NEW SUFFICIENT CONDITIONS FOR GLOBAL ASYMPTOTIC STABILITY OF A KIND OF NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. This paper addresses the stability study for nonlinear neutral differential equations. Thanks to a new technique based on the fixed point theory, we find some new sufficient conditions ensuring the global asymptotic stability of the solution. In this work we extend and improve some related results presented in recent works of literature. Two examples are exhibited to show the effectiveness and advantage of the results proved.

Keywords: contraction mapping principle; asymptotic stability; neutral differential equation

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1. Introduction

It is well-known that the theory of neutral functional differential equations has attracted many types of research due to its wide and great applications in many fields of mathematical science and engineering such as neural networks, population dynamics, control theory, and many other phenomena. For appropriate literature, we can refer to the books [15], [16], [18], [19]. A neutral delay differential equation is a kind of delay differential equation where the delay argument occurs in the highest order derivative of the state, which can be used to describe many real-world

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phenomena that arise in the areas, for example, of lossless transmission lines, theory of automatic control and others, we refer the reader to the references Brayton [5], Hale [15], Kuang [19], Kolmanovskii and Myshkis [18], and the sources related.

Until today, significant progress has been made in the qualitative theory (e.g. oscillation theory, periodicity, stability, existence of a positive periodic solution, asymptotic behavior, boundedness, instability, etc.) of neutral delay differential equations. For more detail, we refer to [1]–[5], [9], [10]–[14], [17], [20], [23]–[35] and the references in these sources. One of the most qualitative concepts in mathematical theory is to determine the stability of a model given. The theory of stability was initiated at the end of the 19th century by Lyapunov. This method is now known as the Lyapunov direct method or Lyapunov function.

For decades, Lyapunov has been developing a method for determining stability in many areas of differential equations without solving the equations themselves. This theory has been proven significantly effective over a century and it has achieved wide applications in various fields of physics and mathematics. Unfortunately, when we try to carry over the principles of the Lyapunov stability theory to special problems, we face a large number of difficulties and it appears that new methods are needed to overcome these obstacles (see [7]–[9]). Luckily, Burton and many other authors have used the fixed point theory as an alternative to studying the stability of deterministic or stochastic systems, where some of these problems of Lyapunov functions have been solved. In the current study, we use this method to address a kind of nonlinear neutral differential equations (see [5], [6], [14], [22]). Moreover, we use this method to address a kind of nonlinear neutral differential equations.

In [17], Jin and Luo studied the asymptotic stability of the scalar nonlinear neutral differential equation of the form

$$u'(t) = -a(t)u(t) + c(t)u'(t-\tau(t)) - b(t)u(t-\tau(t)), \quad t \geqslant 0 \tag{1.1}$$

in the space C^0. The work [17] by Jin and Luo requires that the delay τ is twice differentiable, $\tau'(t) \neq 1$ for $t \geqslant 0$ and c is differentiable. However, there are many interesting examples where these conditions are not satisfied. It is our aim in this paper to remove these restrictive conditions by studying the global stability in the space C^1.

As is known, there are a few papers [1], [3], [4] and [21], [33] where the authors discuss the global asymptotic stability of solutions of neutral differential equations in C^1. For example, Liu and Yang in [21] were the first to establish necessary and sufficient conditions for the asymptotic stability in C^1 for the equation

$$u'(t) = -a(t)u(t) + c(t)u'(t-\tau_1(t)) + Q(t, u(t), u(t-\tau_2(t))), \tag{1.2}$$

where Q is a Lipschitz continuous function in u. Liu and Yang were able, in their work, to avoid the derivative of the coefficient c, and they also did not need that the delay τ is twice differentiable and $\tau'(t) \neq 1$ for $t \geqslant 0$.

Recently, by the same method of Liu and Yang (see [21]), Ardjouni and Djoudi (see [3]) have addressed a more general form than (1.2) like

$$\begin{aligned} u'(t) = & -a(t)u(t) + g(t, u'(t-\tau_1(t)), u'(t-\tau_2(t)), \ldots, u'(t-\tau_n(t))) \\ & + f(t, u(t-\tau_1(t)), u(t-\tau_2(t)), \ldots, u(t-\tau_n(t))), \end{aligned} \tag{1.3}$$

where $f(t, u_1, \ldots, u_n)$ and $g(t, u_1, \ldots, u_n)$ are continuous and satisfy the Lipschitz condition in $u_1, \ldots, u_n$, respectively. However, the case in which one considers all the terms of the equation (1.3) to be nonlinear, still remains unexplored which is the main reason for the analysis we perform in the current paper.

In 2020, Zaid et al. (see [23]) obtained stability results in C^0 about the zero solution of the standard form of the totally nonlinear delay differential equation

$$u'(t) = -\sum_{i=1}^{N} a_i(t, u_t)u(t) + f(t, u_t), \quad t \geqslant t_0. \tag{1.4}$$

In the case $N = 1$, equation (1.4) reduces to that in [12]. With the previous motivation, in this paper we extend the results of [23] to the totally nonlinear neutral differential equation presented in (2.1) (see below). More precisely, we study the stability in the space C^1 (as described in more detail below) which is a stronger and much richer concept of stability than the usual one in C^0. By applying the fixed point theory, we state new and more applicable stability criteria in C^1. The sufficient conditions obtained are quite practical and we no longer need the delay to be twice differentiable or coefficients to be differentiable, which required some previous relevant works, see [1], [2], [11], [17], [35]. This new feature makes the asymptotic behavior in C^1 more important and more useful as well. Our work extends and improves the results of [3], [12], [17], [21], [23]. In addition, two examples are exhibited to test the feasibility and advantage of the results proved.

2. Notations and preliminaries

Let $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^-$ denote $(-\infty,\infty)$, $[0,\infty)$, and $(-\infty,0]$, respectively. In the current paper, we aim at discussing the asymptotic stability in C^1 for a standard form of neutral differential equations

$$u'(t) = -\sum_{i=1}^{N} a_i(t,u_t)u(t) + g(t,u'_t) + f(t,u_t), \quad t \geqslant t_0, \tag{2.1}$$

where $f, g \in C(\mathbb{R}^+ \times B, \mathbb{R})$ and $a_i \in C(\mathbb{R}^+ \times B, \mathbb{R})$, $i = \overline{1,N}$, with

$$B = \{\phi \in C(\mathbb{R}^-,\mathbb{R})\colon\ \phi \text{ bounded}\}$$

and with the norm $\|\phi\|_\circ := \sup\limits_{\theta\in(-\infty,0]} |\phi(\theta)|$. Put also

$$C_L = \{\xi \in C\colon\ \|\xi\|_\circ \leqslant L\} \quad \text{and} \quad C^1_{L'} = \{\xi \in C^1\colon\ \|\xi'\|_\circ \leqslant L'\}.$$

Let $u \in C^1(\mathbb{R},\mathbb{R})$ be bounded and $t \geqslant 0$ a fixed number, let $u_t, u'_t \in C$ be defined by

$$u_t(\theta) = u(t+\theta) \quad \text{and} \quad u'_t(\theta) = u'(t+\theta) \text{ for } \theta \in \mathbb{R}^-. \tag{2.2}$$

We put

$$\|x\|^{[s,t]} := \sup_{\xi\in[s,t]} |x(\xi)|$$

for a function $x\colon\ \mathbb{R} \to \mathbb{R}$.

Before stating the main result of this paper, we impose the following assumptions.

(**A1**) There exists a constant $L > 0$ and a function $b_1 \in C(\mathbb{R},\mathbb{R}^+)$ such that for all $\phi, \psi \in C_L$ and for all $t \geqslant 0$,

$$|f(t,\phi) - f(t,\psi)| \leqslant |b_1(t)|\|\phi - \psi\|_\circ. \tag{2.3}$$

(**A2**) There exists a constant $L' > 0$ and a function $b_2 \in C(\mathbb{R},\mathbb{R}^+)$ such that for all $\phi, \psi \in C^1_{L'}$ and for all $t \geqslant 0$,

$$|g(t,\phi') - g(t,\psi')| \leqslant |b_2(t)|\|\phi' - \psi'\|_\circ. \tag{2.4}$$

(**A3**) For all $\varepsilon > 0$ and $t_1 \geqslant 0$, there exists a $t_2 > t_1$ such that $[t \geqslant t_2, u_t \in C_L]$ implies

$$|f(t,u_t)| \leqslant |b_1(t)|(\varepsilon + \|u\|^{[t_1,t]}). \tag{2.5}$$

(**A4**) For all $\varepsilon > 0$ and $t_1 \geqslant 0$, there exists a $t_3 > t_1$ such that $[t \geqslant t_3, u' \in C^1_{L'}]$ implies

$$|g(t,u'_t)| \leqslant |b_2(t)|(\varepsilon + \|u'\|^{[t_1,t]}). \tag{2.6}$$

(**A5**) There exist $\alpha_1, \alpha_2 \in C(\mathbb{R}, \mathbb{R})$ (α_2 bounded) such that

$$\alpha_1(t) \leqslant \sum_{i=1}^{N} a_i(t, u_t) \leqslant \alpha_2(t).$$

(**A6**) Assume furthermore that

$$f(t,0) = g(t,0) = 0 \quad \forall\, t \geqslant t_0, \tag{2.7}$$

which guarantees that (2.1) possesses a trivial solution $u(t) = 0$.

For each $t_0 \in [0, \infty)$, put $C^1_{t_0} = C^1(] - \infty, t_0], \mathbb{R})$ with the norm defined by

$$|u|_{t_0} := \max_{t \in (-\infty, t_0]} \{|u(t)|, |u'(t)|\}$$

for $u \in C^1_{t_0} = C^1((-\infty, t_0], \mathbb{R})$. In addition, Φ_{t_0} denotes the set

$$\Phi_{t_0} = \left\{ \varphi \in C^1_{t_0} \colon \varphi'_-(t_0) = -\sum_{i=1}^{N} a_i(t_0, \varphi_{t_0}) \varphi(t_0) + g(t_0, \varphi'_{t_0}) + f(t_0, \varphi_{t_0}) \right\}.$$

For each $t_0 \in [0, \infty)$, we choose initial functions for equation (2.1) of the type $\varphi \in \Phi_{t_0}$.

The definitions of stability in C^1 as well as the necessary notation for our study are borrowed from paper [21], but the nonlinearities in our model and the fact that we consider a neutral term make our study nontrivial and meaningful.

We now recall some basic information.

Definition 2.1. For each initial value $(t_0, \varphi) \in [0, \infty) \times \Phi_{t_0}$, u is called a solution of (2.1) associated to (t_0, φ), if $u \in C^1((-\infty, \infty), \mathbb{R})$ satisfies equation (2.1) for almost $t \geqslant t_0$ and $u = \varphi$ for $t \leqslant t_0$. Such a solution is denoted by $u(t) = u(t, t_0, \varphi)$.

We now recall definitions concerning the asymptotic stability in C^1 for the solutions to (2.1).

Definition 2.2. The trivial solution of (2.1) is said to be:

(i) stable in C^1, if for any $\varepsilon > 0$ and $t \geqslant t_0$, there is a scalar $\delta = \delta(\varepsilon, t_0) > 0$ such that, for any initial function $\varphi \in \Phi_{t_0}$ satisfying $|\varphi|_{t_0} < \delta$, we have for the corresponding solution that

$$\max_{s \in (-\infty, t]} \{|u(s, t_0, \varphi)|, |u'(s, t_0, \varphi)|\} < \varepsilon \quad \text{for } t \geqslant t_0,$$

(ii) asymptotically stable in C^1, if $u(t)$ is stable in C^1 and for any initial function $\varphi \in \Phi_{t_0}$ we have for the solution that

$$\lim_{t \to \infty} u(t, t_0, \varphi) = \lim_{t \to \infty} u'(t, t_0, \varphi) = 0.$$

At light of Definition 2.1, sensible conditions are imposed on the initial value of equation (2.1).

Since our model (2.1) involves the nonlinear term $\sum\limits_{i=1}^{N} a_i(t, u_t)u(t)$, it turns out more complex and different than those of [3], [12], [17], [21], [23], which also implies some difficulties in the mathematical analysis. That means we study how the asymptotic behavior property in C^1 is when (1.4) is added to the perturbed nonlinear neutral term $g(t, u_t')$. Motivated by the previously cited literature related to the fixed point approach [9], [10], [11], [12], [14], the Banach fixed point theorem is used to obtain some new sufficient conditions ensuring the global asymptotic stability results in C^1 to equation (2.1). Finally, two examples are given to illustrate the real interest and importance of the results proposed.

3. Stability by contraction mapping

It is well-known that studying the stability of an equation by Banach's fixed point method is based on three essential points: a complete metric space, a variation of parameters formula, and the formulation of an appropriate contraction mapping. The advantage of this method is that the fixed point argument leads to the existence, uniqueness, boundedness, and stability of the equation, all at once. Up to date, no work has considered equation (2.1) to establish sufficient conditions for the global asymptotic behavior in C^1. Let us begin to explore this issue of stability.

In this section, we discuss the asymptotic stability in C^1 for equation (2.1).

Theorem 3.1. *Assume hypotheses* (**A1**)–(**A6**) *hold and for any* $t \geqslant t_0$, *there exists* $\eta \in (0, \frac{1}{2})$ *such that*

$$\liminf_{t\to\infty} \int_{t_0}^{t} \alpha_1(s)\,\mathrm{d}s > -\infty, \tag{3.1}$$

$$\int_{t_0}^{t} \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\,\mathrm{d}s \leqslant \eta, \tag{3.2}$$

$$|\alpha_2(t)| \int_{t_0}^{t} \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\,\mathrm{d}s + (|b_1(t)| + |b_2(t)|) \leqslant \eta, \tag{3.3}$$

$$\int_{0}^{t} \alpha_1(s)\,\mathrm{d}s \to \infty \quad \text{as } t \to \infty. \tag{3.4}$$

Then, the trivial solution to equation (2.1) *is asymptotically stable in* C^1.

P r o o f. First, suppose that $\int_0^t \alpha_1(s)\,\mathrm{d}s \to \infty$ as $t \to \infty$. For each $t_0 \in [0,\infty)$, let $\varphi \in C((-\infty, t_0], \mathbb{R})$ be a fixed initial function. We define S as the space

$$S = \Big\{u \in C^1(\mathbb{R},\mathbb{R})\colon\ \lim_{t\to\infty} u(t) = \lim_{t\to\infty} u'(t) = 0\Big\}$$

with the metric defined by

$$\|u\| := \max_{t\in\mathbb{R}}\{|u(t)|, |u'(t)|\}.$$

Then S is a complete metric space.

Next, we put for any $\varphi \in \Phi_{t_0}$

$$D^l_\varphi = \Big\{u \in S\colon\ u_{t_0} = \varphi \text{ and } \max_{t\geqslant t_0}\{\|u_t\|_\circ, \|u'_t\|_\circ\} \leqslant l\Big\}.$$

Obviously, D^l_φ is a closed convex and bounded subset of S, where $l = \max\{L, L'\}$.

We can use the variation of parameter formula to write equation (2.1) as an integral equation suitable for Banach's fixed point theorem. The expression of the mapping $\mathcal{P}$ below can be deduced as in [21]. Hence, we omit the details.

Put $\mathcal{P}(u)\colon\ \mathbb{R}\to\mathbb{R}$ with $(\mathcal{P}u)(t) = \varphi(t)$ for $t \in (-\infty, t_0]$ and

(3.5)
$$(\mathcal{P}u)(t) = \mathrm{e}^{-\int_{t_0}^t \sum_{i=1}^N a_i(s,u_s)\,\mathrm{d}s}\varphi(t_0) + \int_{t_0}^t \mathrm{e}^{-\int_s^t \sum_{i=1}^N a_i(z,u_z)\,\mathrm{d}z}(g(s,u'_s) + f(s,u_s))\,\mathrm{d}s$$

for $t \geqslant t_0$. It is not difficult to see that $\mathcal{P}(u)\colon\ \mathbb{R}\to\mathbb{R}$ is continuous.

Initially, we show that $\mathcal{P}\colon\ D^l_\varphi \to D^l_\varphi$. In view of (3.5), we can derive

$$\begin{aligned}(3.6)\qquad (\mathcal{P}u)'(t) = &-\varphi(t_0)\sum_{i=1}^N a_i(t,u_t)\mathrm{e}^{-\int_{t_0}^t \sum_{i=1}^N a_i(s,u_s)\,\mathrm{d}s} + g(t,u'_t) + f(t,u_t)\\ &-\sum_{i=1}^N a_i(t,u_t)\int_{t_0}^t \mathrm{e}^{-\int_s^t \sum_{i=1}^N a_i(z,u_z)\,\mathrm{d}z}(g(s,u'_s)+f(s,u_s))\,\mathrm{d}s\\ = &-\sum_{i=1}^N a_i(t,u_t)(\mathcal{P}u)(t) + g(t,u'_t) + f(t,u_t)\end{aligned}$$

for $t \geqslant t_0$. By the definition of Φ_{t_0}, (3.6) yields

$$(\mathcal{P}u)'_+(t_0) = -\sum_{i=1}^N a_i(t_0,u_{t_0})\varphi(t_0) + g(t_0,u'_{t_0}) + f(t_0,u_{t_0}) = \varphi'_-(t_0).$$

Hence, $\mathcal{P}u \in C^1(\mathbb{R})$ for $u \in D^l_\varphi$.

Next, we verify that $\max_{t\geqslant t_0}\{\|(\mathcal{P}u)_t'\|_\circ, \|(\mathcal{P}u)_t\|_\circ\} < l$. Let

$$A = \sup_{t\geqslant t_0}\{|\alpha_2(t)|\} \quad \text{and} \quad K = \sup_{t\geqslant t_0} \mathrm{e}^{-\int_{t_0}^t \alpha_1(s)\,\mathrm{d}s}.$$

By conditions (3.4) and (3.1), $K, A \in [0,\infty)$ for a given small bounded initial function φ with $|\varphi|_{t_0} < \delta_0$, where $\delta_0 > 0$ satisfies

$$\delta_0 < l\min\Big\{1, \frac{1-\eta}{K}, \frac{1-2\eta}{KA}\Big\}. \tag{3.7}$$

Let $u \in D_\varphi^l$, then $\max_{t\geqslant t_0}\{\|u_t'\|_\circ, \|u_t\|_\circ\} \leqslant l$. Due to conditions (2.3), (2.4), (3.7), and (3.2),

$$\begin{aligned}|(\mathcal{P}u)(t)| &\leqslant |\varphi(t_0)|\mathrm{e}^{-\int_{t_0}^t \alpha_1(s)\,\mathrm{d}s} + \int_{t_0}^t \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}|b_2(s)|\|u_s'\|_\circ\,\mathrm{d}s\\ &\quad + \int_{t_0}^t \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}|b_1(s)|\|u_s\|_\circ\,\mathrm{d}s\\ &\leqslant K\delta_0 + \eta l < l.\end{aligned}$$

Now, (3.6) and (2.3), (2.4), (2.7) and (3.2), (3.3), (3.7) imply that

$$\begin{aligned}|(\mathcal{P}u)'(t)| &\leqslant |\varphi(t_0)|\sum_{i=1}^N |a_i(t,u_t)|\mathrm{e}^{-\int_{t_0}^t \sum_{i=1}^N a_i(s,u_s)\,\mathrm{d}s} + |g(t,u_t')| + |f(t,u_t)|\\ &\quad + \sum_{i=1}^N |a_i(t,u_t)|\int_{t_0}^t \mathrm{e}^{-\int_s^t \sum_{i=1}^N a_i(z,u_z)\,\mathrm{d}z}(|g(s,u_s')| + |f(s,u_s)|)\,\mathrm{d}s\\ &\leqslant KA\delta_0 + |g(t,u_t') - g(t,0)| + |f(t,u_t) - f(t,0)|\\ &\quad + l|\alpha_2(t)|\int_{t_0}^t \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\,\mathrm{d}s\\ &\leqslant KA\delta_0 + l(|b_1(t)| + |b_2(t)|) + \eta l\\ &\leqslant KA\delta_0 + 2\eta l < l\end{aligned}$$

by the choice of δ_0. This implies $\max_{t\geqslant t_0}\{|(\mathcal{P}u)(t)|, |(\mathcal{P}u)'(t)|\} < l$. We now show that $(\mathcal{P}u)(t)$ approaches zero as $t\to\infty$.

Owing to condition (3.4), we have

$$\lim_{t\to\infty} \mathrm{e}^{-\int_{t_0}^t \alpha_1(z)\,\mathrm{d}z} = 0.$$

Therefore, it is obvious that the first term of $(\mathcal{P}u)(t)$ tends to zero as $t\to\infty$ because of condition (3.4). Next, we show that the last term of $(\mathcal{P}u)(t)$ tends to zero, too. Since $\lim_{t\to\infty} u(t) = \lim_{t\to\infty} u'(t) = 0$, we can find $T_1 > t_0$ such that for all $t \geqslant T_1$, $\max\{|u(t)|, |u'(t)|\} < \varepsilon$, and the fact $u \in D^l_\varphi$ implies that for all $t \geqslant t_0$, $\max\{\|u_t\|_\circ, \|u'_t\|_\circ\} < l$. Therefore, it follows from (2.5) and (2.6) that we can find $t_2 > T_1$ such that

$$|f(t, u_t)| \leqslant |b_1(t)|(\varepsilon + \|u\|^{[T_1,t]})$$

and

$$|g(t, u'_t)| \leqslant |b_2(t)|(\varepsilon + \|u'\|^{[T_1,t]})$$

for $t \geqslant t_2$. Hence for $t \geqslant t_2$ we have

$$\begin{aligned}
&\left|\int_{t_0}^{t} \mathrm{e}^{-\int_s^t \sum_{i=1}^N a_i(z,u_z)\,\mathrm{d}z}(g(s,u'_s) + f(s,u_s))\,\mathrm{d}s\right| \\
&\qquad\leqslant \int_{t_0}^{t_2} \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}|g(s,u'_s) + f(s,u_s)|\,\mathrm{d}s \\
&\qquad\quad + \int_{t_2}^{t} \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}|g(s,u'_s) + f(s,u_s)|\,\mathrm{d}s \\
&\qquad\leqslant \int_{t_0}^{t_2} \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\max_{s\geqslant t_0}\{\|u'_s\|_\circ, \|u_s\|_\circ\}\,\mathrm{d}s \\
&\qquad\quad + \int_{t_2}^{t} \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}|b_1(s)|(\varepsilon + \|u'\|^{[T_1,s]})\,\mathrm{d}s \\
&\qquad\quad + \int_{t_2}^{t} \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}|b_2(s)|(\varepsilon + \|u\|^{[T_1,s]})\,\mathrm{d}s
\end{aligned}$$

since $\max\{\|u\|^{[T_1,t]}, \|u'\|^{[T_1,t]}\} \leqslant \varepsilon$ for $t \geqslant t_2$. Then,

$$\begin{aligned}
&\int_{t_0}^{t_2} \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\max_{s\geqslant t_0}\{|u'_s|, |u_s|\}\,\mathrm{d}s \\
&\qquad + 2\varepsilon\int_{t_2}^{t} \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\,\mathrm{d}s \\
&\qquad\leqslant l\int_{t_0}^{t_2} \mathrm{e}^{-\int_s^{t_2} \alpha_1(z)\,\mathrm{d}z}\mathrm{e}^{-\int_{t_2}^t \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\,\mathrm{d}s + 2\eta\varepsilon.
\end{aligned}$$

Using condition (3.4), we can find $T \geqslant t_2$ such that for $t \geqslant T$ we get

$$l\mathrm{e}^{-\int_T^t \alpha_1(z)\,\mathrm{d}z}\int_{t_0}^{t_2} \mathrm{e}^{-\int_s^T \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\,\mathrm{d}s \leqslant \varepsilon.$$

This yields $\lim_{t\to\infty} (\mathcal{P}u)(t) = 0$ for $u \in D^l_\varphi$.

Moreover, for each $u \in D^l_\varphi$, $\lim_{t\to\infty} u(t) = \lim_{t\to\infty} u'(t) = 0$, given $\varepsilon > 0$ there exists $T_2 > t_0$ such that for all $t \geqslant T_2$, $\max\{|u(t)|, |u'(t)|\} < \varepsilon$. By conditions (2.5), (2.6) we can find a $T' > T_2$ such that for $t \geqslant T'$ we have

$$|g(t, u'_t)| \leqslant |b_1(t)|(\varepsilon + \|u'\|^{[T_2,t]})$$

and

$$|f(t, u_t)| \leqslant |b_2(t)|(\varepsilon + \|u\|^{[T_2,t]}).$$

For $t \geqslant T'$, we have from (3.6)

$$\begin{aligned} |(\mathcal{P}u)'(t)| &\leqslant \sum_{i=1}^{N} |a_i(t, u_t)||(\mathcal{P}u)(t)| + |g(t, u'_t)| + |f(t, u_t)| \\ &\leqslant \sum_{i=1}^{N} |a_i(t, u_t)||(\mathcal{P}u)(t)| + |b_1(t)|(\varepsilon + \|u'\|^{[T_2,t]}) + |b_2(t)|(\varepsilon + \|u\|^{[T_2,t]}) \\ &\leqslant |\alpha_2(t)||(\mathcal{P}u)(t)| + 2\eta\varepsilon. \end{aligned}$$

This together with (3.1)–(3.3), leads to $\lim_{t\to\infty} (\mathcal{P}u)'(t) = 0$ for $u \in D^l_\varphi$. Therefore, $\mathcal{P}u \in D^l_\varphi$ for $u \in D^l_\varphi$, i.e. $\mathcal{P}\colon D^l_\varphi \to D^l_\varphi$.

We now show that $\mathcal{P}\colon D^l_\varphi \to D^l_\varphi$ is contractive. To this end, suppose that $u, y \in D^l_\varphi$. By conditions (2.3), (2.4), (3.2), (3.3), (3.6), then for $t \geqslant t_0$,

$$\begin{aligned} (3.8)\quad & |(\mathcal{P}u)(t) - (\mathcal{P}y)(t)| \\ &\leqslant \int_{t_0}^{t} e^{-\int_s^t \sum_{i=1}^N a_i(z,u_z)\,dz} (|g(s, u'_s) - g(s, y'_s)| + |f(s, u_s) - f(s, y_s)|)\,ds \\ &\leqslant \int_{t_0}^{t} e^{-\int_s^t \alpha_1(z)\,dz} |b_1(s)|\|u'_s - y'_s\|_\circ\,ds \\ &\quad + \int_{t_0}^{t} e^{-\int_s^t \alpha_1(z)\,dz} |b_2(s)|\|u_s - y_s\|_\circ\,ds \\ &\leqslant \int_{t_0}^{t} e^{-\int_s^t \alpha_1(z)\,dz} (|b_1(s)| + |b_2(s)|) \max_{s\geqslant t_0}\{\|u_s - y_s\|_\circ, \|u'_s - y'_s\|_\circ\}\,ds \\ &\leqslant \eta\|u - y\|. \end{aligned}$$

In addition,

$$\begin{aligned} (3.9)\quad & |(\mathcal{P}u)'(t) - (\mathcal{P}y)'(t)| \\ &\leqslant |\alpha_2(t)||(\mathcal{P}u)(t) - (\mathcal{P}y)(t)| + |g(t, u'_t) - g(t, y'_t)| + |f(t, u_t) - f(t, y_t)| \\ &\leqslant \|u - y\|\Big(|\alpha_2(t)| \int_{t_0}^{t} e^{-\int_s^t \alpha_1(z)\,dz} (|b_1(s)| + |b_2(s)|)\,ds + |b_1(t)| + |b_2(t)|\Big) \\ &\leqslant \eta\|u - y\|. \end{aligned}$$

From (3.8) and (3.9), as $0 < \eta < \frac{1}{2}$, $\mathcal{P}\colon\ D^l_\varphi \to D^l_\varphi$ is a contraction mapping. Hence there exists a unique fixed point u in D^l_φ which means u is a solution of (2.1) with the initial value (t_0, φ), bounded by l, and $\lim\limits_{t\to\infty} u(t) = \lim\limits_{t\to\infty} u'(t) = 0$ as $t \to \infty$.

The following step represents another way in which we can establish the stability of (2.1) by using Banach's fixed point method as the main tool. For comprehensive works published on the stability of some particular cases of the equation mentioned, the readers are referred to the papers by Raffoul [9] and Burton [10]. Let $\varepsilon > 0$ be given. By proceeding now in a different way than before, that is, replacing l by ε in D^l_φ, we obtain the existence of a sufficiently small $\delta > 0$ such that (3.7) is satisfied with $\delta_0 = \delta$. For $|\varphi| < \delta$ it leads to the unique solution u of (2.1) with $u_{t_0} = \varphi$ on $(-\infty, t_0]$ that satisfies $\max\limits_{t\geqslant t_0}\{|u(t)|, |u'(t)|\} < \varepsilon$. Moreover, $\lim\limits_{t\to\infty} u(t, t_0, \varphi) = \lim\limits_{t\to\infty} u'(t, t_0, \varphi) = 0$. We can therefore conclude that the trivial solution of (2.1) is asymptotically stable in C^1.

In the end, we proceed to show the asymptotic stability in C^1 of the trivial solution to equation (2.1). For all $\varepsilon > 0$, let $\delta > 0$ be such that

$$\delta < \varepsilon \min\Big\{1, \frac{1-\eta}{K}, \frac{1-\eta}{KA}\Big\}.$$

If $u(t) = u(t, t_0, \varphi)$ is a solution to equation (2.1) with $|\varphi|_{t_0} < \delta$, then $u(t) = (\mathcal{P}u)(t)$ on $[t_0, \infty)$. We claim that $\|u\| < \varepsilon$. Otherwise, there would exist $t^* > t_0$ such that

$$\max\{|u(t^*, t_0, \varphi)|, |u'(t^*, t_0, \varphi)|\} = \varepsilon \quad \text{and} \quad \max\{|u(t, t_0, \varphi)|, |u'(t, t_0, \varphi)|\} < \varepsilon$$

for $t \leqslant t^*$. If $|u(t^*, t_0, \varphi)| = \varepsilon$, then it follows from (3.5) and (2.3), (2.4), (3.2) that

$$\begin{aligned}
|u(t^*, t_0, \varphi)| = \bigg|&\varphi(t_0)\mathrm{e}^{-\int_{t_0}^{t^*}\sum_{i=1}^N a_i(s, u_s)\,\mathrm{d}s} \\
&+ \int_{t_0}^{t^*} \mathrm{e}^{-\int_{t_0}^{t^*}\sum_{i=1}^N a_i(z, u_z)\,\mathrm{d}z}(g(s, u'_s) + f(s, u_s))\,\mathrm{d}s\bigg| \\
\leqslant\ & |\varphi(t_0)|\mathrm{e}^{-\int_{t_0}^{t^*}\alpha_1(z)\,\mathrm{d}z} \\
&+ \int_{t_0}^{t^*} \mathrm{e}^{-\int_s^{t^*}\alpha_1(z)\,\mathrm{d}z}(|g(s, u'_s) - g(s, 0)| + |f(s, u_s) - f(s, 0)|)\,\mathrm{d}s \\
\leqslant\ & \delta_0 \mathrm{e}^{-\int_{t_0}^{t^*}\alpha_1(z)\,\mathrm{d}z} + \int_{t_0}^{t^*} \mathrm{e}^{-\int_s^{t^*}\alpha_1(z)\,\mathrm{d}z}(|b_1(s)|\|u'_s\|_\circ + |b_2(s)|\|u_s\|_\circ)\,\mathrm{d}s \\
\leqslant\ & K\delta + \eta\varepsilon < \varepsilon,
\end{aligned}$$

which contradicts $|u(t^*, t_0, \varphi)| = \varepsilon$.

If $|u'(t^*,t_0,\varphi)|=\varepsilon$, then it follows from (3.6), (2.3), (2.4), (3.3) that

$$\begin{aligned}|u'(t^*,t_0,\varphi)| &\leqslant |\varphi(t_0)||\alpha_2(t^*)|\mathrm{e}^{-\int_{t_0}^{t^*}\sum_{i=1}^N a_i(s,u_s)\,\mathrm{d}s}+|g(t^*,u'_{t^*})|+|f(t^*,u_{t^*})|\\ &\quad+|\alpha_2(t^*)|\int_{t_0}^{t^*}\mathrm{e}^{-\int_s^{t^*}\alpha_1(z)\,\mathrm{d}z}(|g(s,u'_s)|+|f(s,u_s)|)\,\mathrm{d}s\\ &\leqslant KA\delta+\varepsilon|\alpha_2(t^*)|\int_{t_0}^{t^*}\mathrm{e}^{-\int_s^{t^*}\alpha_1(z)\,\mathrm{d}z}(|b_1(s)|+|b_2(s)|)\,\mathrm{d}s\\ &\quad+|b_1(t^*)|+|b_2(t^*)|\\ &\leqslant KA\delta+\eta\varepsilon<\varepsilon,\end{aligned}$$

which contradicts $|u'(t^*,t_0,\varphi)|=\varepsilon$ as well. Thus, $\max\{|u(t)|,|u'(t)|\}<\varepsilon$ for all $t\geqslant t_0$ and the zero solution of equation (2.1) is stable in C^1. Combined with the fact that

$$\lim_{t\to\infty}u(t)=\lim_{t\to\infty}u'(t)=0,$$

the zero solution of (2.1) is asymptotically stable in C^1 if (3.4) holds. □

Theorem 3.2. *Suppose that conditions* (2.3)–(2.7) *and* (3.1)–(3.3) *hold for* (2.1). *If the trivial solution of* (2.1) *is globally asymptotically stable in* C^1, *then*

$$\lim_{t\to\infty}\int_0^t\alpha_2(s)\,\mathrm{d}s=\infty. \tag{3.10}$$

P r o o f. Arguing by contradiction, suppose condition (3.10) fails. Then (3.1) implies that $\liminf_{t\to\infty}\int_0^t\alpha_2(s)\,\mathrm{d}s>-\infty$ and we find a sequence $\{t_n\}\subset[0,\infty)$, $t_n\to\infty$ as $n\to\infty$, such that

$$\lim_{n\to\infty}\int_0^{t_n}\alpha_2(s)\,\mathrm{d}s=F\quad\text{for each }F\in\mathbb{R}^+.$$

We can also select a constant $q\in\mathbb{R}^+$ such that

$$-q\leqslant\int_0^{t_n}\alpha_2(s)\,\mathrm{d}s\leqslant+q,\quad n=1,2,\ldots$$

Set

$$\overset{0}{K}=\sup_{t\geqslant t_0}\mathrm{e}^{-\int_{t_0}^t\alpha_1(s)\,\mathrm{d}s}\quad\text{and}\quad\overset{0}{A}=\sup_{t\geqslant t_0}\{|\alpha_2(t)|\},\quad J=\liminf_{t\to\infty}\int_0^t\alpha_1(s)\,\mathrm{d}s.$$

Hence, it follows from (3.1) that $J\in\mathbb{R}$, $\overset{0}{K},\overset{0}{A}\in\mathbb{R}^+$.

Since (3.10) fails, then the statement that $\int_0^t \alpha_1(s)\,\mathrm{d}s$ tends to ∞ as $t \to \infty$ fails, too. By (3.1), for the sequence $\{t_n\}$ defined above, one can choose $J \in \mathbb{R}^+$ such that

$$-J \leqslant \int_0^{t_n} \alpha_1(s)\,\mathrm{d}s \leqslant +J, \quad n = 1, 2, \ldots \tag{3.11}$$

Put

$$I_n = \int_0^{t_n} \mathrm{e}^{\int_0^s \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\,\mathrm{d}s, \quad n = 1, 2, \ldots$$

But, in view of condition (3.2) we have

$$I_n = \int_0^{t_n} \mathrm{e}^{\int_0^s \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\,\mathrm{d}s \leqslant \eta.$$

From (3.11), it then follows that

$$I_n = \mathrm{e}^{\int_0^{t_n} \alpha_1(z)\,\mathrm{d}z} \int_0^{t_n} \mathrm{e}^{\int_0^s \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\,\mathrm{d}s \leqslant \eta \mathrm{e}^{\int_0^{t_n} \alpha_1(z)\,\mathrm{d}z} < \mathrm{e}^J.$$

Therefore the sequence $\{I_n\}$ is bounded. Thus, the sequence $\{I_n\}$ has a convergent subsequence. Without loss of generality, we can assume that

$$\lim_{n\to\infty} \int_0^{t_n} \mathrm{e}^{\int_0^s \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\,\mathrm{d}s = \mu \quad \text{for some } \mu \in \mathbb{R}^+.$$

Let m be an integer such that

$$\int_{t_m}^{t_n} \mathrm{e}^{\int_0^s \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\,\mathrm{d}s < \frac{1-\eta}{4B\mathrm{e}^{2q}(\mathrm{e}^{-J}+1)} \tag{3.12}$$

and

$$\mathrm{e}^{-\int_{t_m}^{t_n} \alpha_1(z)\,\mathrm{d}z} > \frac{1}{2}, \quad \mathrm{e}^{-\int_0^{t_n} \alpha_1(z)\,\mathrm{d}z} < \mathrm{e}^{-J} + 1, \quad \mathrm{e}^{\int_0^{t_m} \alpha_1(z)\,\mathrm{d}z} < \mathrm{e}^J + 1 \tag{3.13}$$

for all $n > m$, where

$$B = \max\bigl\{\overset{0}{K}(\mathrm{e}^J + 1), \overset{0}{K}\overset{0}{A}(\mathrm{e}^J + 1), 1\bigr\}.$$

For any $\delta_0 > 0$, we consider $u(t) = u(t, t_m, \varphi)$ to be the solution of (2.1) with $|\varphi|_{t_m} < \delta_0$ and $|\varphi(t_m)| > \delta_0/2$ for $t < t_m$. It therefore follows from (3.5), (3.6), (3.13) and (3.1)–(3.3), that for $t \in [t_m, \infty)$,

$$\begin{aligned}
|u(t)| &\leqslant \delta_0 \mathrm{e}^{-\int_{t_m}^t \alpha_1(s)\,\mathrm{d}s} + \int_{t_m}^t \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}(|g(s, u_s')| + |f(s, u_s)|)\,\mathrm{d}s \\
&\leqslant \overset{0}{K}(\mathrm{e}^J + 1)\delta_0 + \|u\|_{t_m} \int_{t_m}^t \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\,\mathrm{d}s \\
&\leqslant B\delta_0 + \eta\|u\|_{t_m}
\end{aligned}$$

and

$$
\begin{aligned}
|u'(t)| &\leqslant |u(t_m)||\alpha_2(t)|\mathrm{e}^{-\int_{t_m}^{t}\alpha_1(s)\,\mathrm{d}s} + |g(t,u'_t)| + |f(t,u_t)| \\
&\quad + |\alpha_2(t)|\int_{t_m}^{t}\mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}(|g(s,u'_s)| + |f(s,u_s)|)\,\mathrm{d}s \\
&\leqslant \overset{0}{K}\overset{0}{A}(\mathrm{e}^{J}+1)\delta_0 \\
&\quad + \|u\|_{t_m}\Big(|\alpha_2(t)|\int_{t_m}^{t}\mathrm{e}^{-\int_s^t\alpha_1(z)\,\mathrm{d}z}(|b_1(s)|+|b_2(s)|)\,\mathrm{d}s + (|b_1(t)|+|b_2(t)|)\Big) \\
&\leqslant B\delta_0 + \eta\|u\|_{t_m}.
\end{aligned}
$$

Hence, $\|u\|_{t_m} \leqslant B\delta_0 + \eta\|u\|_{t_m}$, thus we have

$$
\|u\|_{t_m} \leqslant \frac{B}{1-\eta}\delta_0 \quad \forall\, t \geqslant t_m. \tag{3.14}
$$

It then follows from (3.5), (3.12)–(3.14) and (2.3), (2.4), (2.7) that for any $n > m$

$$
\begin{aligned}
|u(t_n)| &\geqslant |\varphi(t_m)|\mathrm{e}^{-\int_{t_m}^{t_n}\alpha_2(s)\,\mathrm{d}s} - \left|\int_{t_m}^{t_n}\mathrm{e}^{-\int_s^{t_n}\sum_{i=1}^{N}a_i(z,u_z)\,\mathrm{d}z}(g(s,u'_s)+f(s,u_s))\,\mathrm{d}s\right| \\
&\geqslant \delta_0\mathrm{e}^{-\int_{t_m}^{t_n}\alpha_2(s)\,\mathrm{d}s} - \int_{t_m}^{t_n}\mathrm{e}^{-\int_s^{t_n}\sum_{i=1}^{N}a_i(z,u_z)\,\mathrm{d}z}|g(s,u'_s)+f(s,u_s)|\,\mathrm{d}s \\
&\geqslant \delta_0\mathrm{e}^{-\int_{t_m}^{t_n}\alpha_2(s)\,\mathrm{d}s} - \|u\|_{t_m}\mathrm{e}^{-\int_0^{t_n}\alpha_1(z)\,\mathrm{d}z}\int_{t_m}^{t_n}\mathrm{e}^{\int_0^s\alpha_1(z)\,\mathrm{d}z}(|b_1(s)|+|b_2(s)|)\,\mathrm{d}s \\
&\geqslant \delta_0\mathrm{e}^{-\int_{t_m}^{t_n}\alpha_2(s)\,\mathrm{d}s} - \|u\|_{t_m}\mathrm{e}^{-\int_0^{t_n}\alpha_1(z)\,\mathrm{d}z}\int_{t_m}^{t_n}\mathrm{e}^{\int_0^s\alpha_1(z)\,\mathrm{d}z}(|b_1(s)|+|b_2(s)|)\,\mathrm{d}s.
\end{aligned}
$$

But

$$
\mathrm{e}^{-\int_{t_m}^{t_n}\alpha_2(z)\,\mathrm{d}s} = \mathrm{e}^{\int_{t_n}^{0}\alpha_2(z)\,\mathrm{d}z}\mathrm{e}^{\int_0^{t_m}\alpha_2(z)\,\mathrm{d}z} = \mathrm{e}^{-\int_0^{t_n}\alpha_2(z)\,\mathrm{d}z}\mathrm{e}^{\int_0^{t_m}\alpha_2(z)\,\mathrm{d}z} \geqslant \mathrm{e}^{-2q}
$$

and $\mathrm{e}^{-\int_0^{t_n}\alpha_1(z)\,\mathrm{d}z} \leqslant \mathrm{e}^{-J}+1$, which implies

$$
|u(t_n)| \geqslant \frac{1}{2}\delta_0\mathrm{e}^{-2q} - \frac{\delta_0 B}{1-\eta}(\mathrm{e}^{-J}+1)\frac{1-\eta}{4B\mathrm{e}^{2q}(\mathrm{e}^{-J}+1)} = \frac{1}{2}\delta_0\mathrm{e}^{-2q}. \tag{3.15}
$$

The facts that $\lim\limits_{n\to\infty} t_n = \infty$ and the trivial solution of (2.1) is asymptotically stable in C^1 imply $\lim\limits_{n\to\infty} u(t,t_n,\varphi) = \lim\limits_{n\to\infty} u'(t,t_n,\varphi) = 0$, which is in contradiction with (3.15). The proof of Theorem 3.2 is complete. □

Corollary 3.1. *Assume that* (**A1**)–(**A6**) *hold, and for any* $t \geqslant t_0$*, if there is an* $\eta \in (0, \frac{1}{2})$ *such that*

$$\liminf_{t\to\infty} \int_{t_0}^{t} \alpha_1(s)\,\mathrm{d}s > -\infty$$

and

$$\int_{t_0}^{t} \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\,\mathrm{d}s \leqslant \eta,$$

then the zero solution to equation (2.1) *is asymptotically stable in* C^0 *if*

$$\int_{t_0}^{t} \alpha_1(s)\,\mathrm{d}s \to \infty \quad \text{as } t \to \infty.$$

For equation (2.1), we also have

Corollary 3.2. *Suppose that* (**A1**)–(**A6**) *and* (3.1), (3.2) *hold. If the trivial solution of* (2.1) *is asymptotically stable in* C^0*, then we get*

$$\int_{t_0}^{t} \alpha_2(s)\,\mathrm{d}s \to \infty \quad \text{as } t \to \infty.$$

Remark 3.1. Regarding Ziad et al. [23], Corollary 3.1 and Corollary 3.2 are natural generalizations of Theorem 3.1 and Theorem 3.2 in [23], respectively. In fact, when $g(t, u_t') = 0$ our conditions reduce to those of Ziad et al. (see [23]).

Now we consider the standard form of totally nonlinear neutral differential equations

$$u'(t) = -h(t, u(t)) + g(t, u_t') + f(t, u_t), \quad t \geqslant t_0. \tag{3.16}$$

Similarly to equation (2.1), if we assume that
(**A7**) $h(t, 0)$ and there exist $\alpha_1, \alpha_2 \in C(\mathbb{R}, \mathbb{R})$ such that

$$\alpha_1(t) \leqslant \frac{\partial h(t, u)}{\partial u} \leqslant \alpha_2(t),$$

then we can get the following theorem.

Theorem 3.3. *Assume that* (**A1**)–(**A7**) *and* (3.1)–(3.4) *hold, then the trivial solution to* (3.16) *is asymptotically stable in* C^1.

Proof. For any $h \in C^1$, since $h(t,0)=0$ it is straightforward to see that

$$h(t,u)=\left(\int_0^1 \frac{\partial h(t,su)}{\partial u}\,\mathrm{d}s\right)u.$$

If we set

$$\sum_{i=1}^N a_i(t,u_t)=\int_0^1 \frac{\partial h(t,su)}{\partial u}\,\mathrm{d}s,$$

then we can rewrite (2.1) as (3.16) with

$$\alpha_1(t)\leqslant \sum_{i=1}^N a_i(t,u_t)\leqslant \alpha_2(t).$$

Then the claim is true thanks to Theorem 3.1. □

In addition, we derive another result for equation (3.16).

Theorem 3.4. *If conditions* (**A1**)–(**A6**) *and* (3.1)–(3.3) *are fulfilled, then the zero solution of* (3.16) *with a small initial function is asymptotically stable in* C^1. *If the zero solution of* (3.16) *is globally asymptotically stable in* C^1, *then*

$$\int_0^t \alpha_2(t)\to\infty \quad \text{as } t\to\infty$$

holds.

Choosing $N=1$ and $a_1(t,u_t)=a(t)$ in Theorem 3.1, we have the following result.

Corollary 3.3. *Assume that* (**A1**)–(**A6**) *hold and for any* $t\geqslant t_0$, *there exists a constant* $\eta\in(0,\frac{1}{2})$ *such that*

$$\liminf_{t\to\infty}\int_{t_0}^t a(s)\,\mathrm{d}s>-\infty, \tag{3.17}$$

$$\int_{t_0}^t \mathrm{e}^{-\int_s^t a(z)\,\mathrm{d}z}(|b_1(s)|+|b_2(s)|)\,\mathrm{d}s\leqslant\eta, \tag{3.18}$$

$$|a(t)|\int_{t_0}^t \mathrm{e}^{-\int_s^t a(z)\,\mathrm{d}z}(|b_1(s)|+|b_2(s)|)\,\mathrm{d}s+(|b_1(t)|+|b_2(t)|)\leqslant\eta. \tag{3.19}$$

Then, the trivial solution to equation (2.1) *is asymptotically stable in* C^1 *if only if*

$$\int_{t_0}^t a(s)\,\mathrm{d}s\to\infty \quad \text{as } t\to\infty.$$

R e m a r k 3.2. Theorem 3.1 remains true if conditions (3.2), (3.3) are fulfilled for all $t \geqslant t_\sigma$ with some $t_\sigma \in \mathbb{R}^+$.

4. Remarks and illustrative examples

Let us discuss two examples to illustrate our abstract theory.

E x a m p l e 4.1. Let us consider the nonlinear neutral differential equation

$$u'(t) = -a(t, u(t-\tau(t)))u(t) + f(t, u(t-\tau(t))) + g(t, u'(t-\tau(t))), \tag{4.1}$$

$t \geqslant 0$, where

$$\begin{aligned} a(t, u(t-\tau(t))) &= \frac{1}{1+t}\Big(1 + \frac{|\sin t|}{1+u^2(t-\tau(t))}\Big), \\ g(t, u'(t-\tau(t))) &= \frac{0.1}{1+t}\sin\frac{u'(t-\tau(t))}{10}, \\ f(t, u(t-\tau(t))) &= 0.4\ln\Big(1 + \frac{|u(t-\tau(t))|}{10(1+t)}\Big). \end{aligned}$$

One can take $\alpha_1(t) = (1+t)^{-1}$ and $\alpha_2(t) = 2|\sin t|/(1+t)$, then $\alpha_1(t) \leqslant a(t, u_t) \leqslant \alpha_2(t)$.It is easy to check that

$$|\alpha_2(t)| < 2 \quad \forall\, t \in [0,\infty), \qquad \int_0^t \alpha_1(s)\,\mathrm{d}s \to \infty \quad \text{as } t \to \infty.$$

By straightforward computation, we can check that conditions (2.2) and (2.3) of Theorem 3.1 hold true, where $\tau, \delta \in C(\mathbb{R}^+, \mathbb{R}^+)$ with

$$t - \tau(t) \to \infty \quad \text{and} \quad t - \delta(t) \to \infty \quad \text{as } t \to \infty. \tag{4.2}$$

Assume that $b_1(t) = 0.1/(2(1+t))$ and $b_2(t) = 0.5/(10(1+t))$. Then (2.3), (2.4) hold. Also assume that $\eta = \frac{1}{3}$, then for $t \in [0,\infty)$

$$\begin{aligned} &\int_0^t \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\,\mathrm{d}s \\ &\qquad \leqslant \int_0^t \mathrm{e}^{-\int_s^t (1+z)^{-1}\,\mathrm{d}z}\frac{1}{10(1+s)}\,\mathrm{d}s = \frac{1}{10} \leqslant \eta \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &|\alpha_2(t)|\int_0^t \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z}(|b_1(s)| + |b_2(s)|)\,\mathrm{d}s + (|b_1(t)| + |b_2(t)|) \\ &\qquad \leqslant 2\int_0^t \mathrm{e}^{-\int_s^t (1+z)^{-1}\,\mathrm{d}z}\frac{1}{10(1+s)}\,\mathrm{d}s + \frac{1}{10(1+t)} = \frac{3}{10} \leqslant \eta. \end{aligned} \tag{4.4}$$

Hence, all the conditions of Theorem 3.1 are fulfilled. Therefore, the zero solution to equation (4.1) is asymptotically stable in C^1.

E x a m p l e 4.2. Consider the following equation in the form (2.1),

$$u'(t) = -\sum_{i=1}^{2} a_i(t, u(t-\tau(t)))u(t) + f(t, u(t-\tau_1(t)), u(t-\tau_2(t))) + g(t, u'(t-\tau_1(t)), u'(t-\tau_2(t))) \tag{4.5}$$

and put

$$a_1(t,u) = \frac{0.5\mathrm{e}^t}{1+\mathrm{e}^t}\Big(1+\frac{|\cos t|}{1+\mathrm{e}^{-u^2}}\Big), \quad a_2(t,u)) = \frac{0.5\mathrm{e}^t}{1+\mathrm{e}^t}\Big(1+\frac{|\sin u|}{2}\Big),$$

$\tau \in C(\mathbb{R}^+, \mathbb{R}^+)$, and $\tau_i \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$t - \tau_i(t) \to \infty \quad \text{as } t \to \infty, \ i = 1, 2. \tag{4.6}$$

By simple calculation, we have

$$\alpha_1(t) := \frac{\mathrm{e}^t}{1+\mathrm{e}^t} \leqslant \sum_{i=1}^{2} a_i(t, u(t-\tau(t))) \leqslant \frac{1.75\mathrm{e}^t}{1+\mathrm{e}^t} =: \alpha_2(t)$$

and it is straightforward to check that

$$|\alpha_2(t)| < 1.75 \quad \forall\, t \in [0,\infty) \quad \text{and} \quad \int_0^t \alpha_1(s)\,\mathrm{d}s \to \infty \quad \text{as } t \to \infty.$$

Let

$$f(t,u_1,u_2) = \ln\Big(1+\frac{5(|u_1|+|u_2|)}{100(1+\mathrm{e}^{-t})}\Big),$$
$$g(t,u_1,u_2) = 0.1\sin\frac{u_1}{5(1+\mathrm{e}^{-t})} + 0.12\sin\frac{u_2}{4(1+\mathrm{e}^{-t})},$$

then we obtain

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leqslant |b_1(t)||u_1-v_1| + |b_2(t)||u_2-v_2|,$$
$$|g(t,u_1,u_2) - g(t,v_1,v_2)| \leqslant |c_1(t)||u_1-v_1| + |c_2(t)||u_2-v_2|,$$

where

$$b_1(t) = b_2(t) = \frac{5}{100(1+\mathrm{e}^{-t})}$$

and

$$c_1(t) = \frac{0.02}{1+\mathrm{e}^{-t}}, \quad c_2(t) = \frac{0.03}{1+\mathrm{e}^{-t}}.$$

Then (**A1**)–(**A6**) hold. In addition, let $\eta = \frac{4}{9}$, then for $t \in [0, \infty)$,

$$\begin{aligned} (4.7) \qquad & \int_0^t \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z} \sum_{j=1}^{2} |b_j(s)| + |c_j(s)|\,\mathrm{d}s \\ & < \int_0^t \mathrm{e}^{-\int_s^t (\mathrm{e}^z/(1+\mathrm{e}^z))\,\mathrm{d}z} \Big(\frac{\mathrm{e}^s}{10(1+\mathrm{e}^s)} + \frac{0.05\mathrm{e}^s}{1+\mathrm{e}^s} \Big)\,\mathrm{d}s < 0.15 < \eta \end{aligned}$$

and

$$\begin{aligned} (4.8) \quad & |\alpha_2(t)| \int_0^t \mathrm{e}^{-\int_s^t \alpha_1(z)\,\mathrm{d}z} \sum_{j=1}^{2} (|b_j(s)| + |c_j(s)|)\,\mathrm{d}s + \sum_{j=1}^{2} (|b_j(t)| + |c_j(t)|) \\ & < 1.75 \times \int_0^t \mathrm{e}^{-\int_s^t \mathrm{e}^z(1+\mathrm{e}^z)^{-1}\,\mathrm{d}z} \Big(\frac{\mathrm{e}^s}{10(1+\mathrm{e}^s)} + \frac{0.05\mathrm{e}^s}{1+\mathrm{e}^s} \Big)\,\mathrm{d}s \\ & \quad + \frac{\mathrm{e}^t}{10(1+\mathrm{e}^t)} + \frac{0.05\mathrm{e}^t}{1+\mathrm{e}^t} \\ & < 1.75 \times 0.15 + \frac{\mathrm{e}^t}{10(1+\mathrm{e}^t)} + \frac{0.05\mathrm{e}^t}{1+\mathrm{e}^t} < 1.75 \times 0.15 + 0.15 = 0.413 \leqslant \eta. \end{aligned}$$

Hence, (3.2) and (3.3) hold. According to Theorem 3.1, the zero solution of equation (4.5) is globally asymptotically stable in C^1.

Remark 4.1. Theorem 3.1 includes and generalizes the result of Ardjouni and Djoudi, see [3]. In fact, when we choose $N = 1$ and $a_1(t, u_t) = a(t)$ (a is bounded), $g(t, u'_t) = g(t, u'(t - \tau_1(t)), u'(t - \tau_2(t)), \ldots, u'(t - \tau_n(t)))$ and $f(t, u_t) = f(t, u(t - \tau_1(t)), u(t - \tau_2(t)), \ldots, u(t - \tau_n(t)))$, our conditions reduce to those of Ardjouni and Djoudi (see [3], Theorem 2.1).

Remark 4.2. It has been noted in [26] that a fading memory condition such as (2.5), (2.6) or (4.6) is necessary for the asymptotic behavior of a general neutral differential equation. This means that the equation representing a physical system has to remember its past, but the memory has to fade over time.

Conclusion

In this work, a standard totally nonlinear neutral differential equation has been studied. Based on the Banach fixed point theorem, some new sufficient conditions ensuring the global asymptotic stability in C^1 of the trivial solution to equation (2.1) have been established. The main contribution of this paper confirms the importance and advantage of using the fixed point theory. The derived stability criteria are easy to apply in practice and do not need the differentiability of the delays or coefficients, which are required in [17]. Moreover, we can easily see that Theorem 3.1 and the corollaries cited above are independent of some restrictive conditions in reference [17]. Up to now, the results derived here have not been published in the corresponding literature. Illustrative examples are given to show the efficiency of the results introduced. Hence, in future, we would like to extend the application of this precise approach to more complex delay models such as the equations with damped stochastic perturbations and other variants.

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