AN INEQUALITY FOR FIBONACCI NUMBERS

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Abstract. We extend an inequality for Fibonacci numbers published by P. G. Popescu and J. L. Díaz-Barrero in 2006.

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1. Introduction

The classical Fibonacci numbers are defined by the linear recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-2} + F_{n-1}, \quad n = 2, 3, \ldots$$

A closed-form expression is given by

$$F_n = \frac{1}{\sqrt{5}}(\varphi^n - (1-\varphi)^n), \tag{1.1}$$

where

$$\varphi = \frac{1}{2}\big(1 + \sqrt{5}\big) = 1.618\ldots \tag{1.2}$$

denotes the golden ratio. A detailed collection of the main properties of the Fibonacci numbers can be found, for instance, in Koshy [1].

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The work on this note was inspired by a paper published by Popescu and Díaz-Barrero in 2006 (see [2]). The authors used Jensen's inequality for convex functions to prove the following elegant inequality for Fibonacci numbers,

$$(F_nF_{n+1})^2 \leqslant \sum_{k=1}^{n} F_k^r \sum_{k=1}^{n} F_k^{4-r}, \quad n \in \mathbb{N},\ r \in \mathbb{Z}. \tag{1.3}$$

The proof reveals that (1.3) is valid for all $n \in \mathbb{N}$ and $r \in \mathbb{R}$. Here, we extend this result and state an open problem in connection with (1.3).

2. Results

The following extension of inequality (1.3) holds.

Theorem 2.1. *Let $r, s \in \mathbb{R}$ with $r + s \geqslant 4$. Then, for all $n \in \mathbb{N}$,*

$$(F_nF_{n+1})^2 \leqslant \sum_{k=1}^{n} F_k^r \sum_{k=1}^{n} F_k^s. \tag{2.1}$$

The sign of equality holds in (2.1) *if and only if $n = 1, 2$ or $n \geqslant 3$, $r = s = 2$.*

P r o o f. Since $F_k \geqslant 1$, $k \geqslant 1$, we obtain $F_k^2 \leqslant F_k^{(r+s)/2}$. Using the Cauchy-Schwarz inequality gives

$$(F_nF_{n+1})^2 = \left(\sum_{k=1}^{n} F_k^2\right)^2 \leqslant \left(\sum_{k=1}^{n} F_k^{r/2} F_k^{s/2}\right)^2 \leqslant \sum_{k=1}^{n} F_k^r \sum_{k=1}^{n} F_k^s.$$

We assume that equality holds in (2.1). Let $n \geqslant 3$. We obtain

$$F_k^2 = F_k^{(r+s)/2}, \quad k = 1, 2, \ldots, n.$$

For $k = 3$, we find $\frac{1}{2}(r + s) = 2$. Moreover, we get

$$F_k^{r/2} = \lambda F_k^{s/2}, \quad k = 1, 2, \ldots, n.$$

For $k = 1$ we obtain $\lambda = 1$ and for $k = 3$ we find

$$2^{r/2} = F_3^{r/2} = \lambda F_3^{s/2} = 2^{s/2}.$$

Thus, $r = s$. It follows that $r = s = 2$. □

This result leads to the problem to determine all real numbers r and s such that (2.1) holds for all integers $n \geqslant 1$. Here, we offer a partial solution.

Theorem 2.2. *Let $r, s \in \mathbb{R}$ with $rs \geqslant 0$. The inequality* (2.1) *holds for all $n \in \mathbb{N}$ if and only if $r + s \geqslant 4$.*

Proof. With regard to Theorem 2.1 it remains to prove that if $rs \geqslant 0$ and $r + s < 4$, then (2.1) is not valid for all n. Therefore, it suffices to show that

$$\lim_{n\to\infty} Q_n = 0, \tag{2.2}$$

where

$$Q_n = (F_n F_{n+1})^{-2} \sum_{k=1}^{n} F_k^r \sum_{k=1}^{n} F_k^s.$$

We consider two cases.

Case 1. $r \leqslant 0$ and $s \leqslant 0$. Then we have $F_k^r \leqslant 1$ and $F_k^s \leqslant 1$, $k \geqslant 1$. Thus,

$$\sum_{k=1}^{n} F_k^r \leqslant n \quad \text{and} \quad \sum_{k=1}^{n} F_k^s \leqslant n.$$

It follows that

$$0 < Q_n \leqslant (F_n F_{n+1})^{-2} n^2 \leqslant \Big(\frac{n}{F_n}\Big)^2. \tag{2.3}$$

Let φ be the number defined in (1.2). Then,

$$\frac{1}{2}\varphi^n \leqslant \varphi^n - (1-\varphi)^n. \tag{2.4}$$

Using (1.1) and (2.4) gives

$$0 < \frac{n}{F_n} = \frac{\sqrt{5}n}{\varphi^n - (1-\varphi)^n} \leqslant 2\sqrt{5}\frac{n}{\varphi^n}. \tag{2.5}$$

From (2.3) and (2.5) we conclude that (2.2) holds.

Case 2. $r \geqslant 0$ and $s \geqslant 0$. Since

$$\sum_{k=1}^{n} F_k^r \leqslant nF_n^r \quad \text{and} \quad \sum_{k=1}^{n} F_k^s \leqslant nF_n^s,$$

we obtain

$$0 < Q_n \leqslant (F_n F_{n+1})^{-2} nF_n^r nF_n^s = \frac{n^2}{F_n^\alpha}\Big(\frac{F_n}{F_{n+1}}\Big)^2 \tag{2.6}$$

with $\alpha = 4 - (r+s) > 0$. From (1.1) and (2.4) we get

$$(2.7) \qquad 0 < \frac{n^2}{F_n^\alpha} = (\sqrt{5})^\alpha \frac{n^2}{(\varphi^n - (1-\varphi)^n)^\alpha} \leqslant (2\sqrt{5})^\alpha \frac{n^2}{a^n}$$

with $a = \varphi^\alpha > 1$. Applying

$$\lim_{n\to\infty} \frac{n^2}{a^n} = 0$$

and the known limit relation

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \frac{1}{\varphi}$$

we conclude from (2.6) and (2.7) that (2.2) holds. □

The following problem still remains open: determine all real parameters r and s with $rs < 0$ such that (2.1) holds for all n. It is tempting to conjecture that in this case the condition $r + s \geqslant 4$ is necessary. We show that this is not true.

Let $\max(r,s) \geqslant 6$. Inequality (2.1) is valid for $n = 1, 2$. Let $n \geqslant 3$. Then,

$$(F_n F_{n+1})^2 \leqslant (F_n \cdot 2F_n)^2 = (F_3 F_n^2)^2 \leqslant (F_n^3)^2 = F_n^6 \leqslant \sum_{k=1}^n F_k^r \sum_{k=1}^n F_k^s.$$

This means that there exist numbers r and s with $rs < 0$ and $r+s < 4$ such that (2.1) is valid for all n.

A c k n o w l e d g e m e n t. We thank the referee for the careful reading of the manuscript.

References

[1] *T. Koshy*: Fibonacci and Lucas Numbers with Applications. Volume I. Pure and Applied Mathematics. A Wiley-Interscience Series of Texts, Monographs, and Tracts. Wiley & Sons, New York, 2001. zbl MR doi

[2] *P. G. Popescu, J. L. Díaz-Barrero*: Certain inequalities for convex functions. JIPAM, J. Inequal. Pure Appl. Math. *7* (2006), Article ID 41, 5 pages. zbl MR

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