ENDOMORPHISM KERNEL PROPERTY FOR FINITE GROUPS

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Abstract. A group G has the endomorphism kernel property (EKP) if every congruence relation θ on G is the kernel of an endomorphism on G. In this note we show that all finite abelian groups have EKP and we show infinite series of finite non-abelian groups which have EKP.

Keywords: endomorphism kernel property; nilpotent group; p-group

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1. Introduction

The concept of the (strong) endomorphism kernel property for an universal algebra has been introduced by Blyth, Fang and Silva in [1] and [3] as follows.

Definition 1.1. An algebra A has the *endomorphism kernel property* (EKP) if every congruence relation θ on A different from the universal congruence $\iota_A = A \times A$ is the kernel of an endomorphism on A.

Let $\theta \in \mathrm{Con}(A)$ be a congruence on A. A mapping $f\colon A \to A$ is said to be *compatible* with θ if $a \equiv b(\theta)$ implies $f(a) \equiv f(b)(\theta)$, it means if it preserves the congruence θ. An endomorphism of A is called *strong* if it is compatible with every congruence $\theta \in \mathrm{Con}(A)$.

The notion of compatibility of functions with congruences has been studied in various contexts by many authors. We refer to the monograph [16] for an overview. Compatible functions are sometimes called "congruence preserving functions" or "functions with substitution property".

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Definition 1.2. An algebra A has the *strong endomorphism kernel property* (SEKP) if every congruence relation θ on A different from the universal congruence ι_A is the kernel of a strong endomorphism on A.

If the algebra A has two or more nullary operations and corresponding elements are different in A, the universal congruence ι_A cannot be the kernel of an endomorphism and that is the reason why the universal congruence ι_A is excluded from the definition of both EKP and SEKP. It is not necessary to exclude it for algebras with one-element subalgebras, like groups.

In the original paper [1] Blyth, Fang and Silva proved that finite Boolean algebras, finite chains as bounded distributive lattices possess EKP, finite bounded distributive lattice has EKP if and only if it is a product of chains. They also proved a full characterisation of finite de Morgan algebras having EKP. EKP for finite Stone algebras has been studied by Gaitan and Cortes in [8], by Guričan in [10]. The main approach in papers [1] and [8] lies in regarding algebras in question as Ockham algebras and using the duality theory of Priestley and Urquhart. Another papers concerning this topic are e.g. [11], [15].

Blyth and Silva considered the case of Ockham algebras and in particular of MS-algebras in [3]. For instance, a Boolean algebra has SEKP if and only if it has exactly two elements. A full characterization of MS-algebras having SEKP is provided in [3]. A full characterization of finite distributive double p-algebras and finite double Stone algebras having SEKP was proved by Blyth, Fang and Wang in [2]. SEKP for distributive p-algebras and Stone algebras has been studied and fully characterized by Fang and Fang in [5]. Semilattices with SEKP were fully described by Fang and Sun in [6]. Guričan and Ploščica described unbounded distributive lattices with SEKP in [13]. Halušková described monounary algebras with SEKP in [14]. Double MS-algebras with SEKP were described by Fang in [4]. Guričan proved in [12] that all finite relative Stone algebras have SEKP. Finite abelian groups with SEKP were described by Fang and Sun in [7].

2. Preliminaries

We shall start with an obvious characterization of EKP.

Theorem 2.1 ([1])**.** *Algebra A has EKP if and only if every homomorphic image of A is isomorphic to a subalgebra of A.*

It means that a group G has EKP if and only if every homomorphic image of G, it means every factor group of a group G, is isomorphic to a subgroup of G. We shall consider only nilpotent groups throughout this paper.

Let G be a finite group, $|G| = p_1^{a_1} \dots p_k^{a_k}$, where $p_1, \dots, p_k$ are pairwise different prime numbers. Then G is nilpotent if and only if

$$G \cong G_1 \times G_2 \times \ldots \times G_k, \tag{2.1}$$

where G_i is (isomorphic to) a Sylow p_i-subgroup of G for every $i \in \{1, \dots, k\}$, it means that $|G_1| = p_1^{a_1}, \dots, |G_k| = p_k^{a_k}$.

We shall use the following well known theorem.

Theorem 2.2. *Let G be a finite nilpotent group written in this way as a product of its Sylow p_i-groups G_i,*

$$G = G_1 \times G_2 \times \ldots \times G_k.$$

Let H be a subgroup of G. Then there exist subgroups H_i of G_i, $i = 1, \dots, k$, such that

$$H = H_1 \times H_2 \times \ldots \times H_k.$$

Moreover, if $H \triangleleft G$, then $H_i \triangleleft G_i$ for $i = 1, \dots, k$.

Using this decomposition, the factor group G/H (in the case when $H \triangleleft G$) can be written as a product of factor groups in the form

$$G/H \cong G_1/H_1 \times \ldots \times G_k/H_k.$$

Combining Theorems 2.1 and 2.2 we get:

Theorem 2.3. *Let each of Sylow subgroups $G_1, \dots, G_k$ of a finite nilpotent group G (written in the form (2.1)) have EKP. Then also G has EKP.*

Proof. Homomorphic image of G is isomorphic to a factor group of G. Using Theorem 2.1, it is enough to prove that for any normal subgroup H of G, the factor group G/H is isomorphic to a subgroup of G.

Let G be a finite nilpotent group, $|G| = p_1^{a_1} \dots p_k^{a_k}$, where $p_1, \dots, p_k$ are pairwise different prime numbers. Without loss of generality we can assume that

$$G = G_1 \times G_2 \times \ldots \times G_k,$$

where G_i, $i = 1, \dots, k$ are isomorphic to Sylow subgroups of G. Let $H \triangleleft G$. By Theorem 2.2 we know that

$$G/H \cong G_1/H_1 \times \ldots \times G_k/H_k$$

for suitable normal subgroups H_i of G_i, $i = 1, \dots, k$. For any $i = 1, \dots, k$, the group G_i is a Sylow subgroup of G and therefore G_i/H_i is isomorpic to a subgroup of G_i by Theorem 2.1. Therefore the product $G_1/H_1 \times \ldots \times G_k/H_k$ is isomorphic to a subgroup of $G_1 \times G_2 \times \ldots \times G_k$. Hence, G has EKP. $\square$

3. Finite abelian groups

Let us consider finite abelian groups now. Every abelian group is nilpotent. Let us start with a special case of homomorphic images of a finite abelian p-group. Cyclic group with n elements will be denoted by Z_n. Let p be a prime number. By a structure theorem for finite abelian groups a finite abelian p-group G can be uniquely written as $G \cong Z_{p^{a_1}} \times \ldots \times Z_{p^{a_n}}$, $a_1 \leqslant \ldots \leqslant a_n$. Numbers $p^{a_1}, \ldots, p^{a_n}$ are called *abelian invariants* of a p-group G. We shall use additive notation for a group operation in this section, it means that for the nth power of a group element g we shall write $n \times g$. The subgroup generated by elements $a_1, \ldots, a_n$ will be denoted by $[a_1, \ldots, a_n]$.

Lemma 3.1. *Let $k \geqslant 1$, $a_1 \leqslant \ldots \leqslant a_k$ and $l_1, \ldots, l_k \in \{1, \ldots, p-1\}$. Then*

$$H = Z_{p^{a_1}} \times \ldots \times Z_{p^{a_k}} / [(l_1 \times p^{a_1 - 1}, \ldots, l_k \times p^{a_k - 1})]$$

is isomorphic to $Z_{p^{a_1 - 1}} \times Z_{p^{a_2}} \times \ldots \times Z_{p^{a_k}}$.

Proof. We shall calculate abelian invariants of a group H. Let $\mathbb{Z}$ be the group of integers, $K = [(l_1 \times p^{a_1 - 1}, \ldots, l_k \times p^{a_k - 1})]$, $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be a k-tuple with just one 1 on the ith coordinate.

Let $\varphi\colon\ \mathbb{Z}^k \to H$ be a homomorphism given by:

$$\begin{aligned} \varphi(e_1) &= (1, 0, \ldots, 0) + K, \\ \varphi(e_2) &= (0, 1, \ldots, 0) + K, \\ &\vdots \\ \varphi(e_k) &= (0, \ldots, 0, 1) + K. \end{aligned}$$

It means

$$\varphi(b_1, \ldots, b_k) = (b_1 \bmod p^{a_1}, \ldots, b_k \bmod p^{a_k}) + K.$$

We use

$$K = \{l \times (l_1 \times p^{a_1 - 1}, \ldots, l_k \times p^{a_k - 1});\ l = 0, \ldots, p-1\}$$

to see that $\varphi(b_1, \ldots, b_k) = (0, \ldots, 0) + K$ if and only if

$$(b_1 \bmod p^{a_1}, \ldots, b_k \bmod p^{a_k}) = l \times (l_1 \times p^{a_1 - 1}, \ldots, l_k \times p^{a_k - 1})$$

for some $l \in \{0, \ldots, p-1\}$.

Let $b_i \bmod p^{a_i} = r_i$, then $(b_1, \ldots, b_k) = (r_1 + q_1 \times p^{a_1}, \ldots, r_k + q_k \times p^{a_k})$ and $(r_1, \ldots, r_k) = l \times (l_1 \times p^{a_1 - 1}, \ldots, l_k \times p^{a_k - 1})$, which means that $(b_1, \ldots, b_k) \in \ker(\varphi)$ if and only if

$$(b_1, \ldots, b_k) = l \times (l_1 \times p^{a_1-1}, \ldots, l_k \times p^{a_k-1}) + (q_1 \times p^{a_1})e_1 + \ldots + (q_k \times p^{a_k})e_k$$

for some integers $l, q_1, \ldots, q_k$ and

$$\ker(\varphi) = [(l_1 \times p^{a_1-1}, l_2 \times p^{a_2-1}, \ldots, l_k \times p^{a_k-1}), p^{a_1}e_1, p^{a_2}e_2, \ldots, p^{a_k}e_k].$$

Therefore we can form a matrix

$$A = \begin{pmatrix} l_1 p^{a_1-1} & l_2 p^{a_2-1} & \ldots & l_k p^{a_k-1} \\ p^{a_1} & 0 & \ldots & 0 \\ 0 & p^{a_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p^{a_k} \end{pmatrix}$$

and if we denote

$$\eta_m(A) = \gcd\{\det(A^{i_1 \ldots i_m}_{j_1 \ldots j_m}) \colon\ 1 \leqslant i_1 < \ldots < i_m \leqslant k+1,\ 1 \leqslant j_1 < \ldots < j_m \leqslant k\},$$

where $A^{i_1 \ldots i_m}_{j_1 \ldots j_m}$ is a submatrix of A consisting of elements from rows $1 \leqslant i_1 < \ldots < i_m \leqslant k+1$ and columns $1 \leqslant j_1 < \ldots < j_m \leqslant k$, then abelian invariants of a factor group H are $d_1, \ldots, d_k$ defined by

$$d_1 = \eta_1(A),\ d_2 = \eta_2(A)/\eta_1(A), \ldots,\ d_k = \eta_k(A)/\eta_{k-1}(A).$$

We know that for any $m = 1, \ldots, k$ the number $\eta_m(A)$ divides

$$\det(A^{2 \ldots m+1}_{1 \ldots m}) = \det \begin{pmatrix} p^{a_1} & 0 & \ldots & 0 \\ 0 & p^{a_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p^{a_m} \end{pmatrix} = p^{a_1 + \ldots + a_m}$$

and therefore $\eta_m(A)$ does not depend on numbers $l_1, \ldots, l_k$, because these numbers are coprime with p.

As $a_1 \leqslant \ldots \leqslant a_k$, it is also clear that for $l = 1, \ldots, k$ the least power of p in $\det(A^{i_1\ldots i_m}_{j_1\ldots j_m})$ has the determinant of the "left upper corner" of A, it means

$$\det(A^{1\ldots m}_{1\ldots m}) = \det \begin{pmatrix} l_1 p^{a_1-1} & l_2 p^{a_2-1} & \ldots & & l_m p^{a_m-1} \\ p^{a_1} & 0 & \ldots & 0 & 0 \\ 0 & p^{a_2} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \ldots & p^{a_{m-1}} & 0 \end{pmatrix}$$
$$= l_1 p^{a_1-1} \cdot 0 + \ldots + l_{m-1} p^{a_{m-1}-1} \cdot 0 + l_m p^{a_m-1} \cdot p^{a_1+\ldots+a_{m-1}}$$
$$= l_m \cdot p^{a_1+\ldots+a_{m-1}+(a_m-1)}$$

and therefore $\eta_m(A) = p^{(a_1-1)+a_2+\ldots+a_m}$.

We get $d_1 = p^{a_1-1}$ and for $i = 2, \ldots, k$ we have

$$d_i = p^{(a_1-1)+a_2+\ldots+a_i} / p^{(a_1-1)+a_2+\ldots+a_{i-1}} = p^{a_i}.$$

It means that abelian invariants of H are $p^{a_1-1}, p^{a_2}, \ldots, p^{a_k}$, therefore

$$H \cong Z_{p^{a_1-1}} \times Z_{p^{a_2}} \times \ldots \times Z_{p^{a_k}}.$$

□

Using this we get:

Lemma 3.2. *Let G be a finite abelian p-group, $G = Z_{p^{a_1}} \times \ldots \times Z_{p^{a_n}}$, K be a subgroup of G, $|K| = p$. Then there exist $1 \leqslant i \leqslant n$ such that*

$$G/K \cong Z_{p^{a_1}} \times \ldots \times Z_{p^{a_{i-1}}} \times Z_{p^{a_i-1}} \times Z_{p^{a_{i+1}}} \times \ldots \times Z_{p^{a_n}},$$

which means that the group G/K is isomorphic to a subgroup of G.

Proof. Let $G = Z_{p^{a_1}} \times \ldots \times Z_{p^{a_n}}$. Let us describe subgroups with p elements first. Let $(g_1, \ldots, g_n) \in K \setminus \{(0, \ldots, 0)\}$. Then $K = [(g_1, \ldots, g_n)]$ and $\operatorname{ord}((g_1, \ldots, g_n)) = p$. It means

$$p \times (g_1, \ldots, g_n) = 0 \quad \text{in } G$$

and therefore for every $i = 1, \ldots, n$

$$p \times g_i = 0 \quad \text{in } Z_{p^{a_i}}.$$

It means that either $g_i = 0$ or $g_i = l_i \times p^{a_i-1}$, $l_i \in \{1, \ldots, p-1\}$.

Now, let $X = \{i \in \{1, \ldots, n\} \colon g_i \neq 0\}$. Suppose that $X = \{i_1, \ldots, i_k\}$, $i_1 < i_2 < \ldots < i_k$. Let $Y = \{1, \ldots, n\} \setminus X$, $Y = \{j_1, \ldots, j_{n-k}\}$, $j_1 < j_2 < \ldots < j_{n-k}$ and $l_{j_1}, \ldots, l_{j_{n-k}} = 0$. Then $(g_1, \ldots, g_n) = (l_1 \times p^{a_1-1}, \ldots, l_n \times p^{a_n-1})$ and

$$\begin{aligned} G/K &\cong Z_{p^{a_1}} \times \ldots \times Z_{p^{a_n}}/[(l_1 \times p^{a_1-1}, \ldots, l_n \times p^{a_n-1})] \\ &\cong Z_{p^{a_{j_1}}} \times \ldots \times Z_{p^{a_{j_{n-k}}}} \\ &\quad \times (Z_{p^{a_{i_1}}} \times \ldots \times Z_{p^{a_{i_k}}}/[(l_{i_1} \times p^{a_{i_1}-1}, \ldots, l_{i_k} \times p^{a_{i_k}-1})]). \end{aligned}$$

Using Lemma 3.1 we see that

$$Z_{p^{a_{i_1}}} \times \ldots \times Z_{p^{a_{i_k}}}/[(l_{i_1} \times p^{a_{i_1}-1}, \ldots, l_{i_k} \times p^{a_{i_k}-1})]$$

is isomorphic to $Z_{p^{a_{i_1}-1}} \times Z_{p^{a_{i_2}}} \times \ldots \times Z_{p^{a_{i_k}}}$, which finishes the proof. $\square$

Theorem 3.3. *Let G be a finite abelian p-group, $|G| = p^n$. Then for any subgroup H of G, the factor group G/H is isomorphic to a subgroup of G.*

Proof. We shall proceed by induction. If $n = 1$, G has no proper subgroups. Let $|G| = p^{n+1}$, H be a subgroup of G. If $|H| = p$, the result follows from Lemma 3.2.

Let $|H| = p^k$, $k \geqslant 2$. There exist a subgroup K of H such that $|K| = p$. By the isomorphism theorem we know that

$$G/H \cong (G/K)/(H/K).$$

The group G/K is a p-group with p^n elements and using the induction assumption, the factor group $(G/K)/(H/K)$ is isomorphic to a subgroup of G/K, which is isomorphic to a subgroup of G by Lemma 3.2. This means that G/H is isomorphic to a subgroup of G and this finishes the proof. $\square$

Using this we get the main result of this section.

Theorem 3.4. *Let G be a finite abelian group. Then G has EKP.*

Proof. The group G is a finite nilpotent group. Sylow subgroups of G are abelian p-groups. Combining Theorem 2.1 and Theorem 3.3 we know that all Sylow subgroups of G have EKP. Hence, G has EKP by Theorem 2.3. $\square$

4. Finite nilpotent groups

We shall show infinitely many finite non-abelian groups with EKP in this section. We suppose that p is a prime in this section. We will use multiplication for a group operation. Let G be a group, $Z(G)$ be the centre of G. Let us start with some well known facts/theorems.

Theorem 4.1.

(1) *Let G be a finite p-group. Then $Z(G)$ is nontrivial.*
(2) *Let G be a group. If $G/Z(G)$ is cyclic, then G is abelian.*
(3) *Let G be a finite p-group, $H \lhd G$, $|H| = p$. Then $H \subseteq Z(G)$.*
(4) *Let G be a group, $|G| = p^2$. Then G is abelian, it means G is either cyclic or $G \cong Z_p \times Z_p$.*

Corollary 4.2. *Let G be a non-abelian group, $|G| = p^3$. Then there is exactly one normal subgroup of G which has p elements. Morever, this normal subgroup is the center $Z(G)$ and*

$$G/Z(G) \cong Z_p \times Z_p.$$

Proof. Let G be a non-abelian group. According to Theorem 4.1 (2), $G/Z(G)$ is not cyclic. We know also that $Z(G)$ is not trivial. It means that $|G/Z(G)| = p^2$ and therefore $|Z(G)| = p$.

Now, let $H \lhd G$, $|H| = p$. By Theorem 4.1 (3), $H \subseteq Z(G)$. But this means that $H = Z(G)$. So $Z(G)$ is the only one normal subgroup of G. Moreover, we know that $G/Z(G)$ is not cyclic and it has p^2 elements, therefore

$$G/Z(G) \cong Z_p \times Z_p$$

by Theorem 4.1 (4). □

The following statement is Corollary 5.3.8 in [17].

Corollary 4.3. *Suppose that G is a p-group all of whose abelian subgroups are cyclic. Then G is cyclic or a quaternion group.*

Hence, as a direct consequence we have:

Theorem 4.4. *Let G be a non-abelian group, $|G| = p^3$, where $p > 2$, or $G \cong D_4$ (dihedral 8 element group). Then G has a non-cyclic abelian subgroup H, it means a subgroup H such that*

$$H \cong Z_p \times Z_p.$$

Lemma 4.5. *Let G be a non-abelian group, $|G| = p^3$.*

(1) *If $p > 2$, then G has EKP.*

(2) *If $p = 2$ and $G \cong D_4$, then G has EKP.*

Proof. We have to show that a homomorphic image of G, it means a factor group of G, is isomorphic to a subgroup of G. Let $H \lhd G$.

If $|H| = p^2$, then $|G/H| = p$ and we know that G contains a subgroup with p elements.

If $|H| = p$, then $H = Z(G)$ and by Corollary 4.2 we have

$$G/H = G/Z(G) \cong Z_p \times Z_p.$$

By Theorem 4.4, group D_4 has a subgroup isomorphic to $Z_2 \times Z_2$, a group G with p^3 elements for an odd prime number p has a subgroup isomorphic to $Z_p \times Z_p$. This finishes the proof. □

Next lemma generalizes the previous one.

Lemma 4.6. *Let P be a non-abelian group, $|P| = p^3$ for an odd prime number p or $P = D_4$. Let $G = Z_p^k \times P$. Then G has EKP.*

Proof. Let $H \lhd G$. Let $(a, b) \in H \subseteq Z_p^k \times P$, it means $a \in Z_p^k, b \in P$. Let us remind that $Z(P)$ is a cyclic group with p elements. We shall consider three cases:

Case 1. $b \notin Z(P)$: There is $g \in P$ such that $gbg^{-1}b^{-1} \neq e$. Also $(a^{-1}, b^{-1}) \in H$ and because H is invariant, also $(a, gbg^{-1}) \in H$ and finally $(e, gbg^{-1}b^{-1}) \in H$. Denote $z = gbg^{-1}b^{-1}$. We have that z is a commutator since $z = [g, b]$.

As $P/Z(P) \cong Z_p \times Z_p$, it is an abelian group. Therefore the commutator subgroup satisfies $[P, P] \subseteq Z(P)$. Group P is not abelian, it means that $[P, P] = Z(P)$, $z \in Z(P)$. We know that $(e, z) \in H$, therefore $\{e\} \times Z(P) = [(e, z)] \lhd H$. Clearly, also $\{e\} \times Z(P) \lhd G$. Now, $G/(\{e\} \times Z(P)) \cong Z_p^k \times Z_p \times Z_p$ and it is isomorphic to a subgroup of G. $G/(\{e\} \times Z(P))$ is an abelian group and therefore by Theorem 3.2, factor group $G/H \cong (G/(\{e\}\times Z(P)))/(H/(\{e\}\times Z(P)))$ is isomorphic to a subgroup of $G/(\{e\} \times Z(P))$ and finally, G/H is isomorphic to a subgroup of G.

In the next two cases we assume that there is no element $(a, b) \in H$ with $b \notin Z(P)$.

Case 2. $b \in Z(P)$ and there exists an element $(a_1, b_1) \in H$ such that for some l we have $b^l = b_1$, $a^l \neq a_1$: We have also $(a_1^{-1}, b_1^{-1}) \in H$, it means $(a^l, b^l) \cdot (a_1^{-1}, b_1^{-1}) = (c, e) \in H$, $c \neq e$. Let $K = [c]$. Then $K \lhd Z_p^k$, we see that $K \times \{e\} \lhd H$ and also $K \times \{e\} \lhd G$. Further, $G/(K \times \{e\}) \cong (Z_p^k/K) \times P$. By Lemma 3.2, Z_p^k/K is isomorphic to Z_p^{k-1}.

We can proceed by induction. For $k = 1$,

$$K = Z_p^1 \quad \text{and} \quad G/(K \times \{e\}) \cong (Z_p^1/K) \times P \cong P$$

and P is isomorphic to a subgroup of $G = Z_p^1 \times P$. Therefore

$$G/H \cong (G/(K \times \{e\}))/(H/(K \times \{e\}))$$

is isomorphic to a subgroup of P and therefore also isomorphic to a subgroup of G.

Now, let the statement be true for all $k' < k$, we shall prove that it is true for the number k. We know that Z_p^k/K is isomorphic to Z_p^m for $m < k$, it means that $G/(K \times \{e\}) \cong Z_p^m \times P$ and therefore

$$G/H \cong (G/(K \times \{e\}))/(H/(K \times \{e\})$$

is isomorphic to a subgroup of $Z_p^m \times P$ by induction. Finally, we get that G/H is isomorphic to a subgroup of $G = Z_p^k \times P$.

Case 3. $b \in Z(P)$ and for an element $(a_1, b_1) \in H$, whenever for some l we have $b^l = b_1$, then also $a^l = a_1$: Let us rename b to z. The group $H = [(a, z)]$ in this case. It is clear that z is a generator of the centre $Z(P)$. Let $a = (l_1, \ldots, l_k) \in Z_p^k$, $X = \{i \in \{1, \ldots, k\};\ l_i \neq 1\} = \{i_1, \ldots, i_m\}$, $Y = \{1, \ldots, k\} \setminus X = \{j_1, \ldots, j_{k-m}\}$. Now, let $C_i = Z_p$. Then

$$(Z_p^k \times P)/[(a, z)] \cong C_{j_1} \times \ldots \times C_{j_{k-m}} \times ((C_{i_1} \times \ldots \times C_{i_m} \times P)/[((l_{i_1}, \ldots, l_{i_m}), z)]).$$

It is enough to prove that $C_{i_1} \times \ldots \times C_{i_m} \times P/[((l_{i_1}, \ldots, l_{i_m}), z)]$ is isomorphic to a subgroup of $C_{i_1} \times \ldots \times C_{i_m} \times P = Z_p^m \times P$. To simplify indexing, we shall prove that for $a = (l_1, \ldots, l_m)$, $l_1, \ldots, l_m \neq 1$, $Z_p^m \times P/[((l_1, \ldots, l_m), z)]$ is isomorphic to a subgroup of $Z_p^m \times P$. As Z_p is a cyclic group with p elements, $Z_p = [l_i]$, therefore we shall represent Z_p^m as $[l_1] \times \ldots \times [l_m]$.

If $m = 1$, we can consider the map $\varphi\colon\ [l_1] \times P \to P$ given by $\varphi(l_1^n, b) = bz^{-n}$. Then $(l_1^n, b) \in \ker(\varphi)$ if and only if $bz^{-n} = e$, it means if and only if $b = z^n$. Therefore $\ker(\varphi) = \{(l_1^n, z^n);\ n = 0, \ldots, p-1\} = [(l_1, z)]$.

It is easy to check that φ is a homomorphism (we shall present the proof for more general case later). The group $[l_1] \times P$ has p^4 elements, $\ker(\varphi)$ has p elements and P has p^3 elements, therefore the map φ is surjective and $[l_1] \times P/[(l_1, z)] \cong P$. It means that $[l_1] \times P/[(l_1, z)]$ is isomorphic to a subgroup of P.

Let us consider a general case now. Let $\varphi\colon\ [l_1] \times \ldots \times [l_m] \times P \to [l_2] \times \ldots \times [l_m] \times P$ be given by

$$\varphi(l_1^{a_1}, \ldots, l_m^{a_m}, b) = (l_2^{a_1 - a_2}, l_3^{a_2 - a_3}, \ldots, l_m^{a_{m-1} - a_m}, bz^{-a_m}).$$

Then $(l_1^{a_1},\dots,l_m^{a_m},b)\in\ker(\varphi)$ if and only if

$$a_1-a_2=0,\ a_2-a_3=0,\dots,a_{m-1}-a_m=0,\ bz^{-a_m}=e,$$

which is true if and only if $a_1=\ldots=a_m$ and $b=z^{a_m}$. Therefore

$$\ker(\varphi)=[(l_1,\dots,l_m,z)].$$

The map φ is a homomorphism:

$$\begin{aligned}
\varphi((l_1^{a_1},&\dots,l_m^{a_m},c)\cdot(l_1^{b_1},\dots,l_m^{b_m},d))\\
&=\varphi(l_1^{a_1+b_1},\dots,l_m^{a_m+b_m},cd)\\
&=(l_2^{(a_1+b_1)-(a_2+b_2)},\dots,l_m^{(a_{m-1}+b_{m-1})-(a_m+b_m)},cd\cdot z^{-a_m-b_m})\\
&=(l_2^{(a_1-a_2)+(b_1-b_2)},\dots,l_m^{(a_{m-1}-a_m)+(b_{m-1}-b_m)},cz^{-a_m}d\cdot z^{-b_m})\\
&=(l_2^{a_1-a_2},\dots,l_m^{a_{m-1}-a_m},cz^{-a_m})\cdot(l_2^{b_1-b_2},\dots,l_m^{b_{m-1}-b_m},dz^{-b_m})\\
&=\varphi((l_1^{a_1},\dots,l_m^{a_m},c))\cdot\varphi((l_1^{b_1},\dots,l_m^{b_m},d)).
\end{aligned}$$

The equalities on the last coordinate are valid because z is an element of the centre of P. By counting elements in $[l_1]\times\ldots\times[l_m]\times P$, $\ker(\varphi)$ and in $[l_2]\times\ldots\times[l_m]\times P$, we see that φ is surjective. Therefore $Z_p^m\times P/[(a,z)]\cong Z_p^{m-1}\times P$, which is isomorphic to a subgroup of $G=Z_p^m\times P$.

We see that in every possible case, G/H is isomorphic to a subgroup of G and therefore G has EKP. $\square$

Using this result and the ideas from the section on abelian groups we get:

Theorem 4.7. *Let G be a finite nilpotent group written in the form* (2.1). *Let each Sylow group G_i be (isomorphic to) one of the following groups:*

(1) *an abelian group,*
(2) $Z_{p_i}^{k_i}\times P_i$, *where $k_i\geqslant 0$, $p_i>2$ and P_i is a non-abelian group of order p_i^3,*
(3) $Z_2^{k_i}\times D_4$, *where $k_i\geqslant 0$ and D_4 is a dihedral 8-element group.*

Then G has EKP.

Remark 4.8. Lemma 4.6 does not provide all non-abelian p-groups which have EKP. Direct computation in GAP (see [9]) shows that for example there are 6 non-abelian groups of order $3^4=81$ which have EKP (GAP identifications returned by `IdSmallGroup()` of these groups are [81,6], [81,7], [81,8], [81,9], [81,12], [81,13]), but only 2 of them are of the form $Z_3\times P$, where P is a non-abelian group of order 3^3. There are 4 non-abelian groups of order 81 which do not have EKP (GAP id's of these groups are [81,3], [81,4], [81,10], [81,14]).

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