NOTE ON THE HILBERT 2-CLASS FIELD TOWER

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Abstract. Let k be a number field with a 2-class group isomorphic to the Klein four-group. The aim of this paper is to give a characterization of capitulation types using group properties. Furthermore, as applications, we determine the structure of the second 2-class groups of some special Dirichlet fields $\Bbbk = \mathbb{Q}(\sqrt{d}, \sqrt{-1})$, which leads to a correction of some parts in the main results of A. Azizi and A. Zekhini (2020).

Keywords: multiquadratic field; fundamental systems of units; 2-class group; 2-class field tower; capitulation

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1. Introduction

Let k be an algebraic number field and let $\mathbb{CL}_2(k)$ denote its 2-class group, that is the 2-Sylow subgroup of the ideal class group $\mathbb{CL}(k)$ of k. Denote by $k^{(1)}$ the first Hilbert 2-class field of k, that is the maximal abelian unramified extension of k such that the degree $[k^{(1)} : k]$ is a power of 2, and by $k^{(2)}$ the Hilbert 2-class field of $k^{(1)}$. Let $G_k = \mathrm{Gal}(k^{(2)}/k)$ be the Galois group of $k^{(2)}/k$ and G'_k be its derived subgroup. Then it is well known, by class field theory, that $\mathrm{Gal}(k^{(1)}/k) \simeq \mathbb{CL}_2(k) \simeq G_k/G'_k$.

The determination of the structure of G_k is a classical and difficult open problem of class field theory that is related to many other problems such as the capitulation and the length of the Hilbert 2-class field tower. Actually, our goal in the present paper is to investigate these problems for fields with 2-class groups of type $(2, 2)$. Note that if $\mathbb{CL}_2(k)$ is of type $(2, 2)$, the Hilbert 2-class field tower of k terminates in at most two steps and the structure of G_k is based on the capitulation problem in unramified quadratic extensions of k. In fact, G_k is isomorphic to one of the groups A, Q_m, D_m or S_m, where A is the Klein four-group, and Q_m, D_m, or S_m denote the

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quaternion, dihedral or semidihedral groups, respectively, of order 2^m, with $m \geqslant 3$ and $m \geqslant 4$ for S_m (cf. [13]).

In this paper, we give a characterization of the capitulation types (see Table 2) using some group properties and as an application, we determine the structure of the second 2-class groups of some special Dirichlet fields $\Bbbk = \mathbb{Q}(\sqrt{d}, \sqrt{-1})$.

If k is a number field, we use the following notations:

$h_2(k)$: the 2-class number of k,
$h_2(d)$: the 2-class number of the quadratic field $\mathbb{Q}(\sqrt{d})$,
ε_d: the fundamental unit of the quadratic field $\mathbb{Q}(\sqrt{d})$,
E_k: the unit group of k,
FSU: abbreviation of "fundamental system of units",
$k^{(1)}$: the Hilbert 2-class field of k,
$k^{(2)}$: the Hilbert 2-class field of $k^{(1)}$,
G_k: the Galois group of $k^{(2)}/k$,
k^+: the maximal real subfield of k,
$q(k) = \left[E_k : \prod_i E_{k_i}\right]$: the unit index of k, if k is multiquadratic and k_i are the quadratic subfields of k,
$N_{k'/k}$: the norm map of an extension k'/k.

2. Preliminaries

Let Q_m, D_m, and S_m denote the quaternion, dihedral, and semidihedral groups, respectively, of order 2^m, where $m \geqslant 3$ and $m \geqslant 4$ for S_m; in addition let A be the Klein four-group. Each of these groups is generated by two elements x and y, and admits the following presentations:

$$\begin{array}{ll} x^2 = y^2 = 1,\ y^{-1}xy = x & \text{for } A, \\ x^{2^{m-2}} = y^2 = a,\ a^2 = 1,\ y^{-1}xy = x^{-1} & \text{for } Q_m, \\ x^{2^{m-1}} = y^2 = 1,\ y^{-1}xy = x^{-1} & \text{for } D_m, \\ x^{2^{m-1}} = y^2 = 1,\ y^{-1}xy = x^{2^{m-2}-1} & \text{for } S_m. \end{array}$$

We recall some well known properties of 2-groups G_k such that G_k/G_k' is of type $(2,2)$, where G_k' denotes the commutator subgroup of G_k. For more details about these properties, we refer the reader to [13], pages 272–273, [7], pages 1467–1469, and [9], Chapter 5.

Let x and y be as above. Note that the commutator subgroup G_k' of G is always cyclic and $G_k' = \langle x^2 \rangle$. The group G_k possesses exactly three subgroups of index 2, which are,

$$H_1 = \langle x \rangle, \quad H_2 = \langle x^2, y \rangle, \quad H_3 = \langle x^2, xy \rangle.$$

Note also that for the two cases Q_3 and A, each H_i is cyclic. For the case D_m with $m > 3$, H_2 and H_3 are also dihedral. For Q_m with $m > 3$, H_2 and H_3 are quaternion. Finally for S_m, H_2 is dihedral whereas H_3 is quaternion. Furthermore, if G_k is isomorphic to A (or Q_3), then the subgroups H_i are cyclic of order 2 (or 4, respectively). If G_k is isomorphic to Q_m with $m > 3$, D_m with $m > 3$ or S_m, then H_1 is cyclic and H_i/H_i' is of type $(2,2)$ for $i \in \{2,3\}$, where H_i' is the commutator subgroup of H_i.

Let F_i be the subfield of $k^{(2)}$ fixed by H_i, where $i \in \{1,2,3\}$. If $k^{(2)} \neq k^{(1)}$, $\langle x^4 \rangle$ is the unique subgroup of G_k' of index 2. Let L (L is defined only if $k^{(2)} \neq k^{(1)}$) be the subfield of $k^{(2)}$ fixed by $\langle x^4 \rangle$. Then F_1, F_2 and F_3 are the three quadratic subextensions of $k^{(1)}/k$ and L is the unique subfield of $k^{(2)}$ such that L/k is a nonabelian Galois extension of degree 8.

Let us recall the definition of Taussky's conditions A and B. Let k' be a cyclic unramified extension of a number field k and j denotes the basic homomorphism $j_{k'/k}\colon \mathbb{CL}(k) \to \mathbb{CL}(k')$, induced by the extension of ideals from k to k'. Thus, we say:

- ▷ k'/k satisfies condition A if and only if $|\ker(j_{k'/k}) \cap N_{k'/k}(\mathbb{CL}(k'))| > 1$.
- ▷ k'/k satisfies condition B if and only if $|\ker(j_{k'/k}) \cap N_{k'/k}(\mathbb{CL}(k'))| = 1$.

Set $j_{F_i/k} = j_i$, $i = 1,2,3$. Then we have:

Theorem 2.1 ([13], Theorem 2)**.**

(1) *If* $k^{(1)} = k^{(2)}$, *then all* F_i *satisfy condition* A, $|\ker(j_i)| = 4$ *for* $i = 1,2,3$ *and* G_k *is abelian of type* $(2,2)$.

(2) *If* $\mathrm{Gal}(L/k) \simeq Q_3$, *then all* F_i *satisfy condition* A *and* $|\ker(j_i)| = 2$ *for* $i = 1,2,3$ *and* $G_k \simeq Q_3$.

(3) *If* $\mathrm{Gal}(L/k) \simeq D_3$, *then* F_2, F_3 *satisfy condition* B *and* $|\ker j_2| = |\ker j_3| = 2$. *Furthermore, if* F_1 *satisfies condition* B, *then* $|\ker j_1| = 2$ *and* $G_k \simeq S_m$, *if* F_1 *satisfies condition* A *and* $|\ker j_1| = 2$, *then* $G_k \simeq Q_m$. *If* F_1 *satisfies condition* A *and* $|\ker j_1| = 4$, *then* $G_k \simeq D_m$.

We summarize these results in Table 1.

$\lvert\ker(j_1)\rvert$	(A/B)	$\lvert\ker(j_2)\rvert$	(A/B)	$\lvert\ker(j_3)\rvert$	(A/B)	G_k
4	A	4	A	4	A	$(2,2)$
2	A	2	A	2	A	Q_3
4	A	2	B	2	B	D_m, $m \geqslant 3$
2	A	2	B	2	B	Q_m, $m > 3$
2	B	2	B	2	B	S_m, $m > 3$

Table 1. Capitulation types.

Therefore, one can easily deduce the following remark.

R e m a r k 2.2. The 2-class groups of the three unramified quadratic extensions of k are cyclic if and only if $k^{(1)} = k^{(2)}$ or $k^{(1)} \neq k^{(2)}$ and $G_k \simeq Q_3$. In the other cases the 2-class group of only one unramified quadratic extension is cyclic and the others are of type $(2, 2)$.

3. Another vision of the capitulation types

Let k be a number field having a 2-class group of type $(2, 2)$. Taussky's conditions A and B give a vision of the capitulation types based on the generators of the 2-class group of k. Therefore, in the case where it is impossible for the 2-class group of k to be given in terms of its generators, the results quoted in the above section do not give the exact type of capitulation. Let F_i/k be a Galois extension for $i = 1, 2, 3$. The next results give another method to deal with this problem without needing the generators of the 2-class groups of k.

Keep the notations of the previous section. Assume always that F_i/k is a Galois extension for $i = 1, 2, 3$. If $h_2(F_1) \geqslant 4$, then the situation is schematized in Figure 1.

$$\begin{tikzcd}
 & F_2^{(2)} = F_3^{(2)} = F_1^{(1)} = F_1^{(2)} = k^{(2)} & \\
 & F_2^{(1)} = F_3^{(1)} \arrow[u, dash, "h_2(F_1)/4"'] & \\
 & k^{(1)} \arrow[u, dash] & \\
F_2 \arrow[ur, dash] & F_1 \arrow[u, dash, "2"'] & F_3 \arrow[ul, dash] \\
 & k \arrow[ul, dash] \arrow[u, dash, "2"'] \arrow[ur, dash] &
\end{tikzcd}$$

Figure 1. The Hilbert 2-class field towers for $h_2(F_1) \geqslant 4$.

If $h_2(F_1) = 2$, then G_k is abelian of type $(2, 2)$ and the 2-class group of F_i is cyclic for all $i = 1, 2, 3$. Therefore, the situation is schematized as follows (see Figure 2).

$$\begin{tikzcd}
 & k^{(1)} = F_2^{(1)} = F_3^{(1)} = F_2^{(2)} = F_3^{(2)} = F_1^{(1)} = F_1^{(2)} = k^{(2)} & \\
F_2 \arrow[ur, dash] & F_1 \arrow[u, dash, "2"'] & F_3 \arrow[ul, dash] \\
 & k \arrow[ul, dash] \arrow[u, dash, "2"'] \arrow[ur, dash] &
\end{tikzcd}$$

Figure 2. The Hilbert 2-class field towers for $h_2(F_1) = 2$.

Theorem 3.1. *Keep the above notations.*

(1) *Assume $h_2(F_1) = 4$. If the 2-class group of F_2 or F_3 is cyclic, then the 2-class group of F_i is cyclic for all $i = 1, 2, 3$. Furthermore, G_k is quaternion. Otherwise, G_k is dihedral.*

(2) *Assume now that $h_2(F_1) > 4$. Then*

- ▷ *G_k is a quaternion group if and only if G_{F_2} and G_{F_3} are quaternion groups.*
- ▷ *G_k is a dihedral group if and only if G_{F_2} and G_{F_3} are dihedral groups.*
- ▷ *G_k is a semi-dihedral group if and only if one of the two groups G_{F_2} and G_{F_3} is quaternion and the other is dihedral.*

Proof. (1) If $h_2(F_1) = 4$, then $|G_k| = 8$. Thus Remark 2.2 gives the first item.

(2) Let $i = 2, 3$. Since $F_i^{(2)} = k^{(2)}$, this implies that each G_{F_i} is a subgroup of index 2 in G_k. Thus the group theoretic properties given in Section 2 complete the proof. □

The results of the second item can be summarized in Table 2.

$\lvert\ker(j_1)\rvert$	$\lvert\ker(j_2)\rvert$	G_{F_2}	$\lvert\ker(j_3)\rvert$	G_{F_3}	G_k
4	2	$(2,2)$	2	$(2,2)$	D_3
4	2	D_{m-1}	2	D_{m-1}	D_m, $m > 3$
2	2	Q_{m-1}	2	Q_{m-1}	Q_m, $m > 3$
2	2	D_{m-1}	2	Q_{m-1}	S_m, $m > 3$

Table 2. Capitulation types for the case $h_2(F_1) > 4$.

4. Applications

Let $d = 2q_1q_2$, where $q_1 \equiv q_2 \equiv -1 \pmod 4$ are two distinct positive prime integers such that

$$\left(\frac{2}{q_j}\right) = -\left(\frac{2}{q_k}\right) = \left(\frac{q_j}{q_k}\right) = -\left(\frac{q_k}{q_j}\right) = 1, \quad 1 \leqslant j \neq k \leqslant 2.$$

Let $\Bbbk = \mathbb{Q}(\sqrt{d}, \sqrt{-1})$ be an imaginary bicyclic biquadratic number field, which is called, by Hilbert (see [11]), a special Dirichlet field, and denote by $\Bbbk^{(1)}$ the Hilbert 2-class field of $\Bbbk$ and $\Bbbk^{(2)}$ the Hilbert 2-class field of $\Bbbk^{(1)}$. Put $G_{\Bbbk} = \mathrm{Gal}(\Bbbk^{(2)}/\Bbbk)$. By [1], the 2-class group of $\Bbbk$ is of type $(2, 2)$. In this section we will apply the results of the above sections to determine the structure of $G_{\Bbbk}$.

4.1. Preliminary results. Let us first collect some results that will be useful in what follows. Let k_j, $1 \leqslant j \leqslant 3$, be the three real quadratic subfields of a biquadratic real number field K_0 and $\varepsilon_j > 1$ be the fundamental unit of k_j. Since the square of any unit of K_0 is in the group generated by the ε_j's, $1 \leqslant j \leqslant 3$, then to determine a fundamental system of units of K_0 it suffices to determine which of the units in $B := \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3, \varepsilon_2\varepsilon_3, \varepsilon_1\varepsilon_2\varepsilon_3\}$ are squares in K_0 (see [14]). Hence, by Dirichlet's unit theorem, a fundamental system of units of K_0 consists of three positive units chosen among $B' := B \cup \{\sqrt{\eta}\colon\ \eta \in B \text{ and } \sqrt{\eta} \in K_0\}$. We need the two following lemmas.

Lemma 4.1 ([5])**.** *Let $d \equiv 1 \pmod 4$ be a positive square free integer and $\varepsilon_d = x + y\sqrt{d}$ be the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Assume $N(\varepsilon_d) = 1$, then*
(1) *$x+1$ and $x-1$ are not squares in $\mathbb{N}$, i.e., $2\varepsilon_d$ is not a square in $\mathbb{Q}(\sqrt{d})$.*
(2) *For every prime p dividing d, $p(x+1)$ and $p(x-1)$ are not squares in $\mathbb{N}$.*

In the following lemma, we state a refinement to Lemma 4.1 above.

Lemma 4.2. *Let $d \equiv 1 \pmod 4$ be a positive square free integer and $\varepsilon_d = \frac{1}{2}(x + y\sqrt{d})$ the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Assume $N(\varepsilon_d) = 1$.*
(1) *If $d \equiv 1 \pmod 8$, then both x and y are even.*
(2) *If $d \equiv 5 \pmod 8$, then x and y can be either even or odd. Moreover, if x and y are odd, then $x+2$ and $x-2$ are not squares in $\mathbb{N}$.*

Proof. (1) Assume $d \equiv 1 \pmod 8$. As $N(\varepsilon_d) = 1$, then $x^2 - 4 = y^2 d$, hence $x^2 - 4 \equiv y^2 \pmod 8$. On the other hand, if we suppose that x and y are odd, then $x^2 \equiv y^2 \equiv 1 \pmod 8$, but this implies the contradiction $-3 \equiv 1 \pmod 8$. Thus x and y are even.

(2) Assume $d \equiv 5 \pmod 8$. To prove the first assertion of (2), it suffices to give examples justifying the existence of the two cases. By the PARI/GP system we have:

d	$d \pmod 8$	$N(\varepsilon_d)$	x	y
21	5	1	5	1
69	5	1	25	3
77	5	1	9	1
93	5	1	29	3
133	5	1	173	15
141	5	1	190	16
381	5	1	2030	104
781	5	1	135212398	4838280

For the second assertion, suppose that $x \pm 2 = y_1^2$, $x \mp 2 = dy_2^2$, then

$$\varepsilon_d = \frac{x + y\sqrt{d}}{2} = \frac{1}{4}\big(y_2\sqrt{d} + y_1\big)^2.$$

This in turn implies that $\sqrt{\varepsilon_d} \in \mathbb{Q}(\sqrt{d})$, which is absurd. □

Now we state a lemma which is very useful for getting a FSU of a real biquadratic subfield or imaginary triquadratic subfield of $\mathbb{Q}(\sqrt{2}, \sqrt{q_1}, \sqrt{q_2}, \sqrt{-1})$.

Lemma 4.3. *Let* $q_1 \equiv 7 \pmod 8$ *and* $q_2 \equiv 3 \pmod 8$ *be two primes such that* $(q_2/q_1) = -1$.

(1) *Let* x *and* y *be two integers or semi-integers such that* $\varepsilon_{q_1q_2} = x + y\sqrt{q_1q_2}$, *then*
- (1a) $2q_1(x+1)$ *is a square in* $\mathbb{N}$,
- (1b) $\sqrt{\varepsilon_{q_1q_2}} = y_1\sqrt{q_1} + y_2\sqrt{q_2}$ *and* $1 = q_1y_1^2 - q_2y_2^2$ *for some integers or semi-integers* y_1 *and* y_2 *such that* $y = 2y_1y_2$.

(2) *Let* a *and* b *be two integers such that* $\varepsilon_{2q_1q_2} = a + b\sqrt{2q_1q_2}$. *Then we have*
- (2a) $2q_1(a+1)$ *is a square in* $\mathbb{N}$,
- (2b) $\sqrt{2\varepsilon_{2q_1q_2}} = b_1\sqrt{2q_1} + b_2\sqrt{q_2}$ *and* $2 = 2q_1b_1^2 - q_2b_2^2$ *for some integers* b_1 *and* b_2 *such that* $b = b_1b_2$.

(3) *Let* c *and* d *be two integers such that* $\varepsilon_{2q_1} = c + d\sqrt{2q_1}$ *and let* α *and* β *be two integers such that* $\varepsilon_{q_1} = \alpha + \beta\sqrt{q_1}$. *Then we have*
- (3a) $\sqrt{2\varepsilon_{q_1}} = \beta_1 + \beta_2\sqrt{q_1}$ *and* $2 = \beta_1^2 - q_1\beta_2^2$ *for some integers* β_1 *and* β_2 *such that* $\beta = \beta_1\beta_2$,
- (3b) $\sqrt{2\varepsilon_{2q_1}} = d_1 + d_2\sqrt{2q_1}$ *and* $2 = d_1^2 - 2q_1d_2^2$ *for some integers* d_1 *and* d_2 *such that* $d = d_1d_2$.

(4) *Let* c *and* d *be two integers such that* $\varepsilon_{2q_2} = c + d\sqrt{2q_2}$ *and let* α *and* β *be two integers such that* $\varepsilon_{q_2} = \alpha + \beta\sqrt{q_2}$. *Then we have*
- (4a) $\sqrt{2\varepsilon_{q_2}} = \beta_1 + \beta_2\sqrt{q_2}$ *and* $2 = -\beta_1^2 + q_2\beta_2^2$ *for some integers* β_1 *and* β_2 *such that* $\beta = \beta_1\beta_2$,
- (4b) $\sqrt{2\varepsilon_{2q_2}} = d_1 + d_2\sqrt{2q_2}$ *and* $2 = -d_1^2 + 2q_2d_2^2$ *for some integers* d_1 *and* d_2 *such that* $d = d_1d_2$.

Proof. Using Lemmas 4.1 and 4.2, we get the statements of this lemma, for more details see [6]. □

4.2. Capitulation. Let $q_1 \equiv q_2 \equiv -1 \pmod 4$ be primes. Without loss of generality, we can assume that q_1 and q_2 satisfy the conditions

$$\left(\frac{2}{q_1}\right) = -\left(\frac{2}{q_2}\right) = \left(\frac{q_1}{q_2}\right) = -\left(\frac{q_2}{q_1}\right) = 1.$$

Then, by [1], the 2-class group of $\Bbbk$ is of type $(2,2)$, so denote by $\mathbb{K}_1 = \Bbbk(\sqrt{q_1}) = \mathbb{Q}(\sqrt{q_1}, \sqrt{2q_2}, \mathrm{i})$, $\mathbb{K}_2 = \Bbbk(\sqrt{q_2}) = \mathbb{Q}(\sqrt{q_2}, \sqrt{2q_1}, \mathrm{i})$ and $\mathbb{K}_3 = \Bbbk(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, \sqrt{q_1q_2}, \mathrm{i})$ the three unramified quadratic extensions, within $\Bbbk_1^{(1)}$, of $\Bbbk$.

Now, we correct the error made in the article [4]. The fundamental systems of units given in [4], Proposition 3.1, for $\mathbb{K}_1^+$ and $\mathbb{K}_1$ are not correct. In fact, the error was committed in the FSU of $\mathbb{K}_1^+$, this affected that of $\mathbb{K}_1$, and thus the main theorem.

Proposition 4.4. *Let q_1 and q_2 be two primes defined as above. Then*

(1) *A FSU of $\mathbb{K}_1^+$ is $\{\varepsilon_{q_1}, \sqrt{\varepsilon_{2q_1q_2}}, \sqrt{\varepsilon_{q_1}\varepsilon_{2q_2}}\}$ and that of $\mathbb{K}_1$ is $\{\sqrt{\varepsilon_{2q_1q_2}}, \sqrt{\varepsilon_{q_1}\varepsilon_{2q_2}}, \sqrt{\mathrm{i}\varepsilon_{2q_2}}\}$.*

(2) *A FSU of $\mathbb{K}_2^+$ is $\{\varepsilon_{2q_1q_2}, \sqrt{\varepsilon_{q_2}\varepsilon_{2q_1q_2}}, \sqrt{\varepsilon_{2q_1}\varepsilon_{2q_1q_2}}\}$ and that of $\mathbb{K}_2$ is $\{\sqrt{\varepsilon_{q_2}\varepsilon_{2q_1q_2}}, \sqrt{\varepsilon_{2q_1}\varepsilon_{2q_1q_2}}, \sqrt{\mathrm{i}\varepsilon_{2q_1q_2}}\}$.*

(3) *A FSU of both $\mathbb{K}_3^+$ and $\mathbb{K}_3$ is $\{\varepsilon_2, \varepsilon_{2q_1q_2}, \sqrt{\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}}\}$.*

Proof. Using Lemma 4.3 and the method described in the beginning of this subsection (page 6), we easily deduce the result for $\mathbb{K}_i^+$, $i = 1, 2, 3$ (we proceed as in the proof of [4], Proposition 3.1). Again Lemma 4.3 and [2], Proposition 2, give the result for $\mathbb{K}_i$, $i = 1, 2, 3$. $\square$

Denote by $\kappa_{\mathbb{K}_j}$ the set of classes of $\Bbbk$ capitulating in $\mathbb{K}_j$. Then proceeding as in the proof of [4], Theorem 3.3, we get the following result.

Theorem 4.5. *Let $\mathbb{K}_j$, $1 \leqslant j \leqslant 3$, be the three unramified quadratic extensions of $\Bbbk$ defined above. Then $|\kappa_{\mathbb{K}_1}| = |\kappa_{\mathbb{K}_2}| = |\kappa_{\mathbb{K}_3}| = 2$.*

Lemma 4.6. *Keep the above notations and conditions satisfied by q_1 and q_2. Then, the 2-class group of $\mathbb{K}_2$ is cyclic and those of $\mathbb{K}_1$ and $\mathbb{K}_3$ are of type $(2, 2)$.*

Proof. Let us compute the class number of $\mathbb{K}_2$. For the values of class numbers of quadratic fields, see [8], [12]. Proposition 4.4 implies that $q(\mathbb{K}_2) = 8$, so by the class number formula (cf. [14]) we obtain

$$\begin{aligned} h_2(\mathbb{K}_2) &= \frac{1}{2^5} q(\mathbb{K}_2) h_2(-1) h_2(q_2) h_2(-q_2) h_2(2q_1) h_2(-2q_1) h(2q_1q_2) h_2(-2q_1q_2) \\ &= \frac{1}{2^5} \cdot 8 \cdot h_2(-2q_1) \cdot 2 \cdot 4 \\ &= 2h_2(-2q_1). \end{aligned}$$

Since, by [8], Corollaries (19.6) and (18.4), $h_2(-2q_1)$ is divisible by 4, so $h_2(\mathbb{K}_2)$ is divisible by 8. Therefore, the 2-class group of $\mathbb{K}_2$ cannot be of type $(2, 2)$. It follows that the 2-class group of $\mathbb{K}_2$ is cyclic and those of $\mathbb{K}_1$ and $\mathbb{K}_3$ are of type $(2, 2)$. $\square$

Lemma 4.7 ([6])**.** *Keep the above hypothesis. The Hilbert 2-class field of $\Bbbk$ is $\Bbbk^{(1)} = \mathbb{Q}(\sqrt{2}, \sqrt{q_1}, \sqrt{q_2}, \sqrt{-1})$ and we have*

$$E_{\Bbbk^{(1)}} = \Big\langle \zeta_8, \varepsilon_2, \sqrt{\varepsilon_{2q_2}}, \sqrt{\varepsilon_{q_1q_2}}, \sqrt{\varepsilon_{2q_1q_2}}, \sqrt[4]{\varepsilon_{q_1}\varepsilon_{q_2}\varepsilon_{2q_2}\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}}, \\ \sqrt[4]{\varepsilon_2^2\varepsilon_{2q_1}\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}}, \sqrt[4]{\zeta_8^2\varepsilon_2^2\varepsilon_{q_1}\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}} \Big\rangle.$$

Lemma 4.8. *Keep the above hypothesis. We have:*

$$\begin{aligned}
N_{\Bbbk^{(1)}/\mathbb{K}_1}(\varepsilon_2) &= -1,\\
N_{\Bbbk^{(1)}/\mathbb{K}_1}\big(\sqrt{\varepsilon_{2q_2}}\big) &= -\varepsilon_{2q_2},\\
N_{\Bbbk^{(1)}/\mathbb{K}_1}\big(\sqrt{\varepsilon_{q_1q_2}}\big) &= 1,\\
N_{\Bbbk^{(1)}/\mathbb{K}_1}\big(\sqrt{\varepsilon_{2q_1q_2}}\big) &= \varepsilon_{2q_1q_2},\\
N_{\Bbbk^{(1)}/\mathbb{K}_1}(\zeta_8) &= -\mathrm{i},\\
N_{\Bbbk^{(1)}/\mathbb{K}_1}\big(\sqrt[4]{\varepsilon_{q_1}\varepsilon_{q_2}\varepsilon_{2q_2}\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}}\big) &= \pm\sqrt{\varepsilon_{q_1}\varepsilon_{2q_2}}\sqrt{\varepsilon_{2q_1q_2}},\\
N_{\Bbbk^{(1)}/\mathbb{K}_1}\Big(\sqrt[4]{\varepsilon_2^2\varepsilon_{2q_1}\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}}\Big) &= \pm\sqrt{\varepsilon_{2q_1q_2}},\\
N_{\Bbbk^{(1)}/\mathbb{K}_1}\Big(\sqrt[4]{\zeta_8^2\varepsilon_2^2\varepsilon_{q_1}\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}}\Big) &= \pm\mathrm{i}\sqrt{\mathrm{i}\varepsilon_{q_1}}\sqrt{\varepsilon_{2q_1q_2}}.
\end{aligned}$$

If $q_2=3$, *we have* $N_{\Bbbk^{(1)}/\mathbb{K}_1}(\zeta_6)=1$.

P r o o f. Assume that $q_1\equiv 7 \pmod 8$, $q_2\equiv 3 \pmod 8$ and $(q_2/q_1)=-1$. By the relations given in Lemma 4.3, we have

$$\begin{aligned}
N_{\Bbbk^{(1)}/\mathbb{K}_1}(\varepsilon_2) &= \big(1+\sqrt{2}\big)\big(1-\sqrt{2}\big) = -1,\\
N_{\Bbbk^{(1)}/\mathbb{K}_1}\big(\sqrt{\varepsilon_{2q_2}}\big) &= \frac{1}{\sqrt{2}}\big(d_1+d_2\sqrt{2q_2}\big)\frac{1}{-\sqrt{2}}\big(d_1+d_2\sqrt{2q_2}\big) = -\varepsilon_{2q_2},\\
N_{\Bbbk^{(1)}/\mathbb{K}_1}\big(\sqrt{\varepsilon_{q_1q_2}}\big) &= \big(y_1\sqrt{q_1}+y_2\sqrt{q_2}\big)\big(y_1\sqrt{q_1}-y_2\sqrt{q_2}\big) = y_1^2q_1-y_2^2q_2=1,\\
N_{\Bbbk^{(1)}/\mathbb{K}_1}\big(\sqrt{\varepsilon_{2q_1q_2}}\big) &= \frac{1}{\sqrt{2}}\big(b_1\sqrt{2q_1}+b_2\sqrt{q_2}\big)\frac{1}{-\sqrt{2}}\big(-b_1\sqrt{2q_1}-b_2\sqrt{q_2}\big) = \varepsilon_{2q_1q_2},\\
N_{\Bbbk^{(1)}/\mathbb{K}_1}\big(\sqrt{\varepsilon_{q_1}}\big) &= \frac{1}{\sqrt{2}}\big(\beta_1+\beta_2\sqrt{q_1}\big)\frac{1}{-\sqrt{2}}\big(\beta_1+\beta_2\sqrt{q_1}\big) = -\varepsilon_{q_1},\\
N_{\Bbbk^{(1)}/\mathbb{K}_1}\big(\sqrt{\varepsilon_{2q_1}}\big) &= \frac{1}{\sqrt{2}}\big(d_1+d_2\sqrt{2q_1}\big)\frac{1}{-\sqrt{2}}\big(d_1-d_2\sqrt{2q_2}\big) = -1,\\
N_{\Bbbk^{(1)}/\mathbb{K}_1}\big(\sqrt{\varepsilon_{q_2}}\big) &= \frac{1}{\sqrt{2}}\big(\beta_1+\beta_2\sqrt{q_2}\big)\frac{1}{-\sqrt{2}}\big(\beta_1-\beta_2\sqrt{q_2}\big) = 1,\\
N_{\Bbbk^{(1)}/\mathbb{K}_1}(\zeta_8) &= N_{\Bbbk^{(1)}/\mathbb{K}_1}\Big(\frac{1+\mathrm{i}}{-\sqrt{2}}\Big) = -\zeta_8^2=-\mathrm{i}.
\end{aligned}$$

So we have

$$N_{\Bbbk^{(1)}/\mathbb{K}_1}\big(\sqrt{\varepsilon_{q_1}\varepsilon_{q_2}\varepsilon_{2q_2}\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}}\big) = (-\varepsilon_{q_1})\cdot 1\cdot(-\varepsilon_{2q_2})\cdot 1\cdot \varepsilon_{2q_1q_2} = \varepsilon_{q_1}\varepsilon_{2q_2}\varepsilon_{2q_1q_2}.$$

Then $N_{\Bbbk^{(1)}/\mathbb{K}_1}\big(\sqrt[4]{\varepsilon_{q_1}\varepsilon_{q_2}\varepsilon_{2q_2}\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}}\big) = \pm\sqrt{\varepsilon_{q_1}\varepsilon_{2q_2}}\sqrt{\varepsilon_{2q_1q_2}}$. We similarly get the rest. □

Lemma 4.9. *Keep the above hypothesis. We have:*

$$\begin{aligned}
N_{\Bbbk^{(1)}/\mathbb{K}_3}(\varepsilon_2) &= \varepsilon_2^2,\\
N_{\Bbbk^{(1)}/\mathbb{K}_3}\big(\sqrt{\varepsilon_{2q_2}}\big) &= -1,\\
N_{\Bbbk^{(1)}/\mathbb{K}_3}\big(\sqrt{\varepsilon_{q_1q_2}}\big) &= -\varepsilon_{q_1q_2},\\
N_{\Bbbk^{(1)}/\mathbb{K}_3}\big(\sqrt{\varepsilon_{2q_1q_2}}\big) &= -\varepsilon_{2q_1q_2},\\
N_{\Bbbk^{(1)}/\mathbb{K}_3}(\zeta_8) &= \mathrm{i},\\
N_{\Bbbk^{(1)}/\mathbb{K}_3}\big(\sqrt[4]{\varepsilon_{q_1}\varepsilon_{q_2}\varepsilon_{2q_2}\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}}\big) &= \pm\sqrt{\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}},\\
N_{\Bbbk^{(1)}/\mathbb{K}_3}\Big(\sqrt[4]{\varepsilon_2^2\varepsilon_{2q_1}\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}}\Big) &= \pm\varepsilon_2\sqrt{\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}},\\
N_{\Bbbk^{(1)}/\mathbb{K}_3}\Big(\sqrt[4]{\zeta_8^2\varepsilon_2^2\varepsilon_{q_1}\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}}\Big) &= \pm\zeta_8\varepsilon_2\sqrt{\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}}.
\end{aligned}$$

If $q_2 = 3$, *we have* $N_{\Bbbk^{(1)}/\mathbb{K}_3}(\zeta_6) = 1$.

Proof. The proof is similar to that of Lemma 4.8. □

From the two above lemmas, Proposition 4.4 and [10] we have:

Corollary 4.10. *Keep the above hypothesis. We have:*

(1) *The number of classes of* $\mathbb{K}_1$ *which capitulate in* $\Bbbk^{(1)}$ *is*

$$[\Bbbk^{(1)} : \mathbb{K}_1][E_{\mathbb{K}_1} : N_{\Bbbk^{(1)}/\mathbb{K}_1}(E_{\Bbbk^{(1)}})] = 2\cdot 1 = 2.$$

(2) *The number of classes of* $\mathbb{K}_3$ *which capitulate in* $\Bbbk^{(1)}$ *is*

$$[\Bbbk^{(1)} : \mathbb{K}_3][E_{\mathbb{K}_3} : N_{\Bbbk^{(1)}/\mathbb{K}_3}(E_{\Bbbk^{(1)}})] = 2\cdot 1 = 2.$$

4.3. Main theorem. We can now state the main result of this section.

Theorem 4.11. *Let* $q_1 \equiv q_2 \equiv -1 \pmod 4$ *be two distinct prime integers such that*

$$\Big(\frac{2}{q_j}\Big) = -\Big(\frac{2}{q_k}\Big) = \Big(\frac{q_j}{q_k}\Big) = -\Big(\frac{q_k}{q_j}\Big) = 1,$$

$1 \leqslant j \neq k \leqslant 2$. *Put* $\Bbbk = \mathbb{Q}\big(\sqrt{2q_1q_2}, \mathrm{i}\big)$. *Note that* $\mathbb{K}_j = \Bbbk\big(\sqrt{q_j}\big) = \mathbb{Q}\big(\sqrt{q_j}, \sqrt{2q_k}, \mathrm{i}\big)$, $\mathbb{K}_k = \Bbbk\big(\sqrt{q_k}\big) = \mathbb{Q}\big(\sqrt{q_k}, \sqrt{2q_j}, \mathrm{i}\big)$ *and* $\mathbb{K}_3 = \Bbbk\big(\sqrt{2}\big) = \mathbb{Q}\big(\sqrt{2}, \sqrt{q_1q_2}, \mathrm{i}\big)$ *are three unramified quadratic extensions of* $\Bbbk$. *Let* $m \geqslant 2$ *such that* $2^m = h_2(-2q_j)$. *Then the 2-class field tower of* $\Bbbk$ *stops at* $\Bbbk^{(2)}$ *with* $\Bbbk^{(1)} \neq \Bbbk^{(2)}$ *and*

$$G_{\mathbb{K}_j} \simeq G_{\mathbb{K}_3} \simeq Q_{m+1}, \quad G_{\Bbbk} \simeq Q_{m+2} \quad \text{and} \quad G_{\mathbb{K}_k} \simeq \mathbb{Z}/2^{m+1}\mathbb{Z}.$$

P r o o f. Recall that $G_{\Bbbk} = \mathrm{Gal}(\Bbbk^{(2)}/\Bbbk)$, where $\Bbbk^{(2)}$ is the second Hilbert 2-class field of $\Bbbk$. Without loss of generality, we may suppose that the primes q_1 and q_2 satisfy

$$q_1 \equiv q_2 \equiv -1 \pmod 4 \quad \text{and} \quad \left(\frac{2}{q_1}\right) = -\left(\frac{2}{q_2}\right) = \left(\frac{q_1}{q_2}\right) = -\left(\frac{q_2}{q_1}\right) = 1.$$

As the 2-class group $\mathbb{CL}_2(\Bbbk)$ of $\Bbbk$ is of type $(2,2)$, then $G_{\Bbbk}/G'_{\Bbbk} \simeq (2,2)$. On the other hand, by Theorem 4.5, there are exactly two classes of $\mathbb{CL}_2(\Bbbk)$ which capitulate in each extension $\mathbb{K}_j$, $1 \leqslant j \leqslant 3$, so Theorem 2.1 implies that $G_{\Bbbk}$ is quaternion or semidihedral and the class field tower of $\Bbbk$ stops at $\Bbbk^{(2)}$ with $\Bbbk^{(1)} \neq \Bbbk^{(2)}$; and thus, again by Theorem 2.1, one of the three quadratic extensions of $\Bbbk$ has cyclic 2-class group and the two others have 2-class groups of type $(2,2)$ which is already proved in Lemma 4.6. The 2-class groups of $\mathbb{K}_1$ and $\mathbb{K}_3$ are of type $(2,2)$. They are both sub-extensions of $\Bbbk^{(1)}$ which has a cyclic 2-class group (since $G'_{\Bbbk} \simeq \mathrm{Gal}(\Bbbk^{(2)}/\Bbbk^{(1)}) \simeq \mathbb{CL}_2(\Bbbk^{(1)})$ is a cyclic group), and there are exactly two classes in $\mathbb{K}_1$ and $\mathbb{K}_3$ capitulating in $\Bbbk^{(1)}$ (Corollary 4.10), so neither $G_{\mathbb{K}_1}$ nor $G_{\mathbb{K}_3}$ is dihedral. Hence the result comes by Table 2. □

R e m a r k 4.12. At the first step in the above proof, we showed that $G_{\Bbbk}$ is quaternion or semidihedral. Note that it is impossible to decide whether $G_{\Bbbk}$ is quaternion or semidihedral by using the usual method given by Kisilevsky (by determining whether $\mathbb{K}_i/\Bbbk$ is of type A or B, $i = 1, 3$). In fact, it is hard to determine the generators of the 2-class groups. For this reason the authors of [3] couldn't decide whether G_k is quaternion or semidihedral with $k = \mathbb{Q}(\sqrt{-2}, \sqrt{pq})$ for two primes $p \equiv 5 \pmod 8$ and $q \equiv 7 \pmod 8$ (see [3], Corollary 17). Using the same techniques described in the general context in Section 3, we gave the answer in [7], Remark 5.7.

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