \leqslant \sum_{j\in J} v_j^2 \|(\Lambda_j \oplus \Gamma_j) P_{W_j\oplus V_j}(f,g)\|^2 \leqslant B_1\|(f,g)\|^2$$

and hence $\Lambda \oplus \Gamma$ is a g-fusion frame for $H \oplus X$ with bounds $A_1 = \min\{A, C\}$ and $B_1 = \max\{B, D\}$. Furthermore, for $(f, g) \in H \oplus X$ we have

$$\begin{aligned}
S_{\Lambda\oplus\Gamma}(f,g) &= \sum_{j\in J} v_j^2 P_{W_j\oplus V_j}(\Lambda_j \oplus \Gamma_j)^*(\Lambda_j \oplus \Gamma_j) P_{W_j\oplus V_j}(f,g) \\
&= \sum_{j\in J} v_j^2 P_{W_j\oplus V_j}(\Lambda_j \oplus \Gamma_j)^*(\Lambda_j \oplus \Gamma_j)(P_{W_j}(f), P_{V_j}(g)) \\
&= \sum_{j\in J} v_j^2 P_{W_j\oplus V_j}(\Lambda_j \oplus \Gamma_j)^*(\Lambda_j P_{W_j}(f), \Gamma_j P_{V_j}(g)) \\
&= \sum_{j\in J} v_j^2 P_{W_j\oplus V_j}(\Lambda_j^* \oplus \Gamma_j^*)(\Lambda_j P_{W_j}(f), \Gamma_j P_{V_j}(g)) \\
&= \sum_{j\in J} v_j^2 P_{W_j\oplus V_j}(\Lambda_j^*\Lambda_j P_{W_j}(f), \Gamma_j^*\Gamma_j P_{V_j}(g)) \\
&= \sum_{j\in J} v_j^2 (P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}(f), P_{V_j}\Gamma_j^*\Gamma_j P_{V_j}(g)) \\
&= \Bigl(\sum_{j\in J} v_j^2 P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}(f), \sum_{j\in J} v_j^2 P_{V_j}\Gamma_j^*\Gamma_j P_{V_j}(g)\Bigr) \\
&= (S_\Lambda(f), S_\Gamma(g)) \\
&= (S_\Lambda \oplus S_\Gamma)(f,g) \quad \forall\, (f,g) \in H \oplus X.
\end{aligned}$$

Hence, $S_{\Lambda\oplus\Gamma} = S_\Lambda \oplus S_\Gamma$. This completes the proof. □

Theorem 5.8. *Let $\Lambda \oplus \Gamma = \{(W_j \oplus V_j, \Lambda_j \oplus \Gamma_j, v_j)\}_{j\in J}$ be a g-fusion frame for $H \oplus X$ with frame operator $S_{\Lambda\oplus\Gamma}$. Then*

$$\Delta' = \{(S_{\Lambda\oplus\Gamma}^{-1/2}(W_j \oplus V_j), (\Lambda_j \oplus \Gamma_j) P_{W_j\oplus V_j} S_{\Lambda\oplus\Gamma}^{-1/2}, v_j)\}_{j\in J}$$

is a Parseval g-fusion frame for $H \oplus X$.

Proof. Since $S_{\Lambda\oplus\Gamma}$ is a positive operator, there exists a unique positive square root $S_{\Lambda\oplus\Gamma}^{1/2}$ (or $S_{\Lambda\oplus\Gamma}^{-1/2}$) and they commute with $S_{\Lambda\oplus\Gamma}$ and $S_{\Lambda\oplus\Gamma}^{-1}$. Therefore, each $(f, g) \in H \oplus X$ can be written as

$$\begin{aligned}
(f,g) &= S_{\Lambda\oplus\Gamma}^{-1/2} S_{\Lambda\oplus\Gamma} S_{\Lambda\oplus\Gamma}^{-1/2}(f,g) \\
&= \sum_{j\in J} v_j^2 S_{\Lambda\oplus\Gamma}^{-1/2} P_{W_j\oplus V_j}(\Lambda_j \oplus \Gamma_j)^*(\Lambda_j \oplus \Gamma_j) P_{W_j\oplus V_j} S_{\Lambda\oplus\Gamma}^{-1/2}(f,g).
\end{aligned}$$

Now, for each $(f,g)\in H\oplus X$ we have

$$\begin{aligned}\|(f,g)\|^2&=\langle (f,g),(f,g)\rangle\\&=\Big\langle\sum_{j\in J}v_j^2S_{\Lambda\oplus\Gamma}^{-1/2}P_{W_j\oplus V_j}(\Lambda_j\oplus\Gamma_j)^*(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1/2}(f,g),(f,g)\Big\rangle\\&=\sum_{j\in J}v_j^2\langle(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1/2}(f,g),(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1/2}(f,g)\rangle\\&=\sum_{j\in J}v_j^2\|(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1/2}(f,g)\|^2\\&=\sum_{j\in J}v_j^2\|(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1/2}P_{(S_{\Lambda\oplus\Gamma}^{-1/2}(W_j\oplus V_j))}(f,g)\|^2\\&\qquad\qquad\text{(by Theorem 2.3).}\end{aligned}$$

This shows that Δ' is a Parseval g-fusion frame for $H\oplus X$. □

Theorem 5.9. *Let $\Lambda\oplus\Gamma=\{(W_j\oplus V_j,\Lambda_j\oplus\Gamma_j,v_j)\}_{j\in J}$ be a g-fusion frame for $H\oplus X$ with bounds A_1,B_1 and $S_{\Lambda\oplus\Gamma}$ be the corresponding frame operator. Then*

$$\Delta=\{(S_{\Lambda\oplus\Gamma}^{-1}(W_j\oplus V_j),(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1},v_j)\}_{j\in J}$$

is a g-fusion frame for $H\oplus X$ with frame operator $S_{\Lambda\oplus\Gamma}^{-1}$.

Proof. For any $(f,g)\in H\oplus X$ we have

$$\begin{aligned}(f,g)&=S_{\Lambda\oplus\Gamma}S_{\Lambda\oplus\Gamma}^{-1}(f,g)\\&=\sum_{j\in J}v_j^2P_{W_j\oplus V_j}(\Lambda_j\oplus\Gamma_j)^*(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1}(f,g).\end{aligned}\tag{5.6}$$

By Theorem 2.3, for any $(f,g)\in H\oplus X$ we have

$$\begin{aligned}&\sum_{j\in J}v_j^2\|(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1}P_{S_{\Lambda\oplus\Gamma}^{-1}(W_j\oplus V_j)}(f,g)\|^2\\&\quad=\sum_{j\in J}v_j^2\|(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1}(f,g)\|^2\\&\quad\leqslant B_1\|S_{\Lambda\oplus\Gamma}^{-1}\|^2\|(f,g)\|^2\quad\text{(since }\Lambda\oplus\Gamma\text{ is }g\text{-fusion frame).}\end{aligned}\tag{5.7}$$

On the other hand, using (5.6), we get

$$
\begin{aligned}
\|(f,g)\|^4 &= |\langle (f,g),(f,g)\rangle|^2 \\
&= \left|\left\langle \sum_{j\in J} v_j^2 P_{W_j\oplus V_j}(\Lambda_j\oplus\Gamma_j)^*(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1}(f,g),(f,g)\right\rangle\right|^2 \\
&= \left|\sum_{j\in J} v_j^2 \langle (\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1}(f,g),(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}(f,g)\rangle\right|^2 \\
&\leqslant \sum_{j\in J} v_j^2 \|(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1}(f,g)\|^2 \sum_{j\in J} v_j^2\|(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}(f,g)\|^2 \\
&\leqslant \sum_{j\in J} v_j^2 \|(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1}(f,g)\|^2 B_1\|(f,g)\|^2 \\
&\qquad\qquad\qquad\text{(as } \Lambda\oplus\Gamma \text{ is } g\text{-fusion frame)} \\
&= B_1\|(f,g)\|^2 \sum_{j\in J} v_j^2 \|(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1}P_{S_{\Lambda\oplus\Gamma}^{-1}(W_j\oplus V_j)}(f,g)\|^2 \\
&\qquad\qquad\qquad\text{(from (5.7)).}
\end{aligned}
$$

Therefore

$$
B_1^{-1}\|(f,g)\|^2 \leqslant \sum_{j\in J} v_j^2 \|(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1}P_{S_{\Lambda\oplus\Gamma}^{-1}(W_j\oplus V_j)}(f,g)\|^2.
$$

Hence, Δ is a g-fusion frame for $H\oplus X$. Let S_Δ be the g-fusion frame operator for Δ and take $\Delta_j=\Lambda_j\oplus\Gamma_j$. Now, for each

$$
\begin{aligned}
&(f,g)\in H\oplus X, S_\Delta(f,g) \\
&= \sum_{j\in J} v_j^2 P_{S_{\Lambda\oplus\Gamma}^{-1}(W_j\oplus V_j)}(\Delta_j P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1})^*(\Delta_j P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1})P_{S_{\Lambda\oplus\Gamma}^{-1}(W_j\oplus V_j)}(f,g) \\
&= \sum_{j\in J} v_j^2 (P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1}P_{S_{\Lambda\oplus\Gamma}^{-1}(W_j\oplus V_j)})^*\Delta_j^*\Delta_j(P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1}P_{S_{\Lambda\oplus\Gamma}^{-1}(W_j\oplus V_j)})(f,g) \\
&= \sum_{j\in J} v_j^2 (P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1})^*\Delta_j^*\Delta_j(P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1})(f,g) \quad \text{(using Theorem 2.3)} \\
&= \sum_{j\in J} v_j^2 S_{\Lambda\oplus\Gamma}^{-1}P_{W_j\oplus V_j}(\Lambda_j\oplus\Gamma_j)^*(\Lambda_j\oplus\Gamma_j)(P_{W_j\oplus V_j}S_{\Lambda\oplus\Gamma}^{-1})(f,g) \\
&= S_{\Lambda\oplus\Gamma}^{-1}\left(\sum_{j\in J} v_j^2 P_{W_j\oplus V_j}(\Lambda_j\oplus\Gamma_j)^*(\Lambda_j\oplus\Gamma_j)P_{W_j\oplus V_j}(S_{\Lambda\oplus\Gamma}^{-1}(f,g))\right) \\
&= S_{\Lambda\oplus\Gamma}^{-1}S_{\Lambda\oplus\Gamma}(S_{\Lambda\oplus\Gamma}^{-1}(f,g)) \quad \text{(by definition of } S_{\Lambda\oplus\Gamma}) \\
&= S_{\Lambda\oplus\Gamma}^{-1}(f,g).
\end{aligned}
$$

Thus, $S_\Delta = S_{\Lambda\oplus\Gamma}^{-1}$. This completes the proof. □

Note 5.10. *Form Theorem 5.9 we can conclude that if $\Lambda \oplus \Gamma$ is a g-fusion frame for $H \oplus K$, then Δ is also a g-fusion frame for $H \oplus K$. The g-fusion frame Δ is a called the canonical dual g-fusion frame of $\Lambda \oplus \Gamma$.*

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