EQUIVALENCE BUNDLES OVER A FINITE GROUP AND STRONG MORITA EQUIVALENCE FOR UNITAL INCLUSIONS OF UNITAL C^*-ALGEBRAS

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Abstract. Let $\mathcal{A} = \{A_t\}_{t\in G}$ and $\mathcal{B} = \{B_t\}_{t\in G}$ be C^*-algebraic bundles over a finite group G. Let $C = \bigoplus\limits_{t\in G} A_t$ and $D = \bigoplus\limits_{t\in G} B_t$. Also, let $A = A_e$ and $B = B_e$, where e is the unit element in G. We suppose that C and D are unital and A and B have the unit elements in C and D, respectively. In this paper, we show that if there is an equivalence $\mathcal{A}-\mathcal{B}$-bundle over G with some properties, then the unital inclusions of unital C^*-algebras $A \subset C$ and $B \subset D$ induced by $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent. Also, we suppose that $\mathcal{A}$ and $\mathcal{B}$ are saturated and that $A' \cap C = \mathbf{C}1$. We show that if $A \subset C$ and $B \subset D$ are strongly Morita equivalent, then there are an automorphism f of G and an equivalence bundle $\mathcal{A} - \mathcal{B}^f$-bundle over G with the above properties, where $\mathcal{B}^f$ is the C^*-algebraic bundle induced by $\mathcal{B}$ and f, which is defined by $\mathcal{B}^f = \{B_{f(t)}\}_{t\in G}$. Furthermore, we give an application.

Keywords: C^*-algebraic bundle; equivalence bundle; inclusions of C^*-algebra; strong Morita equivalence

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1. Introduction

Let $\mathcal{A} = \{A_t\}_{t\in G}$ be a C^*-algebraic bundle over a finite group G. Let $C = \bigoplus\limits_{t\in G} A_t$ and $A_e = A$, where e is the unit element in G. We suppose that C is unital and that A has the unit element in C. Then we obtain a unital inclusion of unital C^*-algebras, $A \subset C$. We call it the *unital inclusion of unital C^*-algebras induced by a C^*-algebraic bundle $\mathcal{A} = \{A_t\}_{t\in G}$*. Let E^A be the canonical conditional expectation from C onto A defined by

$$E^A(x) = x_e \quad \text{for all } x = \sum_{t\in G} x_i \in C.$$

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Definition 1.1. Let $\mathcal{A} = \{A_t\}_{t\in G}$ be a C^*-algebraic bundle over a finite group G. We say that $\mathcal{A}$ is *saturated* if $\overline{A_t A_t^*} = A$ for all $t \in G$.

Since A is unital, in our case we do not need to take the closure in Definition 1.1. If $\mathcal{A}$ is saturated, by [9], Corollary 3.2, E^A is of index-finite type and its Watatani index $\mathrm{Ind}_W(E^A) = |G|$, where $|G|$ is the order of G.

Let $\mathcal{B} = \{B_t\}_{t\in G}$ be another C^*-algebraic bundle over G. Let $D = \bigoplus_{t\in G} B_t$ and $B = B_e$. Also, we suppose that $\mathcal{B}$ has the same conditions as $\mathcal{A}$. Let $B \subset D$ be the unital inclusion of unital C^*-algebras induced by $\mathcal{B}$.

Let $\mathcal{X} = \{X_t\}_{t\in G}$ be an $\mathcal{A} - \mathcal{B}$-equivalence bundle defined by Abadie and Ferraro (see [1], Definition 2.2). Moreover, we suppose that

$$ {}_C\langle X_t, X_s\rangle = A_{ts^{-1}}, \quad \langle X_t, X_s\rangle_D = B_{t^{-1}s} $$

for any $t, s \in G$, where ${}_C\langle X_t, X_s\rangle$ means the linear span of the set

$$ \{{}_C\langle x, y\rangle \in A_{ts^{-1}} : \ x \in X_t, \ y \in X_s\} $$

and $\langle X_t, X_s\rangle_D$ means the linear span of the similar set to the above. The above two properties are stronger than properties (7R) and (7L) in [1], Definition 2.1.

In the present paper, we show that if there is an $\mathcal{A} - \mathcal{B}$-equivalence bundle $\mathcal{X} = \{X_t\}_{t\in G}$ such that ${}_C\langle X_t, X_s\rangle = A_{ts^{-1}}$ and $\langle X_t, X_s\rangle_D = B_{t^{-1}s}$ for any $t, s \in G$, then the unital inclusions of unital C^*-algebras $A \subset C$ and $B \subset D$ induced by $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent. Also, we suppose that $\mathcal{A}$ and $\mathcal{B}$ are saturated and that $A' \cap C = \mathbf{C}1$. We show that if $A \subset C$ and $B \subset D$ are strongly Morita equivalent, then there are an automorphism f of G and an $\mathcal{A} - \mathcal{B}^f$-equivalence bundle $\mathcal{X} = \{X_t\}_{t\in G}$ such that ${}_C\langle X_t, X_s\rangle = A_{ts^{-1}}$ and $\langle X_t, X_s\rangle_D = B_{f(t^{-1}s)}$ for any $t, s \in G$, where $\mathcal{B}^f$ is the C^*-algebraic bundle induced by $\mathcal{B} = \{B_t\}_{t\in G}$ and f, which is defined by $\mathcal{B}^f = \{B_{f(t)}\}_{t\in G}$.

Let A and B be unital C^*-algebras and X an $A - B$-equivalence bimodule. Then we denote its left A-action and right B-action on X by $a \cdot x$ and $x \cdot b$ for any $a \in A$, $b \in B$ and $x \in X$, respectively. Also, we mean by the words "Hilbert C^*-bimodules" Hilbert C^*-bimodules in the sense of Brown, Mingo and Shen, see [3].

2. Equivalence bundles over a finite group

Let $\mathcal{A} = \{A_t\}_{t\in G}$ and $\mathcal{B} = \{B_t\}_{t\in G}$ be C^*-algebraic bundles over a finite group G. Let e be the unit element in G. Let $C = \bigoplus_{t\in G} A_t$, $D = \bigoplus_{t\in G} B_t$ and $A = A_e$, $B = B_e$. We suppose that C and D are unital and that A and B have the unit elements in C and D, respectively. Let $\mathcal{X} = \{X_t\}_{t\in G}$ be an $\mathcal{A}-\mathcal{B}$-equivalence bundle over G such that

$$ {}_C\langle X_t, X_s\rangle = A_{ts^{-1}}, \quad \langle X_t, X_s\rangle_D = B_{t^{-1}s} $$

for any $t, s \in G$. Let $Y = \bigoplus_{t\in G} X_t$ and $X = X_e$. Then Y is a $C - D$-equivalence bimodule by Abadie and Ferraro (see [1], Definitions 2.1 and 2.2). Also, X is an $A - B$-equivalence bimodule since ${}_C\langle X, X\rangle = A$ and $\langle X, X\rangle_D = B$.

Proposition 2.1. *Let $\mathcal{A} = \{A_t\}_{t\in G}$ and $\mathcal{B} = \{B_t\}_{t\in G}$ be C^*-algebraic bundles over a finite group G. Let $C = \bigoplus_{t\in G} A_t$ and $D = \bigoplus_{t\in G} B_t$. Also, let $A = A_e$ and $B = B_e$, where e is the unit element in G. We suppose that C and D are unital and that A and B have the unit elements in C and D, respectively. Also, we suppose that there is an $\mathcal{A}-\mathcal{B}$-equivalence bundle $\mathcal{X} = \{X_t\}_{t\in G}$ over G such that*

$$ {}_C\langle X_t, X_s\rangle = A_{ts^{-1}}, \quad \langle X_t, X_s\rangle_D = B_{t^{-1}s} $$

for any $t, s \in G$. Then the unital inclusions of unital C^*-algebras $A \subset C$ and $B \subset D$ are strongly Morita equivalent.

Proof. Let $Y = \bigoplus_{t\in G} X_t$ and $X = X_e$. By the above discussions and [10], Definition 2.1, we only have to show that

$$ {}_C\langle Y, X\rangle = C, \quad \langle Y, X\rangle_D = D. $$

Let $x \in X$ and $y = \sum_{t\in G} y_t \in Y$, where $y_t \in X_t$ for any $t \in G$. Then

$$ {}_C\langle y, x\rangle = \sum_{t\in G} {}_C\langle y_t, x\rangle, \quad \langle y, x\rangle_D = \sum_{t\in G} \langle y_t, x\rangle_D. $$

We note that ${}_C\langle y_t, x\rangle \in A_t$ and $\langle y_t, x\rangle_D \in B_t$ for any $t \in G$. Since ${}_D\langle X_t, X_s\rangle = A_{ts^{-1}}$ and $\langle X_t, X_s\rangle_D = B_{t^{-1}s}$ for any $t, s \in G$, by the above computations, we can see that

$$ {}_C\langle Y, X\rangle = C, \quad \langle Y, X\rangle_D = D. $$

Therefore we obtain the conclusion. $\square$

Next, we give an example of an equivalence bundle $\mathcal{X} = \{X_t\}_{t\in G}$ over G satisfying the above properties. In order to do this, we prepare a lemma. Let $\mathcal{A} = \{A_t\}_{t\in G}$ and $\mathcal{B} = \{B_t\}_{t\in G}$ be as above. Let $\mathcal{X} = \{X_t\}_{t\in G}$ be a complex Banach bundle over G with the maps defined by

$$\begin{aligned} &(y, d) \in Y \times D \mapsto y \cdot d \in Y, \quad (y, z) \in Y \times Y \mapsto \langle y, z\rangle_D \in D, \\ &(c, y) \in C \times Y \mapsto c \cdot y \in Y, \quad (y, z) \in Y \times Y \mapsto {}_C\langle y, z\rangle \in C, \end{aligned}$$

where $Y = \bigoplus\limits_{t\in G} X_t$.

Lemma 2.2. *With the above notation, we suppose that by the above maps, Y is a $C - D$-equivalence bimodule satisfying that*

$${}_C\langle X_t, X_s\rangle = A_{ts^{-1}}, \quad \langle X_t, X_s\rangle_D = B_{t^{-1}s}$$

for any $t, s \in G$. If $\mathcal{X}$ satisfies Conditions (1R)–(3R) *and* (1L)–(3L) *in* [1], *Definition* 2.1, *then $\mathcal{X}$ is an $\mathcal{A} - \mathcal{B}$-equivalence bundle.*

Proof. Since Y is a $C - D$-equivalence bimodule, $\mathcal{X}$ has Conditions (4R)–(6R) and (4L)–(6L) in [1], Definition 2.1 except that X_t is complete with the norms $\|\langle\cdot,\cdot\rangle_D\|^{1/2} = \|{}_C\langle\cdot,\cdot\rangle\|^{1/2}$ for any $t \in G$. But we know that if Y is complete with two different norms, then the two norms are equivalent. Hence, X_t is complete with the norms $\|\langle\cdot,\cdot\rangle_D\|^{1/2} = \|{}_C\langle\cdot,\cdot\rangle\|^{1/2}$ for any $t \in G$. Furthermore, since

$${}_C\langle X_t, X_s\rangle = A_{ts^{-1}}, \quad \langle X_t, X_s\rangle_D = B_{t^{-1}s}$$

for any $t, s \in G$, $\mathcal{X}$ has Conditions (7R) and (7L) in [1], Definition 2.1. Therefore we obtain the conclusion. $\square$

We give an example of an $\mathcal{A} - \mathcal{B}$-equivalence bundle $\mathcal{X} = \{X_t\}_{t\in G}$ such that

$${}_C\langle X_t, X_s\rangle = A_{ts^{-1}}, \quad \langle X_t, X_s\rangle_D = B_{t^{-1}s}$$

for any $t, s \in G$.

Example 2.3. Let G be a finite group. Let α be an action of G on a unital C^*-algebra A. Let u_t be implementing unitary elements of α, that is, $\alpha_t = \mathrm{Ad}(u_t)$ for any $t \in G$. Then the crossed product of A by α, $A \rtimes_\alpha G$ is

$$A \rtimes_\alpha G = \left\{\sum_{t\in G} a_t u_t \colon\ a_t \in A \text{ for any } t \in G\right\}.$$

Let $A_t = Au_t$ for any $t \in G$. By routine computations, we see that $\mathcal{A}_\alpha = \{A_t\}_{t\in G}$ is a C^*-algebraic bundle over G. We call $\mathcal{A}_\alpha$ the *C^*-algebraic bundle over G induced by an action α*. Let β be an action of G on a unital C^*-algebra B and let $\mathcal{A}_\beta = \{B_t\}_{t\in G}$

induced by β, where $B_t = Bv_t$ for any $t \in G$ and v_t are implementing unitary elements of β. We suppose that α and β are strongly Morita equivalent with respect to an action λ of G on an $A-B$-equivalence bimodule X. Let $X \rtimes_\lambda G$ be the crossed product of X by λ defined by Kajiwara and Watatani (see [5], Definition 1.4), that is, the direct sum of n-copies of X as a vector space, where n is the order of G. And its elements are written as formal sums so that

$$X \rtimes_\lambda G = \left\{ \sum_{t\in G} x_t w_t \colon\ x_t \in X \text{ for any } t \in G \right\},$$

where w_t are indeterminates for all $t \in G$. Let $C = A \rtimes_\alpha G$, $D = B \rtimes_\beta G$ and $Y = X \rtimes_\lambda G$. Then by [5], Proposition 1.7, Y is a $C-D$-equivalence bimodule, where we define the left C-action and the right D-action on Y by

$$(au_t)\cdot(xw_s) = (a\cdot\lambda_t(x))w_{ts}, \quad (xw_s)\cdot(bv_t) = (x\cdot\beta_s(b))v_{st}$$

for any $a \in A$, $b \in B$, $x \in X$ and $t, s \in G$ and we define the left C-valued inner product and the right D-valued inner product on Y by extending linearly the following:

$${}_C\langle xw_t, yw_s\rangle = {}_A\langle x, \lambda_{ts^{-1}}(y)\rangle u_{ts^{-1}}, \quad \langle xw_t, yw_s\rangle_D = \beta_{t^{-1}}(\langle x, y\rangle_B)v_{t^{-1}s}$$

for any $x, y \in X$, $t, s \in G$. Let $X_t = Xw_t$ for any $t \in G$ and $\mathcal{X}_\lambda = \{X_t\}_{t\in G}$. Then $Y = \bigoplus_{t\in G} X_t$. Also, $\mathcal{X}_\lambda$ has Conditions (1R)–(3R) and (1L)–(3L) in [1], Definition 2.1. Furthermore, X is an $A-B$-equivalence bimodule and $\mathcal{X}_\lambda$ satisfies

$${}_C\langle X_t, X_s\rangle = A_{ts^{-1}}, \quad \langle X_t, X_s\rangle_D = B_{t^{-1}s}$$

for any $t, s \in G$. Therefore $\mathcal{X}_\lambda$ is an $\mathcal{A}_\alpha - \mathcal{A}_\beta$-equivalence bundle by Lemma 2.2.

3. Saturated C^*-algebraic bundles over a finite group

Let $\mathcal{A} = \{A_t\}_{t\in G}$ be a saturated C^*-algebraic bundle over a finite group G. Let e be the unit element in G. Let $C = \bigoplus_{t\in G} A_t$ and $A = A_e$. We suppose that C is unital and that A has the unit element in C. Let E^A be the canonical conditional expectation from C onto A defined in Section 1, which is of Watatani index-finite type. Let C_1 be the C^*-basic construction of C and e_A the Jones' projection for E^A. By [9], Lemma 3.7, there is an action $\alpha^{\mathcal{A}}$ of G on C_1 induced by $\mathcal{A}$ defined as follows: Since $\mathcal{A}$ is saturated and A is unital, there is a finite set $\{x_i^t\}_{i=1}^{n_t} \subset A_t$ such that $\sum_{i=1}^{n_t} x_i^t x_i^{t*} = 1$ for any $t \in G$. Let $e_t = \sum_{i=1}^{n_t} x_i^t e_A x_i^{t*}$ for all $t \in G$. Then by [9], Lemmas 3.3, 3.5 and Remark 3.4, $\{e_t\}_{t\in G}$ are mutually orthogonal projections in $A'\cap C_1$, which are independent of the choice of $\{x_i^t\}_{i=1}^{n_t}$, with $\sum\limits_{t\in G} e_t = 1$ such that C and e_t generate the C^*-algebra C_1 for all $t\in G$. We define $\alpha^{\mathcal{A}}$ by $\alpha_t^{\mathcal{A}}(c)=c$ and $\alpha_t^{\mathcal{A}}(e_A)=e_{t^{-1}}$ for any $t\in G$, $c\in C$. Let $\mathcal{A}_1=\{Y_{\alpha_t^{\mathcal{A}}}\}_{t\in G}$ be the C^*-algebraic bundle over G induced by the action $\alpha^{\mathcal{A}}$ of G which is defined in [9], Sections 5, 6, that is, let $Y_{\alpha_t^{\mathcal{A}}}=e_AC_1\alpha_t^{\mathcal{A}}(e_A)=e_AC_1e_{t^{-1}}$ for any $t\in G$. The product $\bullet$ and the involution $\sharp$ in $\mathcal{A}_1$ are defined as follows:

$$(x,y)\in Y_{\alpha_t^{\mathcal{A}}}\times Y_{\alpha_s^{\mathcal{A}}}\mapsto x\bullet y = x\alpha_t^{\mathcal{A}}(y)\in Y_{\alpha_{ts}^{\mathcal{A}}},$$
$$x\in Y_{\alpha_t^{\mathcal{A}}}\mapsto x^\sharp=\alpha_{t^{-1}}^{\mathcal{A}}(x^*)\in Y_{\alpha_{t^{-1}}^{\mathcal{A}}}.$$

Lemma 3.1. *With the above notation, $\mathcal{A}$ and $\mathcal{A}_1$ are isomorphic as C^*-algebraic bundles over G.*

Proof. Since $C_1=Ce_AC$, for any $t\in G$

$$Y_{\alpha_t^{\mathcal{A}}}=e_ACe_ACe_{t^{-1}}=e_AACe_{t^{-1}}=e_ACe_{t^{-1}}.$$

Let x be any element in C. Then we can write that $x=\sum\limits_{s\in G}x_s$, where $x_s\in A_s$. Hence

$$\begin{aligned}
e_Axe_{t^{-1}} &= \sum_{s,i} e_Ax_sx_i^{t^{-1}}e_Ax_i^{t^{-1}*}=\sum_{s,i}E^A(x_sx_i^{t^{-1}})e_Ax_i^{t^{-1}*}\\
&=\sum_i x_tx_i^{t^{-1}}e_Ax_i^{t^{-1}*}=e_Ax_t\sum_i x_i^{t^{-1}}x_i^{t^{-1}*}=e_Ax_t.
\end{aligned}$$

Thus, $Y_{\alpha_t^{\mathcal{A}}}=e_ACe_{t^{-1}}=e_AA_t$ for any $t\in G$. Let π_t be the map from A_t to $Y_{\alpha_t^{\mathcal{A}}}$ defined by $\pi_t(x)=e_Ax$ for any $x\in A_t$ and $t\in G$. By the above discussions π_t is a linear map from A_t onto $Y_{\alpha_t^{\mathcal{A}}}$. Then

$$\|\pi_t(x)\|^2=\|e_Axx^*e_A\|=\|E^A(xx^*)e_A\|=\|E^A(xx^*)\|=\|xx^*\|=\|x\|^2.$$

Hence, π_t is injective for any $t\in G$. Thus, $A_t\cong e_AC_1\alpha_t^{\mathcal{A}}(e_A)$ as Banach spaces for any $t\in G$. Also, for any $x\in A_t$, $y\in A_s$, $t,s\in G$,

$$\begin{aligned}
\pi_t(x)\bullet\pi_s(y) &= e_Ax\alpha_t^{\mathcal{A}}(e_Ay)=e_Axe_{t^{-1}}y=e_A\sum_i xx_i^{t^{-1}}e_Ax_i^{t^{-1}*}y\\
&=e_A\sum_i xx_i^{t^{-1}}x_i^{t^{-1}*}y=e_Axy=\pi_{ts}(xy),\\
\pi_t(x)^\sharp &= \alpha_{t^{-1}}^{\mathcal{A}}(\pi_t(x)^*)=\alpha_{t^{-1}}^{\mathcal{A}}((e_Ax)^*)=\alpha_{t^{-1}}^{\mathcal{A}}(x^*e_A)=x^*e_t\\
&=\sum_i x^*x_i^te_Ax_i^{t*}=e_A\sum_i x^*x_i^tx_i^{t*}=e_Ax^*=\pi_{t^{-1}}(x^*).
\end{aligned}$$

Therefore $\mathcal{A}=\{A_t\}_{t\in G}$ and $\mathcal{A}_1=\{Y_{\alpha_t^{\mathcal{A}}}\}_{t\in G}$ are isomorphic as C^*-algebraic bundles over G. $\square$

4. Strong Morita equivalence for unital inclusions of unital C^*-algebras

Let $\mathcal{A} = \{A_t\}_{t\in G}$ and $\mathcal{B} = \{B_t\}_{t\in G}$ be saturated C^*-algebraic bundles over a finite group G. Let e be the unit element in G. Let $C = \bigoplus_{t\in G} A_t$, $D = \bigoplus_{t\in G} B_t$ and $A = A_e$, $B = B_e$. We suppose that C and D are unital and that A and B have the unit elements in C and D, respectively. Let E^A and E^B be the canonical conditional expectations from C and D onto A and B defined in Section 1, respectively. They are of Watatani index-finite type. Let $A \subset C$ and $B \subset D$ be the unital inclusions of unital C^*-algebras induced by $\mathcal{A}$ and $\mathcal{B}$, respectively. We suppose that $A \subset C$ and $B \subset D$ are strongly Morita equivalent with respect to a $C - D$-equivalence bimodule Y and its closed subspace X. Also, we suppose that $A' \cap C = \mathbf{C}1$. Then by [10], Lemma 10.3, $B' \cap D = \mathbf{C}1$ and by [7], Lemma 4.1 and its proof, there is a unique conditional expectation E^X from Y onto X with respect to E^A and E^B.

Let C_1 and D_1 be the C^*-basic constructions of C and D and e_A and e_B the Jones' projections for E^A and E^B, respectively. Let $\alpha^{\mathcal{A}}$ and $\alpha^{\mathcal{B}}$ be actions of G on C_1 and D_1 induced by $\mathcal{A}$ and $\mathcal{B}$, respectively. Furthermore, let C_2 and D_2 be the C^*-basic constructions of C_1 and D_1 for the dual conditional expectations E^C of E^A and E^D of E^B, which are isomorphic to $C_1 \rtimes_{\alpha^{\mathcal{A}}} G$ and $D_1 \rtimes_{\alpha^{\mathcal{B}}} G$, respectively. We identify C_2 and D_2 with $C_1 \rtimes_{\alpha^{\mathcal{A}}} G$ and $D_1 \rtimes_{\alpha^{\mathcal{B}}} G$, respectively. By [10], Corollary 6.3, the unital inclusions $C_1 \subset C_2$ and $D_1 \subset D_2$ are strongly Morita equivalent with respect to a $C_2 - D_2$-equivalence bimodule Y_2 and its closed subspace Y_1, where Y_1 and Y_2 are the $C_1 - D_1$-equivalence bimodule and the $C_2 - D_2$-equivalence bimodule defined in [10], Section 6, respectively, and Y_1 is regarded as a closed subspace of Y_2 in the same way as in [10], Section 6. Also, $C_1' \cap C_2 = \mathbf{C}1$ by the proof of Watatani (see [13], Proposition 2.7.3) since $A' \cap C = \mathbf{C}1$. Hence, by [11], Corollary 6.5, there are an automorphism f of G, a $C_1 - D_1$-equivalence bimodule Z and an action λ of G on Z such that $\alpha^{\mathcal{A}}$ and β, the action of G on D_1 defined by $\beta_t(d) = \alpha^{\mathcal{B}}_{f(t)}(d)$ for any $t \in G$, $d \in D$, are strongly Morita equivalent with respect to λ.

Let $\mathcal{A}_1 = \{Y_{\alpha_t^{\mathcal{A}}}\}_{t\in G}$ and $\mathcal{B}_1 = \{Y_{\alpha_t^{\mathcal{B}}}\}_{t\in G}$ be the C^*-algebraic bundles over G induced by the actions $\alpha^{\mathcal{A}}$ and $\alpha^{\mathcal{B}}$, which are defined in Section 3. Furthermore, let $\mathcal{B}^f = \{B_{f(t)}\}_{t\in G}$ be the C^*-algebraic bundle over G induced by $\mathcal{B}$ and f and let $\mathcal{B}_1^f = \{Y_{\beta_t}\}_{t\in G}$ be the C^*-algebraic bundle over G induced by the action β, which is defined in Section 3. We construct an $\mathcal{A}_1 - \mathcal{B}_1^f$-equivalence bundle $\mathcal{Z} = \{Z_t\}_{t\in G}$ over G.

Let $Z_t = e_A \cdot Z \cdot \beta_t(e_B)$ for any $t \in G$ and let $W = \bigoplus_{t\in G} Z_t$. Also, by Lemma 3.1 and its proof, $\bigoplus_{t\in G} Y_{\alpha_t^{\mathcal{A}}} \cong C$ and $\bigoplus_{t\in G} Y_{\beta_t} \cong D$ as C^*-algebras. We identify $\bigoplus_{t\in G} Y_{\alpha_t^{\mathcal{A}}}$ and $\bigoplus_{t\in G} Y_{\beta_t}$ with C and D, respectively. We define the left C-action $\diamond$ and the left

C-valued inner product ${}_C\langle\cdot,\cdot\rangle$ on W by

$$\begin{aligned} e_A x\alpha_t^{\mathcal{A}}(e_A)\diamond[e_A\cdot z\cdot\beta_s(e_B)] &\overset{\text{def}}{=} e_A x\alpha_t^{\mathcal{A}}(e_A)\cdot\lambda_t(e_A\cdot z\cdot\beta_s(e_B)) \\ &= e_A\cdot[x\alpha_t^{\mathcal{A}}(e_A)\cdot\lambda_t(z)]\cdot\beta_{ts}(e_B), \\ {}_C\langle e_A\cdot w\cdot\beta_t(e_B),\ e_A\cdot z\cdot\beta_s(e_B)\rangle &\overset{\text{def}}{=} {}_{C_1}\langle e_A\cdot w\cdot\beta_t(e_B),\ \lambda_{ts^{-1}}(e_A\cdot z\cdot\beta_s(e_B))\rangle \\ &= e_A{}_{C_1}\langle w\cdot\beta_t(e_B),\ \lambda_{ts^{-1}}(z)\cdot\beta_t(e_B)\rangle\alpha_{ts^{-1}}^{\mathcal{A}}(e_A), \end{aligned}$$

where $e_A x\alpha_t^{\mathcal{A}}(e_A)\in e_A C_1\alpha_t^{\mathcal{A}}(e_A)$, $e_A\cdot z\cdot\beta_s(e_B)\in Z_s$, $e_A\cdot w\cdot\beta_t(e_B)\in Z_t$. Also, we define the right D-action, which is also denoted by the same symbol $\diamond$ and the D-valued inner product $\langle\cdot,\cdot\rangle_D$ on W by

$$\begin{aligned} [e_A\cdot z\cdot\beta_t(e_B)]\diamond e_B x\beta_s(e_B) &\overset{\text{def}}{=} e_A\cdot z\cdot\beta_t(e_B)\beta_t(x)\beta_{ts}(e_B) \\ &= e_A\cdot[z\cdot\beta_t(e_B)\beta_t(x)]\cdot\beta_{ts}(e_B), \\ \langle e_A\cdot z\cdot\beta_t(e_B), e_A\cdot w\cdot\beta_s(e_B)\rangle_D &\overset{\text{def}}{=} \beta_{t^{-1}}(\langle e_A\cdot z\cdot\beta_t(e_B), e_A\cdot w\cdot\beta_s(e_B)\rangle_{D_1}) \\ &= e_B\beta_{t^{-1}}(\langle e_A\cdot z, e_A\cdot w\rangle_{D_1})\beta_{t^{-1}s}(e_B), \end{aligned}$$

where $e_B x\beta_s(e_B)\in e_B D_1\beta_s(e_B)$, $e_A\cdot z\cdot\beta_t(e_B)\in Z_t$, $e_A\cdot w\cdot\beta_s(e_B)\in Z_s$. By the above definitions, $\mathcal{Z}$ has Conditions (1R)–(3R) and (1L)–(3L) in [1], Definition 2.1. We show that $\mathcal{Z}$ has Conditions (4R) and (4L) in [1], Definition 2.1 and that $\mathcal{Z}$ is an $\mathcal{A}_1-\mathcal{B}_1^f$-bundle in the same way as in Example 2.3.

Lemma 4.1. *With the above notation, $\mathcal{Z}$ has Conditions* (4R) *and* (4L) *in* [1], *Definition* 2.1.

Proof. Let $e_A\cdot z\cdot\beta_t(e_B)\in Z_t$, $e_A\cdot w\cdot\beta_s(e_B)\in Z_s$ and $e_B x\beta_r(e_B)\in e_B D_1\beta_r(e_B)$, where $t,s,r\in G$. Then by routine computations, we can see that

$$\begin{aligned} &\langle e_A\cdot z\cdot\beta_t(e_B), [e_A\cdot w\cdot\beta_s(e_B)]\diamond e_B x\beta_r(e_B)\rangle_D \\ &\qquad = \langle e_A\cdot z\cdot\beta_t(e_B), e_A\cdot w\cdot\beta_s(e_B)\rangle_D\bullet e_B x\beta_r(e_B) \end{aligned}$$

and that

$$\langle e_A\cdot z\cdot\beta_t(e_B), e_A\cdot w\cdot\beta_s(e_B)\rangle_D^{\sharp} = \langle e_A\cdot w\cdot\beta_s(e_B), e_A\cdot z\cdot\beta_t(e_B)\rangle_D.$$

Hence, $\mathcal{Z}$ has Condition (4R) in [1], Definition 2.1. Next, let $e_A\cdot z\cdot\beta_t(e_B)\in Z_t$, $e_A\cdot w\cdot\beta_s(e_B)\in Z_s$ and $e_A x\alpha_r^{\mathcal{A}}(e_A)\in e_A C_1\alpha_r^{\mathcal{A}}(e_A)$, where $t,s,r\in G$. Then by routine computations, we can see that

$$\begin{aligned} &{}_C\langle e_A x\alpha_r^{\mathcal{A}}(e_A)\diamond[e_A\cdot z\cdot\beta_t(e_B)], e_A\cdot w\cdot\beta_s(e_B)\rangle \\ &\qquad = e_A x\alpha_r^{\mathcal{A}}(e_A)\bullet{}_C\langle e_A\cdot z\cdot\beta_t(e_B), e_A\cdot w\cdot\beta_s(e_B)\rangle, \\ &{}_C\langle e_A\cdot z\cdot\beta_t(e_B), e_A\cdot w\cdot\beta_s(e_B)\rangle^{\sharp} = {}_C\langle e_A\cdot w\cdot\beta_s(e_B), e_A\cdot z\cdot\beta_t(e_B)\rangle. \end{aligned}$$

Hence, $\mathcal{Z}$ has Condition (4L) in [1], Definition 2.1. □

By Lemma 4.1, W is a $C-D$-bimodule having Properties (1)–(6) in [5], Lemma 1.3. In order to prove that $\mathcal{Z}$ has Conditions (5R), (6R) and (5L), (6L) in [1], Definition 2.1 using [5], Lemma 1.3, we show that W has Properties (7)–(10) in [5], Lemma 1.3.

Lemma 4.2. *With the above notation, W has the following:*

(1) $(e_A x\alpha_t^{\mathcal{A}}(e_A) \diamond [e_A \cdot z \cdot \beta_s(e_B)]) \diamond e_B y\beta_r(e_B) = e_A x\alpha_t^{\mathcal{A}}(e_A) \diamond ([e_A \cdot z \cdot \beta_s(e_B)] \diamond e_B y\beta_r(e_B))$,

(2) $\langle e_A x\alpha_t^{\mathcal{A}}(e_A) \diamond [e_A \cdot z \cdot \beta_s(e_B)], e_A \cdot w \cdot \beta_r(e_B)\rangle_D = \langle e_A \cdot z \cdot \beta_s(e_B), (e_A x\alpha_t^{\mathcal{A}}(e_A))^\sharp \diamond [e_A \cdot w \cdot \beta_r(e_B)]\rangle_D$,

(3) ${}_C\langle e_A \cdot z \cdot \beta_s(e_B), [e_A \cdot w \cdot \beta_r(e_B)] \diamond e_B y\beta_t(e_B)\rangle = {}_C\langle [e_A \cdot z \cdot \beta_s(e_B)] \diamond (e_B y\beta_t(e_B))^\sharp, e_A \cdot w \cdot \beta_r(e_B)\rangle$,

where $x \in C_1$, $y \in D_1$, $z, w \in Z$, $t, s, r \in G$.

Proof. We can show the lemma by routine computations. $\square$

By Lemma 4.2, W has Properties (7), (8) in [5], Lemma 1.3.

Lemma 4.3. *With the above notation, there are finite subsets $\{u_i\}_i$ and $\{v_j\}_j$ of W such that*

$$\sum_i u_i \diamond \langle u_i, x\rangle_D = x = \sum_j {}_C\langle x, v_j\rangle \diamond v_j \quad \textit{for any } x \in W.$$

Proof. Since Z is a $C_1 - D_1$-equivalence bimodule, there are finite subsets $\{z_i\}_i$ and $\{w_j\}_j$ of Z such that

$$\sum_i z_i \cdot \langle z_i, z\rangle_{D_1} = z = \sum_j {}_{C_1}\langle z, w_j\rangle \cdot w_j$$

for any $z \in Z$. Then for any $z \in Z$, $s \in G$,

$$\begin{aligned}
&\sum_{i,t} [e_A \cdot z_i \cdot \beta_t(e_B)] \diamond \langle e_A \cdot z_i \cdot \beta_t(e_B), e_A \cdot z \cdot \beta_s(e_B)\rangle_D \\
&\quad = \sum_{i,t} e_A \cdot z_i \cdot \beta_t(e_B) \diamond \beta_{t^{-1}}(\langle e_A \cdot z_i \cdot \beta_t(e_B), e_A \cdot z \cdot \beta_s(e_B)\rangle_{D_1}) \\
&\quad = \sum_{i,t} e_A \cdot z_i \cdot \beta_t(e_B) \diamond e_B \beta_{t^{-1}}(\langle e_A \cdot z_i, e_A \cdot z\rangle_{D_1})\beta_{t^{-1}s}(e_B) \\
&\quad = \sum_{i,t} e_A \cdot z_i \cdot \beta_t(e_B)\langle e_A \cdot z_i, e_A \cdot z\rangle_{D_1}\beta_s(e_B) \\
&\quad = \sum_{i,t} e_A \cdot [z_i \cdot \langle z_i \cdot \beta_t(e_B), e_A \cdot z\rangle_{D_1}] \cdot \beta_s(e_B) \\
&\quad = \sum_i e_A \cdot [z_i \cdot \langle z_i, e_A \cdot z\rangle_{D_1}] \cdot \beta_s(e_B) = e_A \cdot z \cdot \beta_s(e_B)
\end{aligned}$$

since $\sum_{t\in G}\beta_t(e_B)=1$ by [9], Remark 3.4. Also, by the same way and the same reason, for any $z\in Z$, $s\in G$,

$$\sum_{j,t} {}_C\langle e_A\cdot z\cdot\beta_s(e_B), e_A\cdot\lambda_t(w_j)\cdot\beta_t(e_B)\rangle\diamond[e_A\cdot\lambda_t(w_j)\cdot\beta_t(e_B)]=e_A\cdot z\cdot\beta_s(e_B).$$

Therefore we obtain the conclusion. □

Remark 4.4. By Lemma 4.2, $\{e_A\cdot z_i\cdot\beta_t(e_B)\}_{i,t}$ is a right D-basis and $\{e_A\cdot\lambda_t(w_j)\cdot\beta_t(e_B)\}_{j,t}$ is a left C-basis of W in the sense of Kajiwara and Watatani (see [6]).

By Lemma 4.2, W has Properties (9), (10) in [5], Lemma 1.3. Hence, by [5], Lemma 1.3, W is a Hilbert $C-D$- bimodule in the sense of [5], Definition 1.1. Thus, $\mathcal{Z}$ has Conditions (5R), (6R) and (5L), (6L) in [1], Definition 2.1.

Proposition 4.5. *With the above notation, $\mathcal{Z}$ is an $\mathcal{A}_1-\mathcal{B}_1^f$-equivalence bundle over G such that*

$${}_{\mathcal{A}_1}\langle Z_t, Z_s\rangle = Y_{\alpha^{\mathcal{A}}_{ts^{-1}}},\quad \langle Z_t, Z_s\rangle_{\mathcal{B}_1^f}=Y_{\beta_{t^{-1}s}}$$

for any $t,s\in G$.

Proof. First, we show that the left C-valued inner product and the right D-valued inner product on W are compatible. Let $y,z,w\in Z$ and $t,s,r\in G$. Since Z is a C_1-D_1-equivalence bimodule, by routine computations, we can see that

$$\begin{aligned}&{}_C\langle e_A\cdot z\cdot\beta_t(e_B), e_A\cdot y\cdot\beta_s(e_B)\rangle\diamond[e_A\cdot w\cdot\beta_r(e_B)]\\&\qquad\qquad=[e_A\cdot z\cdot\beta_t(e_B)]\diamond\langle e_A\cdot y\cdot\beta_s(e_B), e_A\cdot w\cdot\beta_r(e_B)\rangle_D.\end{aligned}$$

Hence, the left C-valued inner product and the right D-valued inner product are compatible. Thus, by Lemmas 4.1–4.3, $\mathcal{Z}$ is an $\mathcal{A}_1-\mathcal{B}_1^f$-equivalence bundle over G. Next, we show that

$${}_{\mathcal{A}_1}\langle Z_t, Z_s\rangle = Y_{\alpha^{\mathcal{A}}_{ts^{-1}}},\quad \langle Z_t, Z_s\rangle_{\mathcal{B}_1^f}=Y_{\beta_{t^{-1}s}}$$

for any $t,s\in G$. Let $t,s\in G$. Since E^B is of Watatani index-finite type, there is a quasi-basis $\{(d_j,d_j^*)\}\subset D\times D$ for E^B. Thus $\sum_j d_je_Bd_j^*=1$. Since Z is a C_1-D_1-equivalence bimodule, there is a finite subset $\{z_i\}$ of Z such that $\sum_i {}_{C_1}\langle z_i, z_i\rangle=1$.

Let $c \in C$. Then

$$\sum_{i,j} {}_C\langle e_A c \cdot \lambda_t(z_i) \cdot \beta_t(d_j e_B), e_A \cdot \lambda_s(z_i) \cdot \beta_s(d_j e_B)\rangle$$
$$= \sum_{i,j} {}_{C_1}\langle e_A c \cdot \lambda_t(z_i) \cdot \beta_t(d_j e_B), \lambda_{ts^{-1}}(e_A \cdot \lambda_s(z_i) \cdot \beta_s(d_j e_B))\rangle$$
$$= \sum_{i,j} {}_{C_1}\langle e_A c \cdot \lambda_t(z_i) \cdot \beta_t(d_j e_B), \alpha^{\mathcal{A}}_{ts^{-1}}(e_A) \cdot \lambda_t(z_i) \cdot \beta_t(d_j e_B)\rangle$$
$$= \sum_{i,j} e_A {}_{C_1}\langle c \cdot \lambda_t(z_i) \cdot \beta_t(d_j e_B d_j^*), \lambda_t(z_i)\rangle \alpha^{\mathcal{A}}_{ts^{-1}}(e_A)$$
$$= \sum_{i} e_A c {}_{C_1}\langle \lambda_t(z_i), \lambda_t(z_i)\rangle \alpha^{\mathcal{A}}_{ts^{-1}}(e_A)$$
$$= \sum_{i} e_A c \alpha^{\mathcal{A}}_t({}_{C_1}\langle z_i, z_i\rangle) \alpha^{\mathcal{A}}_{ts^{-1}}(e_A) = e_A c \alpha^{\mathcal{A}}_{ts^{-1}}(e_A).$$

Hence, we obtain that ${}_C\langle Z_t, Z_s\rangle = Y_{\alpha^{\mathcal{A}}_{ts^{-1}}}$ for any $t, s \in G$. Also, since E^A is of Watatani index-finite type, there is a quasi-basis $\{(c_j, c_j^*)\} \subset C \times C$ for E^A. Thus $\sum_j c_j e_A c_j^* = 1$. Since Z is a $C_1 - D_1$-equivalence bimodule, there is a finite subset $\{w_i\}$ of Z such that $\sum_i \langle w_i, w_i\rangle_{D_1} = 1$. In the same way as above, for any $d \in D_1$,

$$\sum_{i,j} \langle e_A c_j^* \cdot w_i \cdot \beta_t(e_B), e_A c_j^* \cdot w_i \cdot d\beta_s(e_B)\rangle_D = e_B \beta_{t^{-1}}(d) \beta_{t^{-1}s}(e_B).$$

Hence, we obtain that $\langle Z_t, Z_s\rangle_D = Y_{\beta_{t^{-1}s}}$ for any $t, s \in G$. Therefore we obtain the conclusion. □

Theorem 4.6. *Let $\mathcal{A} = \{A_t\}_{t\in G}$ and $\mathcal{B} = \{B_t\}_{t\in G}$ be saturated C^*-algebraic bundles over a finite group G. Let e be the unit element in G. Let $C = \bigoplus_{t\in G} A_t$, $D = \bigoplus_{t\in G} B_t$ and $A = A_e$, $B = B_e$. We suppose that C and D are unital and that A and B have the unit elements in C and D, respectively. Let $A \subset C$ and $B \subset D$ be the unital inclusions of unital C^*-algebras induced by $\mathcal{A}$ and $\mathcal{B}$, respectively. Also, we suppose that $A' \cap C = \mathbf{C}1$. If $A \subset C$ and $B \subset D$ are strongly Morita equivalent, then there are an automorphism f of G and an $\mathcal{A} - \mathcal{B}^f$-equivalence bundle $\mathcal{Z} = \{Z_t\}_{t\in G}$ satisfying that*

$${}_C\langle Z_t, Z_s\rangle = A_{ts^{-1}}, \quad \langle Z_t, Z_s\rangle_D = B_{f(t^{-1}s)}$$

for any $t, s \in G$, where $\mathcal{B}^f$ is the C^*-algebraic bundle over G induced by $\mathcal{B}$ and f defined by $\mathcal{B}^f = \{B_{f(t)}\}_{t\in G}$.

P r o o f. This is immediate by Lemma 3.1 and Proposition 4.5. □

5. Application

Let A and B be unital C^*-algebras and X a Hilbert $A-B$-bimodule. Let $\widetilde{X}$ be its dual Hilbert $B-A$-bimodule. For any $x \in X$, $\widetilde{x}$ denotes the element in $\widetilde{X}$ induced by $x \in X$.

Lemma 5.1. *Let A, B and C be unital C^*-algebras. Let X be a Hilbert $A-B$-bimodule and Y a Hilbert $B-C$-bimodule. Then $\widetilde{X \otimes_B Y} \cong \widetilde{Y} \otimes_B \widetilde{X}$ as Hilbert $C-A$-bimodules.*

P r o o f. Let π be the map from $\widetilde{X \otimes_B Y}$ to $\widetilde{Y} \otimes_B \widetilde{X}$ defined by $\pi(\widetilde{x \otimes y}) = \widetilde{y} \otimes \widetilde{x}$ for any $x \in X$, $y \in Y$. Then by routine computations, we can see that π is a Hilbert $C-A$-bimodule isomorphism of $\widetilde{X \otimes_B Y}$ onto $\widetilde{Y} \otimes_B \widetilde{X}$. $\square$

We identify $\widetilde{X \otimes_B Y}$ with $\widetilde{Y} \otimes_B \widetilde{X}$ by the isomorphism π defined in the proof of Lemma 5.1. Next, we give the definition of an involutive Hilbert $A-A$-bimodule modifying [8].

Definition 5.2. We say that a Hilbert $A-A$-bimodule X is *involutive* if there exists a conjugate linear map $x \in X \mapsto x^\natural \in X$ such that

(1) $(x^\natural)^\natural = x$, $x \in X$,

(2) $(a \cdot x \cdot b)^\natural = b^* \cdot x^\natural \cdot a^*$, $x \in X$, $a, b \in A$,

(3) ${}_A\langle x, y^\natural\rangle = \langle x^\natural, y\rangle_A$, $x, y \in X$.

We call the above conjugate linear map $\natural$ an *involution* on X. If X is full with the both inner products, X is an involutive $A-A$-equivalence bimodule. For each involutive Hilbert $A-A$-bimodule, let L_X be the linking C^*-algebra induced by X and C_X the C^*-subalgebra of L_X, which is defined in [8], that is,

$$C_X = \left\{ \begin{bmatrix} a & x \\ \widetilde{x}^\natural & a \end{bmatrix} : \; a \in A, \; x \in X \right\}.$$

We note that C_X acts on $X \oplus A$ (see Brown, Green and Rieffel [2] and Rieffel [12]). The norm of C_X is defined as the operator norm on $X \oplus A$.

Let A be a unital C^*-algebra and X an involutive Hilbert $A-A$-bimodule. Let $\widetilde{X}$ be its dual Hilbert $A-A$-bimodule. We define the map $^\natural$ on $\widetilde{X}$ by $(\widetilde{x})^\natural = \widetilde{(x^\natural)}$ for any $\widetilde{x} \in \widetilde{X}$.

Lemma 5.3. *With the above notation, the above map $^\natural$ is an involution on $\widetilde{X}$.*

P r o o f. This is immediate by direct computations. $\square$

For each involutive Hilbert $A-A$-bimodule X, we regard $\widetilde{X}$ as an involutive $A-A$-bimodule in the same manner as in Lemma 5.3.

Let $\mathbf{Z}_2=\mathbf{Z}/2\mathbf{Z}$ and we suppose that $\mathbf{Z}_2$ consists of the unit element 0 and another element 1. Let X be an involutive Hilbert $A-A$-bimodule. We construct a C^*-algebraic bundle over $\mathbf{Z}_2$ induced by X. Let $A_0=A$ and $A_1=X$. Let $\mathcal{A}_X=\{A_t\}_{t\in\mathbf{Z}_2}$. We define a product $\bullet$ and an involution $\sharp$ as follows:

(1) $a\bullet b=ab$, $a,b\in A$,

(2) $a\bullet x=a\cdot x$, $x\bullet a=x\cdot a$, $a\in A$, $x\in X$,

(3) $x\bullet y={}_A\langle x,y^\natural\rangle=\langle x^\natural,y\rangle_A$, $x,y\in X$,

(4) $a^\sharp=a^*$, $a\in A$,

(5) $x^\sharp=x^\natural$, $x\in X$.

Then $A\oplus X$ is a $*$-algebra and by routine computations, $A\oplus X$ is isomorphic to C_X as $*$-algebras. We identify $A\oplus X$ with C_X as $*$-algebras. We define a norm of $A\oplus X$ as the operator norm on $X\oplus A$. Hence, $\mathcal{A}_X$ is a C^*-algebraic bundle over $\mathbf{Z}_2$. Thus, we obtain a correspondence from the involutive Hilbert $A-A$-bimodules to the C^*-algebraic bundles over $\mathbf{Z}_2$. Next, let $\mathcal{A}=\{A_t\}_{t\in\mathbf{Z}_2}$ be a C^*-algebraic bundle over $\mathbf{Z}_2$. Then A_1 ia an involutive Hilbert $A-A$-bimodule. Hence, we obtain a correspondence from the C^*-algebraic bundles over $\mathbf{Z}_2$ to the involutive Hilbert $A-A$-bimodules. Clearly, the above two correspondences are the inverse correspondences of each other. Furthermore, the inclusion of unital C^*-algebras $A\subset C_X$ induced by X and the inclusion of unital C^*-algebras $A\subset A\oplus X$ induced by the C^*-algebraic bundle $\mathcal{A}_X$ coincide.

Lemma 5.4. *Let X and Y be involutive Hilbert $A-A$-bimodules and $\mathcal{A}_X$ and $\mathcal{A}_Y$ the C^*-algebraic bundles over $\mathbf{Z}_2$ induced by X and Y, respectively. Then $\mathcal{A}_X\cong\mathcal{A}_Y$ as C^*-algebraic bundles over $\mathbf{Z}_2$ if and only if $X\cong Y$ as involutive Hilbert $A-A$-bimodules.*

Proof. We suppose that $\mathcal{A}_X\cong\mathcal{A}_Y$ as C^*-algebraic bundles over $\mathbf{Z}_2$. Then there is a C^*-algebraic bundle isomorphism $\{\pi_t\}_{t\in\mathbf{Z}_2}$ of $\mathcal{A}_X$ onto $\mathcal{A}_Y$. We identify A with $\pi_0(A)$. Then π_1 is an involutive Hilbert $A-A$-bimodule isomorphism of X onto Y. Next, we suppose that there is an involutive Hilbert $A-A$-bimodule isomorphism π of X onto Y. Let $\pi_0=\mathrm{id}_A$ and $\pi_1=\pi$. Then $\{\pi_t\}_{t\in\mathbf{Z}_2}$ is a C^*-algebraic bundle isomorphism $\mathcal{A}_X$ onto $\mathcal{A}_Y$. □

Lemma 5.5. *Let X be an involutive Hilbert $A-A$-bimodule and $\mathcal{A}_X$ the C^*-algebraic bundle over $\mathbf{Z}_2$ induced by X. Then X is full with the both inner products if and only if $\mathcal{A}_X$ is saturated.*

P r o o f. We suppose that X is full with the both inner products. Then

$$A_1 \bullet A_1^\sharp = {}_A\langle X, X\rangle = A = A_0.$$

Also,

$$A_0 \bullet A_1^\sharp = A \cdot X^\natural = A \cdot X = X = A_1,$$
$$A_1 \bullet A_0^\sharp = X \cdot A^* = X \cdot A = X = A_1$$

by [3], Proposition 1.7. Clearly $A_0 \bullet A_0 = AA = A = A_0$. Hence $\mathcal{A}_X$ is saturated. Next, we suppose that $\mathcal{A}_X$ is saturated. Then

$${}_A\langle X, X\rangle = A_1 \bullet A_1^\sharp = A_1 = A, \quad \langle X, X\rangle_A = {}_A\langle X^\natural, X^\natural\rangle = {}_A\langle X, X\rangle = A.$$

Thus, X is full with the both inner products. $\square$

R e m a r k 5.6. Let X be an involutive Hilbert $A-A$-bimodule. Then by the above proof, we see that X is full with the left A-valued inner product if and only if X is full with the right A-valued inner product.

Lemma 5.7. *Let A and B be unital C^*-algebras and M an $A-B$-equivalence bimodule. Let X be an involutive Hilbert $A-A$-bimodule. Then $\widetilde{M} \otimes_A X \otimes_A M$ is an involutive Hilbert $B-B$-bimodule whose involution $\natural$ is defined by*

$$(\widetilde{m} \otimes x \otimes n)^\natural = \widetilde{n} \otimes x^\natural \otimes m$$

for any $m, n \in M$, $x \in X$.

P r o o f. This is immediate by routine computations. $\square$

Let A, B, X and M be as in Lemma 5.7. Let Y be an involutive Hilbert $B-B$-bimodule. We suppose that there is an involutive Hilbert $B-B$-bimodule isomorphism Φ of $\widetilde{M} \otimes_A X \otimes_A M$ onto Y. Let $\widetilde{\Phi}$ be the linear map from $\widetilde{M} \otimes_A \widetilde{X} \otimes_A M$ onto $\widetilde{Y}$ defined by

$$\widetilde{\Phi}(\widetilde{m} \otimes \widetilde{x} \otimes n) = \widetilde{\Phi}((\widetilde{n} \otimes x \otimes m)^\sim) = [\Phi(\widetilde{n} \otimes x \otimes m)]^\sim$$

for any $m, n \in M$, $x \in X$.

Lemma 5.8. *With the above notation, $\widetilde{\Phi}$ is an involutive Hilbert $B-B$-bimodule isomorphism of $\widetilde{M} \otimes_A \widetilde{X} \otimes_A M$ onto $\widetilde{Y}$.*

P r o o f. This is immediate by routine computations. $\square$

Again, let A, B, X and M be as in Lemma 5.7. Let Y be an involutive Hilbert $B-B$-bimodule. We suppose that there is an involutive Hilbert $B-B$-bimodule isomorphism Φ of $\widetilde{M}\otimes_A X\otimes_A M$ onto Y. We identify A and X with $M\otimes_B\widetilde{M}$ and $A\otimes_A X$ by the isomorphisms defined by

$$m\otimes n\in M\otimes_B\widetilde{M}\mapsto {}_A\langle m,n\rangle\in A,\quad a\otimes x\in A\otimes_A X\mapsto a\cdot x\in X,$$

respectively. Since M is an $A-B$-equivalence bimodule, there is a finite subset $\{u_i\}$ of M with $\sum_i {}_A\langle u_i,\, u_i\rangle=1$. Let $x\in X$, $m\in M$. Then

$$x\otimes m=1_A\cdot x\otimes m=\sum_i {}_A\langle u_i,u_i\rangle\cdot x\otimes m=\sum_i u_i\otimes\widetilde{u}_i\otimes x\otimes m.$$

Hence, there is the linear map Ψ from $X\otimes_A M$ to $M\otimes_B Y$ defined by

$$\Psi(x\otimes m)=\sum_i u_i\otimes\Phi(\widetilde{u_i}\otimes x\otimes m)$$

for any $x\in X$, $m\in M$. By the definition of Ψ, we can see that Ψ is a Hilbert $A-B$-bimodule isomorphism of $X\otimes_A M$ onto $M\otimes_B Y$.

Lemma 5.9. *With the above notation, the Hilbert $A-B$-bimodule isomorphism Ψ of $X\otimes_A M$ onto $M\otimes_B Y$ is independent of the choice of a finite subset $\{u_i\}$ of M with $\sum_i {}_A\langle u_i,u_i\rangle=1$.*

Proof. Let $\{v_j\}$ be another finite subset of M with $\sum_j {}_A\langle v_j,v_j\rangle=1$. Then for any $x\in X$, $m\in M$,

$$\begin{aligned}\sum_i & u_i\otimes\Phi(\widetilde{u_i}\otimes x\otimes m)\\ &=\sum_{i,j} {}_A\langle v_j,v_j\rangle\cdot u_i\otimes\Phi(\widetilde{u_i}\otimes x\otimes m)=\sum_{i,j} v_j\cdot\langle v_j,u_i\rangle_B\otimes\Phi(\widetilde{u_i}\otimes x\otimes m)\\ &=\sum_{i,j} v_j\otimes\Phi([u_i\cdot\langle u_i,v_j\rangle_B]^{\sim}\otimes x\otimes m)=\sum_j v_j\otimes\Phi(\widetilde{v_j}\otimes x\otimes m).\end{aligned}$$

Therefore, we obtain the conclusion. □

Similarly, let $\widetilde{\Psi}$ be the Hilbert $A-B$-bimodule isomorphism of $\widetilde{X}\otimes_A M$ onto $M\otimes_B\widetilde{Y}$ defined by

$$\widetilde{\Psi}(\widetilde{x}\otimes m)=\sum_i u_i\otimes\widetilde{\Phi}(\widetilde{u}_i\otimes\widetilde{x}\otimes m)$$

for any $x \in X$, $m \in M$. We construct the inverse map of Ψ, which is a Hilbert $A-B$-bimodule isomorphism of $M \otimes_B Y$ onto $X \otimes_A M$. Let Θ be the linear map from $M \otimes_B Y$ to $X \otimes_A M$ defined by

$$\Theta(m \otimes y) = m \otimes \Phi^{-1}(y)$$

for any $m \in M$, $y \in Y$, where we identify $M \otimes_B \widetilde{M} \otimes_A X \otimes_A M$ with $X \otimes_A M$ as Hilbert $A-B$-bimodules by the map

$$m \otimes \widetilde{n} \otimes x \otimes m_1 \in M \otimes_B \widetilde{M} \otimes_A X \otimes_A M \mapsto {}_A\langle m, n\rangle \cdot x \otimes m_1 \in X \otimes_A M.$$

Lemma 5.10. *With the above notation, Θ is the Hilbert $A-B$-bimodule isomorphism of $M \otimes_B Y$ onto $X \otimes_A M$ such that $\Theta \circ \Psi = \mathrm{id}_{X\otimes_A M}$ and $\Psi \circ \Theta = \mathrm{id}_{M\otimes_B Y}$.*

Proof. Let $m, m_1 \in M$, $y, y_1 \in Y$. Then

$$\begin{aligned}
{}_A\langle \Theta(m \otimes y), \Theta(m_1 \otimes y_1)\rangle &= {}_A\langle m \otimes \Phi^{-1}(y), m_1 \otimes \Phi^{-1}(y_1)\rangle \\
&= {}_A\langle m \cdot {}_B\langle \Phi^{-1}(y), \Phi^{-1}(y_1)\rangle, m_1\rangle \\
&= {}_A\langle m \cdot {}_B\langle y, y_1\rangle, m_1\rangle = {}_A\langle m \otimes y, m_1 \otimes y_1\rangle.
\end{aligned}$$

Hence, Θ preserves the left A-valued inner products. Similarly, we can see that Θ preserves the right B-valued inner products. Furthermore, for any $x \in X$, $m \in M$,

$$\begin{aligned}
(\Theta \circ \Psi)(x \otimes m) &= \sum_i \Theta(u_i \otimes \Phi(\widetilde{u}_i \otimes x \otimes m)) = \sum_i u_i \otimes \widetilde{u}_i \otimes x \otimes m \\
&= \sum_i {}_A\langle u_i, u_i\rangle \cdot x \otimes m = x \otimes m
\end{aligned}$$

since we identify $M \otimes \widetilde{M}$ with A as $A-A$-equivalence bimodules by the map $m \otimes \widetilde{n} \in M \otimes_B \widetilde{M} \mapsto {}_A\langle m, n\rangle \in A$. Hence, $\Theta \circ \Psi = \mathrm{id}_{X\otimes_A M}$. Hence, $\Psi \circ \Theta \circ \Psi = \Psi$ on $X \otimes_A M$. Since Ψ is surjective, $\Psi \circ \Theta = \mathrm{id}_{M\otimes_B Y}$. Therefore, by the remark after [4], Definition 1.1.18, Θ is a Hilbert $A-B$-bimodule isomorphism of $M \otimes_B Y$ onto $X \otimes_A M$ such that $\Theta \circ \Psi = \mathrm{id}_{X\otimes_A M}$ and $\Psi \circ \Theta = \mathrm{id}_{M\otimes_B Y}$. □

Similarly, we see that the inverse map of $(\widetilde{\Psi})^{-1}$ is defined by

$$(\widetilde{\Psi})^{-1}(m \otimes \widetilde{y}) = m \otimes (\widetilde{\Phi})^{-1}(\widetilde{y})$$

for any $m \in M$, $y \in Y$, where we identify $M \otimes_B \widetilde{M} \otimes_A \widetilde{X} \otimes_A M$ with $\widetilde{X} \otimes_A M$ as Hilbert $A-B$-bimodules by the map

$$m \otimes \widetilde{n} \otimes \widetilde{x} \otimes m_1 \in M \otimes_B \widetilde{M} \otimes_A \widetilde{X} \otimes_A M \mapsto {}_A\langle m, n\rangle \cdot \widetilde{x} \otimes m_1 \in \widetilde{X} \otimes_A M.$$

We prepare some lemmas in order to show Proposition 5.14.

Lemma 5.11. *Let A and B be unital C^*-algebras. Let X and Y be an involutive Hilbert $A-A$-bimodule and an involutive Hilbert $B-B$-bimodule, respectively. Let $\mathcal{A}_X = \{A_t\}_{t\in\mathbf{Z}_2}$ and $\mathcal{A}_Y = \{B_t\}_{t\in\mathbf{Z}_2}$ be C^*-algebraic bundles over $\mathbf{Z}_2$ induced by X and Y, respectively. We suppose that there is an $\mathcal{A}_X - \mathcal{A}_Y$-equivalence bundle $\mathcal{M} = \{M_t\}_{t\in\mathbf{Z}_2}$ over $\mathbf{Z}_2$ such that*

$$ {}_C\langle M_t, M_s\rangle = A_{ts^{-1}}, \quad \langle M_t, M_s\rangle_D = B_{t^{-1}s} $$

for any $t, s \in \mathbf{Z}_2$, where $C = A \oplus X$ and $D = B \oplus Y$. Then there is an $A-B$-equivalence bimodule M such that $Y \cong \widetilde{M} \otimes_A X \otimes_A M$ as involutive Hilbert $B-B$-bimodules.

P r o o f. By the assumptions, M_0 is an $A-B$-equivalence bimodule. Let $M = M_0$. Then by Lemma 5.7, $\widetilde{M} \otimes_A X \otimes_A M$ is an involutive Hilbert $B-B$-bimodule whose involution is defined by $(\widetilde{m}\otimes x\otimes n)^\natural = \widetilde{n}\otimes x^\natural \otimes m$ for any $m, n \in M$, $x \in X$. We show that $Y \cong \widetilde{M} \otimes_A X \otimes_A M$ as involutive Hilbert $B-B$-bimodules. Let Φ be the map from $\widetilde{M} \otimes_A X \otimes_A M$ to Y defined by

$$ \Phi(\widetilde{m} \otimes x \otimes n) = \langle m, x \cdot n\rangle_D $$

for any $m, n \in M$, $x \in X$. Since $A_1 = X$ and $M = M_0$, $X \cdot M_0 \subset M_1$. And $\langle M_0, M_1\rangle_D \in B_1 = Y$. Hence, Φ is a map from $\widetilde{M} \otimes_A X \otimes_A M$ to Y. Clearly, Φ is a linear and $B-B$-bimodule map. We show that Φ is surjective. Indeed,

$$ X \cdot M = A_1 \cdot M_0 = {}_C\langle M_1, M_0\rangle \cdot M_0 = M_1 \cdot \langle M_0, M_0\rangle_D = M_1 \cdot B = M_1 $$

by [3], Proposition 1.7. Hence, $\langle M, X \cdot M\rangle_D = \langle M, M_1\rangle_D = Y$. Thus, Φ is surjective. Let $m, n, m_1, n_1 \in M$, $x, x_1 \in X$. Then

$$ \begin{aligned} & {}_B\langle \widetilde{m} \otimes x \otimes n, \widetilde{m}_1 \otimes x_1 \otimes n_1\rangle \\ &\quad = {}_B\langle \widetilde{m} \cdot {}_A\langle x \otimes n, x_1 \otimes n_1\rangle, \widetilde{m}_1\rangle = {}_B\langle [{}_A\langle x_1 \otimes n_1, x \otimes n\rangle \cdot m]^\sim, \widetilde{m}_1\rangle \\ &\quad = \langle {}_A\langle x_1 \otimes n_1, x \otimes n\rangle \cdot m, m_1\rangle_B = \langle {}_A\langle x_1 \cdot {}_A\langle n_1, n\rangle, x\rangle \cdot m, m_1\rangle_B \\ &\quad = \langle [(x_1 \bullet {}_C\langle n_1, n\rangle) \bullet x^\natural] \cdot m, m_1\rangle_B = \langle [{}_C\langle x_1 \cdot n_1, n\rangle \bullet x^\natural] \cdot m, m_1\rangle_B \\ &\quad = \langle {}_C\langle [x_1 \cdot n_1], n\rangle \cdot [x^\natural \cdot m], m_1\rangle_B = \langle [x_1 \cdot n_1] \cdot \langle n, x^\natural \cdot m\rangle_D, m_1\rangle_B \\ &\quad = \langle x^\natural \cdot m, n\rangle_D \bullet \langle x_1 \cdot n_1, m_1\rangle_D = \langle m, x \cdot n\rangle_D \bullet \langle m_1, x_1 \cdot n_1\rangle_D^\sharp \\ &\quad = {}_B\langle \langle m, x \cdot n\rangle_D, \langle m_1, x_1 \cdot n_1\rangle_D\rangle = {}_B\langle \Phi(\widetilde{m} \otimes x \otimes n), \Phi(\widetilde{m}_1 \otimes x_1 \otimes n_1)\rangle. \end{aligned} $$

Hence, Φ preserves the left B-valued inner products. Also, similarly we can see that Φ preserves the right B-valued inner products. Furthermore,

$$ \Phi(\widetilde{m} \otimes x \otimes n)^\natural = \langle m, x \cdot n\rangle_Y^\natural = \langle m, x \cdot n\rangle_D^\sharp = \langle x \cdot n, m\rangle_D = \langle x \cdot n, m\rangle_Y. $$

On the other hand,

$$\Phi((\widetilde{m}\otimes x\otimes n)^{\natural}) = \Phi(\widetilde{n}\otimes x^{\natural}\otimes m) = \langle n, x^{\natural}\cdot m\rangle_Y = \langle x\cdot n, m\rangle_Y = \Phi(\widetilde{m}\otimes x\otimes n)^{\natural}.$$

Hence, Φ preserves the involutions $\natural$. Therefore $Y\cong \widetilde{M}\otimes_A X\otimes_A M$ as involutive Hilbert $B-B$-bimodules. $\square$

Let A and B be unital C^*-algebras. Let X and Y be an involutive Hilbert $A-A$-bimodule and an involutive Hilbert $B-B$-bimodule and let $\mathcal{A}_X$ and $\mathcal{A}_Y$ be the C^*-algebraic bundles over $\mathbf{Z}_2$ induced by X and Y, respectively. We suppose that there is an $A-B$-equivalence bimodule M such that

$$Y\cong \widetilde{M}\otimes_A X\otimes_A M$$

as involutive Hilbert $B-B$-bimodules. We construct an $\mathcal{A}_X-\mathcal{A}_Y$-equivalence bundle $\mathcal{M}=\{M_t\}_{t\in\mathbf{Z}_2}$ over $\mathbf{Z}_2$ such that

$${}_C\langle M_t, M_s\rangle = A_{ts^{-1}}, \quad \langle M_t, M_s\rangle_D = B_{t^{-1}s}$$

for any $t, s\in\mathbf{Z}_2$, where $C=A\oplus X$ and $D=B\oplus Y$.

Let Φ be an involutive Hilbert $B-B$-bimodule isomorphism of $\widetilde{M}\otimes_A X\otimes_A M$ onto Y. Then by the above discussions, there are the Hilbert $A-B$-bimodule isomorphisms Ψ of $X\otimes_A M$ onto $M\otimes_B Y$ and $\widetilde{\Psi}$ of $\widetilde{X}\otimes_A M$ onto $M\otimes_B\widetilde{Y}$, respectively. We construct a C_X-C_Y-equivalence bimodule C_M from M. Let C_M be the linear span of the set

$${}^X C_M = \left\{\begin{bmatrix} m_1 & x\otimes m_2 \\ \widetilde{x}^{\natural}\otimes m_2 & m_1\end{bmatrix} : m_1, m_2\in M, x\in X\right\}.$$

We define the left C_X-action on C_M by

$$\begin{bmatrix} a & z \\ \widetilde{z}^{\natural} & a\end{bmatrix}\cdot\begin{bmatrix} m_1 & x\otimes m_2 \\ \widetilde{x}^{\natural}\otimes m_2 & m_1\end{bmatrix} = \begin{bmatrix} a\otimes m_1 + z\otimes\widetilde{x}^{\natural}\otimes m_2 & a\otimes x\otimes m_2 + z\otimes m_1 \\ \widetilde{z}^{\natural}\otimes m_1 + a\otimes\widetilde{x}^{\natural}\otimes m_2 & \widetilde{z}^{\natural}\otimes x\otimes m_2 + a\otimes m_1\end{bmatrix}$$

for any $a\in A$, $m_1, m_2\in M$, $x, z\in X$, where we regard the tensor product as a left C_X-action on C_M in the formal manner. But we identify $A\otimes_A M$ and $X\otimes_A\widetilde{X}$, $\widetilde{X}\otimes_A X$ with M and closed two-sided ideals of A by the isomorphism and the monomorphisms defined by

$$a\otimes m\in A\otimes_A M\mapsto a\cdot m\in M, \quad x\otimes\widetilde{z}\in X\otimes_A\widetilde{X}\mapsto {}_A\langle x, z\rangle\in A,$$
$$\widetilde{x}\otimes z\in\widetilde{X}\otimes_A X\mapsto\langle x, z\rangle_A\in A.$$

Hence, we obtain that

$$\begin{aligned}
&\begin{bmatrix} a & z \\ \widetilde{z}^\natural & a \end{bmatrix} \cdot \begin{bmatrix} m_1 & x \otimes m_2 \\ \widetilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix} \\
&\qquad = \begin{bmatrix} a \cdot m_1 + {}_A\langle z, x^\natural \rangle \cdot m_2 & a \cdot x \otimes m_2 + z \otimes m_1 \\ \widetilde{z}^\natural \otimes m_1 + \widetilde{(a \cdot x)}^\natural \otimes m_2 & \langle \widetilde{z}^\natural, x \rangle_A \cdot m_2 + a \cdot m_1 \end{bmatrix} \in C_M.
\end{aligned}$$

We define the right C_Y-action on C_M by

$$\begin{bmatrix} m_1 & x \otimes m_2 \\ \widetilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix} \cdot \begin{bmatrix} b & y \\ \widetilde{y}^\natural & b \end{bmatrix} = \begin{bmatrix} m_1 \otimes b + x \otimes m_2 \otimes \widetilde{y}^\natural & m_1 \otimes y + x \otimes m_2 \otimes b \\ \widetilde{x}^\natural \otimes m_2 \otimes b + m_1 \otimes \widetilde{y}^\natural & \widetilde{x}^\natural \otimes m_2 \otimes y + m_1 \otimes b \end{bmatrix}$$

for any $b \in B$, $x \in X$, $y \in Y$, $m_1, m_2 \in M$, where we regard the tensor product as a right C_Y-action on C_M in the formal manner. But we identify $X \otimes_A M$ and $\widetilde{X} \otimes_A M$ with $M \otimes_B Y$ and $M \otimes_B \widetilde{Y}$ by Ψ and $\widetilde{\Psi}$, respectively. Hence, we obtain that

$$\begin{aligned}
&\begin{bmatrix} m_1 & x \otimes m_2 \\ \widetilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix} \cdot \begin{bmatrix} b & y \\ \widetilde{y}^\natural & b \end{bmatrix} \\
&\quad = \begin{bmatrix} m_1 \otimes b + x \otimes (\widetilde{\Psi})^{-1}(m_2 \otimes \widetilde{y}^\natural) & \Psi^{-1}(m_1 \otimes y) + x \otimes m_2 \otimes b \\ \widetilde{x}^\natural \otimes m_2 \otimes b + (\widetilde{\Psi})^{-1}(m_1 \otimes \widetilde{y}^\natural) & \widetilde{x}^\natural \otimes \Psi^{-1}(m_2 \otimes y) + m_1 \otimes b \end{bmatrix}.
\end{aligned}$$

Furthermore, we identify $M \otimes_B B$ and $Y \otimes_B \widetilde{Y}$, $\widetilde{Y} \otimes_B Y$ with M and closed two-sided ideals of B by the isomorphism and the monomorphisms defined by

$$\begin{aligned}
m \otimes b \in M \otimes_B B &\mapsto m \cdot b \in M, \\
y \otimes \widetilde{z} \in Y \otimes_B \widetilde{Y} &\mapsto {}_B\langle y, z \rangle \in B, \\
\widetilde{y} \otimes z \in \widetilde{Y} \otimes_B Y &\mapsto \langle y, z \rangle_B \in B,
\end{aligned}$$

respectively. Then $x \otimes (\widetilde{\Psi})^{-1}(m_2 \otimes y^\natural) = \widetilde{x}^\natural \otimes \Psi^{-1}(m_2 \otimes y)$ and we see that

$$\begin{bmatrix} m_1 & x \otimes m_2 \\ \widetilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix} \cdot \begin{bmatrix} b & y \\ \widetilde{y}^\natural & b \end{bmatrix} \in C_M.$$

Indeed, for any $\varepsilon > 0$, there are finite sets $\{n_k\}, \{l_k\} \subset M$ and $\{z_k\} \subset X$ such that

$$\left\| \Phi^{-1}(y) - \sum_k \widetilde{n}_k \otimes z_k \otimes l_k \right\| < \varepsilon.$$

Also,

$$\begin{aligned}
\left\| (\widetilde{\Phi})^{-1}(\widetilde{y}^\natural) - \left[\left(\sum_k \widetilde{n}_k \otimes z_k \otimes l_k \right)^\natural \right]^\sim \right\| &= \left\| [\Phi^{-1}(y)^\natural]^\sim - \left[\left(\sum_k \widetilde{n}_k \otimes z_k \otimes l_k \right)^\natural \right]^\sim \right\| \\
&= \left\| \Phi^{-1}(y) - \sum_k \widetilde{n}_k \otimes z_k \otimes l_k \right\| < \varepsilon.
\end{aligned}$$

Thus

$$\left\|x \otimes (\widetilde{\Psi})^{-1}(m_2 \otimes \widetilde{y}^\natural) - x \otimes m_2 \otimes \left[\left(\sum_k \widetilde{n}_k \otimes z_k \otimes l_k\right)^\natural\right]^\sim\right\|$$
$$= \left\|x \otimes m_2 \otimes (\widetilde{\Phi})^{-1}(\widetilde{y}^\natural) - x \otimes m_2 \otimes \sum_k [(\widetilde{n}_k \otimes z_k \otimes l_k)^\natural]^\sim\right\| \leqslant \|x\|\|m_2\|\varepsilon$$

and

$$\left\|\widetilde{x}^\natural \otimes \Psi^{-1}(m_2 \otimes y) - \widetilde{x}^\natural \otimes m_2 \otimes \sum_k \widetilde{n}_k \otimes z_k \otimes l_k\right\|$$
$$= \left\|\widetilde{x}^\natural \otimes m_2 \otimes \Phi^{-1}(y) - \widetilde{x}^\natural \otimes m_2 \otimes \sum_k \widetilde{n}_k \otimes z_k \otimes l_k\right\| \leqslant \|x\|\|m_2\|\varepsilon.$$

Furthermore, we can see that

$$\begin{aligned} x \otimes m_2 \otimes & \left[\left(\sum_k \widetilde{n}_k \otimes \widetilde{z}_k \otimes l_k\right)^\natural\right]^\sim \\ &= \sum_k x \otimes m_2 \otimes \widetilde{n}_k \otimes \widetilde{z_k}^\natural \otimes l_k = \sum_k {}_A\langle x \cdot {}_A\langle m_2, n_k\rangle, z_k^\natural\rangle \cdot l_k \\ &= \sum_k \widetilde{x}^\natural \otimes m_2 \otimes \widetilde{n}_k \otimes z_k \otimes l_k = \widetilde{x}^\natural \otimes m_2 \otimes \sum_k \widetilde{n}_k \otimes z_k \otimes l_k, \end{aligned}$$

where we identify $A \otimes_A M$ and $X \otimes_A \widetilde{X}$, $\widetilde{X} \otimes_A X$ with M and closed two-sided ideals of A by the isomorphism and the monomorphisms defined by

$$\begin{aligned} a \otimes m \in A \otimes_A M &\mapsto a \cdot m \in M, \\ x \otimes \widetilde{z} \in X \otimes_A \widetilde{X} &\mapsto {}_A\langle x, z\rangle \in A, \\ \widetilde{x} \otimes z \in \widetilde{X} \otimes_A X &\mapsto \langle x, z\rangle_A \in A. \end{aligned}$$

Hence

$$x \otimes m_2 \otimes \left[\left(\sum_k \widetilde{n}_k \otimes \widetilde{z}_k \otimes l_k\right)^\natural\right]^\sim = \widetilde{x}^\natural \otimes m_2 \otimes \sum_k \widetilde{n}_k \otimes z_k \otimes l_k.$$

It follows that

$$\|x \otimes (\widetilde{\Psi})^{-1}(m_2 \otimes y^\natural) - \widetilde{x}^\natural \otimes \Psi^{-1}(m_2 \otimes y)\| \leqslant 2\|x\|\,\|m_2\|\varepsilon.$$

Since ε is arbitrary, we can see that $x \otimes (\widetilde{\Psi})^{-1}(m_2 \otimes y^\natural) = \widetilde{x}^\natural \otimes \Psi^{-1}(m_2 \otimes y)$ and that

$$\begin{bmatrix} m_1 & x \otimes m_2 \\ \widetilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix} \cdot \begin{bmatrix} b & y \\ \widetilde{y}^\natural & b \end{bmatrix} \in C_M.$$

Before we define a left C_X-valued inner product and a right C_Y-valued inner product on C_M, we define a conjugate linear map on C_M,

$$\begin{bmatrix} m_1 & x\otimes m_2 \\ \widetilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix} \in C_M \mapsto \begin{bmatrix} m_1 & x\otimes m_2 \\ \widetilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix}^\sim \in C_M$$

by

$$\begin{bmatrix} m_1 & x\otimes m_2 \\ \widetilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix}^\sim = \begin{bmatrix} \widetilde{m}_1 & (\widetilde{x}^\natural \otimes m_2)^\sim \\ (x\otimes m_2)^\sim & \widetilde{m}_1 \end{bmatrix}$$

for any $m_1, m_2 \in M$, $x \in X$. Since we identify $\widetilde{X \otimes_A M}$ and $\widetilde{\widetilde{X} \otimes_A M}$ with $\widetilde{M} \otimes_A \widetilde{X}$ and $\widetilde{M} \otimes_A X$ by Lemma 5.1, respectively, we obtain that

$$\begin{bmatrix} m_1 & x\otimes m_2 \\ \widetilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix}^\sim = \begin{bmatrix} \widetilde{m}_1 & \widetilde{m}_2 \otimes x^\natural \\ \widetilde{m}_2 \otimes \widetilde{x} & \widetilde{m}_1 \end{bmatrix}.$$

We define the left C_X-valued inner product on C_M by

$$\begin{aligned}
&{}_{C_X}\left\langle \begin{bmatrix} m_1 & x\otimes m_2 \\ \widetilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix}, \begin{bmatrix} n_1 & z\otimes n_2 \\ \widetilde{z}^\natural \otimes n_2 & n_1 \end{bmatrix} \right\rangle \\
&\quad = \begin{bmatrix} m_1 & x\otimes m_2 \\ \widetilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix} \cdot \begin{bmatrix} n_1 & z\otimes n_2 \\ \widetilde{z}^\natural \otimes n_2 & n_1 \end{bmatrix}^\sim \\
&\quad = \begin{bmatrix} {}_A\langle m_1, n_1\rangle + {}_A\langle x \cdot {}_A\langle m_2, n_2\rangle, z\rangle & {}_A\langle m_1, n_2\rangle, \cdot z^\natural + x \cdot {}_A\langle m_2, n_1\rangle \\ \widetilde{x}^\natural \cdot {}_A\langle m_2, n_1\rangle + {}_A\langle m_1, n_2\rangle \cdot \widetilde{z} & {}_A\langle x \cdot {}_A\langle m_2, n_2\rangle, z\rangle + {}_A\langle m_1, n_1\rangle \end{bmatrix}
\end{aligned}$$

for any $m_1, m_2, n_1, n_2 \in M$, $x, z \in X$, where we regard the tensor product as a product in C_M in the formal manner and identify in the same way as above. Similarly, we define the right C_Y-valued inner product on C_M by

$$\begin{aligned}
&\left\langle \begin{bmatrix} m_1 & x\otimes m_2 \\ \widetilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix}, \begin{bmatrix} n_1 & z\otimes n_2 \\ \widetilde{z}^\natural \otimes n_2 & n_1 \end{bmatrix} \right\rangle_{C_Y} \\
&\quad = \begin{bmatrix} m_1 & x\otimes m_2 \\ \widetilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix}^\sim \cdot \begin{bmatrix} n_1 & z\otimes n_2 \\ \widetilde{z}^\natural \otimes n_2 & n_1 \end{bmatrix} \\
&\quad = \begin{bmatrix} \langle m_1, n_1\rangle_B + \langle m_2, \langle x, z\rangle_A \cdot n_2\rangle_B & \widetilde{m}_1 \otimes \Psi(z\otimes n_2) + \widetilde{m}_2 \otimes \Psi(x^\natural \otimes n_1) \\ \widetilde{m}_2 \otimes \widetilde{\Psi}(\widetilde{x} \otimes n_1) + \widetilde{m}_1 \otimes \widetilde{\Psi}(\widetilde{z}^\natural \otimes n_2) & \langle m_2, \langle x, z\rangle_A \cdot n_2\rangle_B + \langle m_1, n_1\rangle_B \end{bmatrix}
\end{aligned}$$

for any $m_1, m_2, n_1, n_2 \in M$, $x, z \in X$, where we regard the tensor product as a product in C_M in the formal manner, identifying in the same way as above and by the isomorphisms Ψ and $\widetilde{\Psi}$. Here, we have to show that the value of the above inner product on C_M exists in C_M. Indeed, by routine computations,

$$\widetilde{m}_1 \otimes \Psi(z\otimes n_2) = \sum_i \widetilde{m}_1 \otimes u_i \otimes \Phi(\widetilde{u}_i \otimes z \otimes n_2) = \Phi(\widetilde{m}_1 \otimes z \otimes n_2) \in Y,$$

$$\widetilde{m}_2 \otimes \Psi(x^\natural \otimes n_1) = \Phi(\widetilde{m}_2 \otimes x^\natural \otimes n_1) \in Y.$$

Also,

$$\widetilde{m}_2 \otimes \widetilde{\Psi}(\widetilde{x} \otimes n_1) = \sum_i \widetilde{m}_2 \otimes u_i \otimes \widetilde{\Phi}(\widetilde{u}_i \otimes \widetilde{x} \otimes n_1) = \Phi(\widetilde{n}_1 \otimes x \otimes m_2)^{\sim} \in \widetilde{Y},$$

$$\widetilde{n}_1 \otimes \widetilde{\Psi}(\widetilde{z^{\natural}} \otimes n_2) = \sum_i \widetilde{m}_1 \otimes u_i \otimes \widetilde{\Phi}(\widetilde{u}_i \otimes \widetilde{z^{\natural}} \otimes n_2) = \sum_i \Phi(\widetilde{n}_2 \otimes z^{\natural} \otimes m_1)^{\sim} \in \widetilde{Y}.$$

Thus

$$\begin{aligned}[\widetilde{m}_2 \otimes \widetilde{\Psi}(\widetilde{x} \otimes n_1) + \widetilde{n}_1 \otimes \widetilde{\Psi}(\widetilde{z^{\natural}} \otimes n_2)]^{\widetilde{\natural}} &= \Phi(\widetilde{n}_1 \otimes x \otimes m_2)^{\natural} + \Phi(\widetilde{n}_2 \otimes z^{\natural} \otimes m_1)^{\natural} \\ &= \Phi(\widetilde{m}_2 \otimes x^{\natural} \otimes n_1) + \Phi(\widetilde{m}_1 \otimes z \otimes n_2) \\ &= \widetilde{m}_1 \otimes \Psi(z \otimes x) + \widetilde{m}_2 \otimes \Psi(x^{\natural} \otimes n_1).\end{aligned}$$

Hence

$$\left\langle \begin{bmatrix} m_1 & x \otimes m_2 \\ \widetilde{x^{\natural}} \otimes m_2 & m_1 \end{bmatrix}, \begin{bmatrix} n_1 & z \otimes n_2 \\ \widetilde{z^{\natural}} \otimes n_2 & n_1 \end{bmatrix} \right\rangle_{C_Y} \in C_Y.$$

By the above definitions, C_M has the left C_X- and the right C_Y-actions and the left C_X-valued and the right C_Y-valued inner products.

Let C_M^1 be the linear span of the set

$$C_M^Y = \left\{ \begin{bmatrix} m_1 & m_2 \otimes y \\ m_2 \otimes \widetilde{y^{\natural}} & m_1 \end{bmatrix} : \, m_1, m_2 \in M, y \in Y \right\}.$$

In the similar way to the above, we define a left C_X- and a right C_Y-actions on C_M^1 and a left C_X-valued and a right C_Y-valued inner products. But identifying $X \otimes_A M$ and $\widetilde{X} \otimes_A M$ with $M \otimes_B Y$ and $M \otimes_B \widetilde{Y}$ by Ψ and $\widetilde{\Psi}$, respectively, we can see that each of them coincides with the other by routine computations. Hence, we obtain the following lemma:

Lemma 5.12. *With the above notation, C_M is a $C_X - C_Y$-equivalence bimodule.*

P r o o f. By the definitions of the left C_X-action and the left C_X-valued inner product on C_M, we can see that Conditions (a)–(d) in [6], Proposition 1.12 hold. By the definitions of the right C_Y-action and the right C_Y-valued inner product on C_M, we can also see that the similar conditions to Conditions (a)–(d) in [6], Proposition 1.12 hold. Furthermore, we can easily see that the associativity of the left C_X-valued inner product and the right C_Y-valued inner product hold. Since M is an $A - B$-equivalence bimodule, there are finite subsets $\{u_i\}_{i=1}^n$ and $\{v_j\}_{j=1}^m$ of M such that

$$\sum_{i=1}^n {}_A\langle u_i, u_i \rangle = 1, \quad \sum_{j=1}^m \langle v_j, v_j \rangle_B = 1.$$

Let $U_i = \begin{bmatrix} u_i & 0 \\ 0 & u_i \end{bmatrix}$ for any i and let $V_j = \begin{bmatrix} v_j & 0 \\ 0 & v_j \end{bmatrix}$ for any j. Then $\{U_i\}$ and $\{V_j\}$ are finite subsets of C_M and

$$\sum_{i=1}^{n} {}_{C_X}\langle U_i, U_i\rangle = \sum_{i=1}^{n} \begin{bmatrix} u_i & 0 \\ 0 & u_i \end{bmatrix} \begin{bmatrix} \widetilde{u}_i & 0 \\ 0 & \widetilde{u}_i \end{bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} {}_A\langle u_i, u_i\rangle & 0 \\ 0 & {}_A\langle u_i, u_i\rangle \end{bmatrix} = 1_{C_X}.$$

Similarly, $\sum_{j=1}^{m} \langle V_j, V_j\rangle_{C_Y} = 1_{C_Y}$. Thus, since the associativity of the left C_X-valued inner product and the right C_Y-valued inner product on C_M holds, we can see that $\{U_i\}$ and $\{V_j\}$ are a right C_Y-basis and a left C_X-basis of C_M, respectively. Hence by [6], Proposition 1.12, C_M is a $C_X - C_Y$-equivalence bimodule. $\square$

Lemma 5.13. *Let A and B be unital C^*-algebras. Let X and Y be an involutive Hilbert $A-A$-bimodule and an involutive Hilbert $B-B$-bimodule, respectively. Let $\mathcal{A}_X = \{A_t\}_{t\in\mathbf{Z}_2}$ and $\mathcal{A}_Y = \{B_t\}_{t\in\mathbf{Z}_2}$ be C^*-algebraic bundles over $\mathbf{Z}_2$ induced by X and Y, respectively. We suppose that there is an $A-B$-equivalence bimodule M such that*

$$Y \cong \widetilde{M} \otimes_A X \otimes_A M$$

as involutive Hilbert $B-B$-bimodules. Then there is an $\mathcal{A}_X - \mathcal{A}_Y$-equivalence bundle $\mathcal{M} = \{M_t\}_{t\in\mathbf{Z}_2}$ over $\mathbf{Z}_2$ such that

$${}_C\langle M_t, M_s\rangle = A_{ts^{-1}}, \quad \langle M_t, M_s\rangle_D = B_{t^{-1}s}$$

for any $t, s \in \mathbf{Z}_2$, where $C = A \oplus X$ and $D = B \oplus Y$.

P r o o f. Let C_M be the $C_X - C_Y$-equivalence bimodule induced by M, which is defined in the above. We identify $M \oplus (X \otimes_A M)$ with C_M as vector spaces over $\mathbf{C}$ by the isomorphism defined by

$$m_1 \oplus (x \otimes m_2) \in M \oplus (X \otimes_A M) \mapsto \begin{bmatrix} m_1 & x \otimes m_2 \\ \widetilde{x^\natural} \otimes m_2 & m_1 \end{bmatrix} \in C_M.$$

Since we identify $C = A \oplus X$ and $D = B \oplus Y$ with C_X and C_Y, respectively, $M \oplus (X \otimes_A M)$ is a $C-D$-equivalence bimodule by above identifications and Lemma 5.12. Let $M_0 = M$ and $M_1 = X \otimes_A M$. We note that $X \otimes_A M$ is identified with $M \otimes_B Y$ by the Hilbert $A-B$-bimodule isomorphism Ψ. Let $\mathcal{M} = \{M_t\}_{t\in\mathbf{Z}_2}$. Then by routine computations, $\mathcal{M}$ is an $\mathcal{A}_X - \mathcal{A}_Y$-equivalence bundle over $\mathbf{Z}_2$ such that

$${}_C\langle M_t, M_s\rangle = A_{ts^{-1}}, \quad \langle M_t, M_s\rangle_D = B_{t^{-1}s}$$

for any $t, s \in \mathbf{Z}_2$. $\square$

Proposition 5.14. *Let A and B be unital C^*-algebras. Let X and Y be an involutive Hilbert $A-A$-bimodule and an involutive Hilbert $B-B$-bimodule, respectively. Let $\mathcal{A}_X = \{A_t\}_{t\in\mathbf{Z}_2}$ and $\mathcal{A}_Y = \{B_t\}_{t\in\mathbf{Z}_2}$ be the C^*-algebraic bundles over $\mathbf{Z}_2$ induced by X and Y, respectively. Then the following conditions are equivalent:*

(1) *There is an $\mathcal{A}_X - \mathcal{A}_Y$-equivalence bundle $\mathcal{M} = \{M_t\}_{t\in\mathbf{Z}_2}$ over $\mathbf{Z}_2$ such that*

$$_C\langle M_t, M_s\rangle = A_{ts^{-1}}, \quad \langle M_t, M_s\rangle_D = B_{t^{-1}s}$$

for any $t, s \in \mathbf{Z}_2$, where $C = A \oplus X$ and $D = B \oplus Y$.

(2) *There is an $A - B$-equivalence bimodule M such that*

$$Y \cong \widetilde{M} \otimes_A X \otimes_A M$$

as involutive Hilbert $B - B$-bimodules.

P r o o f. This is immediate by Lemmas 5.11 and 5.13. □

Theorem 5.15. *Let A and B be unital C^*-algebras. Let X and Y be an involutive Hilbert $A-A$-bimodule and an involutive Hilbert $B-B$-bimodule, respectively. Let $A \subset C_X$ and $B \subset C_Y$ be the unital inclusions of unital C^*-algebras induced by X and Y, respectively. Then the following hold:*

(1) *If there is an $A - B$-equivalence bimodule M such that*

$$\widetilde{M} \otimes_A X \otimes_A M \cong Y$$

as involutie Hilbert $B - B$-bimodules, then the unital inclusions $A \subset C_X$ and $B \subset C_Y$ are strongly Morita equivalent.

(2) *We suppose that X and Y are full with the both inner products and that $A' \cap C_X = \mathbf{C}1$. If the unital inclusions $A \subset C_X$ and $B \subset C_Y$ are strongly Morita equivalent, then there is an $A - B$-equivalence bimodule M such that*

$$\widetilde{M} \otimes_A X \otimes_A M \cong Y$$

as involutive Hilbert $B - B$-bimodules.

P r o o f. Let $\mathcal{A}_X = \{A_t\}_{t\in\mathbf{Z}_2}$ and $\mathcal{A}_Y = \{B_t\}_{t\in\mathbf{Z}_2}$ be the C^*-algebraic bundles over $\mathbf{Z}_2$ induced by X and Y, respectively. We prove (1). We suppose that there is an $A - B$-equivalence bimodule M such that

$$\widetilde{M} \otimes_A X \otimes_A M \cong Y$$

as involutive Hilbert $B-B$-bimodules. Then by Proposition 5.14, there is an $\mathcal{A}_X - \mathcal{A}_Y$-equivalence bundle $\mathcal{M} = \{M_t\}_{t\in\mathbf{Z}_2}$ over $\mathbf{Z}_2$ such that

$$ {}_C\langle M_t, M_s\rangle = A_{ts^{-1}}, \quad \langle M_t, M_s\rangle_D = B_{t^{-1}s} $$

for any $t, s \in \mathbf{Z}_2$, where $C = A \oplus X$ and $D = B \oplus Y$. Hence, by Proposition 2.1, the unital inclusions of unital C^*-algebras $A \subset C$ and $B \subset D$ are strongly Morita equivalent. Since we identify $A \subset C$ and $B \subset D$ with $A \subset C_X$ and $B \subset C_Y$, respectively, $A \subset C_X$ and $B \subset C_Y$ are strongly Morita equivalent. Next, we prove (2). We suppose that X and Y are full with the both inner products and that $A' \cap C_X = \mathbf{C}1$. Also, we suppose that $A \subset C_X$ and $B \subset C_Y$ are strongly Morita equivalent. Then $\mathcal{A}_X$ and $\mathcal{A}_Y$ are saturated by Lemma 5.5. Since the identity map $\mathrm{id}_{\mathbf{Z}_2}$ is the only automorphism of $\mathbf{Z}_2$, by Theorem 4.6 there is an $\mathcal{A}_X - \mathcal{A}_Y$-equivalence bundle $\mathcal{M} = \{M_t\}_{t\in\mathbf{Z}_2}$ such that

$$ {}_C\langle M_t, M_s\rangle = A_{ts^{-1}}, \quad \langle M_t, M_s\rangle_D = B_{t^{-1}s} $$

for any $t, s \in \mathbf{Z}$. Hence, from Proposition 5.14, there is an $A-B$-equivalence bimodule M such that

$$ Y \cong \widetilde{M} \otimes_A X \otimes_A M $$

as involutive Hilbert $B-B$-bimodules. □

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