# EXISTENCE OF WEAK SOLUTIONS FOR STEADY FLOWS OF ELECTRORHEOLOGICAL FLUID WITH NAVIER-SLIP TYPE BOUNDARY CONDITIONS

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*Abstract.* We prove the existence of weak solutions for steady flows of electrorheological fluids with homogeneous Navier-slip type boundary conditions provided $p(x) > 2n/(n+2)$. To prove this, we show Poincaré- and Korn-type inequalities, and then construct Lipschitz truncation functions preserving the zero normal component in variable exponent Sobolev spaces.

*Keywords*: existence of weak solutions; electrorheological fluid; Lipschitz truncation; variable exponent

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## 1. Introduction

In this paper, we are concerned with the problem

$$\begin{aligned} -\operatorname{div} S(x,\mathcal{D}u) + (u\cdot\nabla)u + \nabla\pi &= \operatorname{div} F && \text{in } \Omega,\\ \operatorname{div} u &= 0 && \text{in } \Omega,\\ u\cdot\nu = 0, \quad (S(x,\mathcal{D}u)\nu)_\tau + \alpha u_\tau &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $u$ is the velocity, $\pi$ the pressure, $F$ prescribed functions and $\mathcal{D}u$ the symmetric part of $\nabla u$, $u_\tau = u - (u\cdot\nu)\nu$ (where $\nu$ is the unit outer normal on $\partial\Omega$), $\alpha \geqslant 0$ and

$$(S(x,\mathcal{D}u)\nu)_\tau := S(x,\mathcal{D}u)\nu - (S(x,\mathcal{D}u)\nu\cdot\nu)\nu.$$

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We assume that $S(x,\zeta)$ is a Carathéodory function satisfying the following hypotheses:

$$\exists\, c_* > 0,\ S(x,\zeta)\colon\ \zeta \geqslant c_*^{-1}(|\zeta|^{p(x)} - 1), \tag{1.1}$$

$$|S(x,\zeta)| \leqslant c_*(1+|\zeta|)^{p(x)-1}, \tag{1.2}$$

$$(S(x,\zeta) - S(x,\xi))\colon\ (\zeta - \xi) > 0 \quad \forall\, \zeta,\xi \in M_{\mathrm{sym}}\ (\zeta \neq \xi), \tag{1.3}$$

where $p(x) > 1$ is a prescribed function and $M_{\mathrm{sym}}$ the set of all symmetric $n \times n$ matrices.

The system (1.1)–(1.2) models the steady motion of incompressible generalized Newtonian fluids, in particular, that of electrorheological fluids with shear-dependent viscosities, which are viscous fluids characterized by remarkable changes in their viscosity when an electromagnetic field is applied. In recent years the study of electrorheological fluids and of PDEs with nonstandard growth has been a very increasing research field (see [1], [5], [10], [11], [14], [33], [35]–[39]).

The existence of a weak solution to the system (1.1)–(1.2) with a homogeneous Dirichlet boundary condition was first proved for $p = \mathrm{const} > 3n/(n+2)$ by Ladyzhenskaya (see [27]) and Lions (see [29]) by means of the monotone operator theory and the compactness method. In [21], [34] the authors independently proved it for $p = \mathrm{const} > 2n/(n+1)$ by using the so-called $L^\infty$-truncation method. This bound $p > 2n/(n+1)$ was improved to $p > 2n/(n+2)$ in [22] by so-called Lipschitz truncation method. In the case of $p \neq \mathrm{const}$, the existence of weak solutions was shown in [35] for $p(x) > 3n/(n+2)$ and in [15] for $p(x) > 2n/(n+2)$. For further results, we refer to [16], [37].

It is well-known that the Navier-slip type boundary conditions (1.3) are an appropriate model for flow problems with free boundaries, for flows past chemically reacting walls, and for many other important flows in the real world, see [3] and the references therein. The boundary condition (1.3) with $\alpha = 0$ is sometimes called the perfect slip one. It is worth noting that homogeneous Dirichlet and perfect slip boundary conditions are the limit cases of Navier-slip boundary conditions. The Navier-Stokes system under the Navier-slip type boundary conditions was studied by many mathematicians (see [26], [28], [32], [40] and the references therein, etc.). There are some papers about the generalized Newtonian fluid described by the $p$-power law, see [4], [9], [19], [25], [30]. In particular, the authors established in [8] the existence of a weak solution for unsteady flows of the generalized Newtonian fluid provided $p = \mathrm{const} > 2n/(n+2)$, based on the parabolic Lipschitz truncation method. But there seems to be no paper concerning fluids described by the $p(x)$-power law under the Navier-slip type boundary conditions.

We show the existence of weak solutions to the problem (1.1)–(1.3) for the critical restriction $p(x) > 2n/(n+2)$ in any dimension $n \geqslant 2$ and for $\alpha \geqslant 0$.

To prove this, we first prove the Poincaré- and Korn-type inequalities in vector-valued variable exponent Sobolev spaces $W_\nu^{1,p(x)}(\Omega)$ with a vanishing normal component on the boundary, which are new in the generality given.

The Poincaré and Korn inequalities

$$\|u\|_p \leqslant c\,\mathrm{diam}(\Omega)\|\nabla u\|_p \quad \forall\, u \in W_0^{1,p}(\Omega),$$
$$\|\nabla u\|_p \leqslant c\|\mathcal{D}u\|_p \quad \forall\, u \in [W_0^{1,p}(\Omega)]^n$$

are well-known and have been widely used in the mathematical study of PDEs, where $\mathcal{D}u := \frac{1}{2}(\nabla u + (\nabla u)^T)$. Note that the Korn inequality has an important application not only in elasticity and hydrodynamics but also in statistical physics, more precisely, in the study of relaxation to equilibrium of rarefied gases modeled by Boltzmann's equation, see [12]. These inequalities continue to hold for variable exponent Sobolev spaces $W_0^{1,p(x)}(\Omega)$ provided that $p(x)$ is log-Hölder continuous on $\Omega$, see [14], Theorems 8.2.4 and 14.3.21, see also [17]. It is also well-known that these inequalities hold for vector-valued usual Sobolev spaces with a vanishing normal component on the boundary, see [2], [12], [13], [24] and [23], Exercise II 5.6.

To prove the existence of weak solutions to the problem (1.1)–(1.3), we next construct Lipschitz truncations such that they can preserve the zero normal component by using the reflection method. This appears to be a new even for a constant $p$. This leads us to the proof of the existence of weak solutions to the problem (1.1)–(1.3) under the critical restriction $p(x) > 2n/(n+2)$. It seems to be still possible to apply the argument from [8] using the parabolic Lipschitz truncation method for the proof of the existence of weak solutions to our problem. But in our viewpoint, our method enables us to give a shorter proof than that from [8] since we do not need the introduction of a cut-off function. As pointed out in [15], Lipschitz truncations of Sobolev functions are used in the existence and regularity theories of PDEs and Calculus of Variations. Note that the Lipschitz truncation method in $W_0^{1,p}(\Omega)$ or $W_0^{1,p(x)}(\Omega)$ is well-known, see [15], [22]. For more detail, we refer to [6], [7], [18].

The paper is organized as follows. In Section 2, we give preliminaries and state the main result. In Section 3, we prove the Poincaré- and Korn-type inequalities and show the Lipschitz truncation method in $W_\nu^{1,p(x)}(\Omega)$. Section 4 is devoted to the proof of the existence of weak solution to the problem (1.1)–(1.3).

## 2. Preliminaries and main result

Let $\Omega \subset \mathbb{R}^n$ be a domain and $p \in L^\infty(\Omega)$, $p \geqslant 1$. As in [14] we introduce the following variable exponent Lebesgue space, equipped with the corresponding Luxemburg norms

$$L^{p(x)}(\Omega) := \left\{ u\colon\ \Omega \to \mathbb{R}\colon\ u \text{ is measurable and } \varrho_{p(x)}(u) := \int_\Omega |u|^{p(x)}\,\mathrm{d}x < \infty \right\},$$
$$\|u\|_{p(x),\Omega} = \|u\|_{L^{p(x)}(\Omega)} := \inf\left\{ \lambda > 0\colon\ \varrho_{p(x)}\left(\frac{u}{\lambda}\right) \leqslant 1 \right\}$$

and the variable exponent Sobolev space

$$W^{k,p(x)}(\Omega) := \{u\colon\ \nabla^\alpha u \in L^{p(x)}(\Omega)\ \forall\, |\alpha| \leqslant k\},$$
$$\|u\|_{k,p(x),\Omega} = \|u\|_{W^{k,p(x)}(\Omega)} := \sum_{|\alpha|\leqslant k} \|\nabla^\alpha u\|_{p(x)}.$$

We denote $\|u\|_{p(x),\Omega}$, $\|u\|_{k,p(x),\Omega}$ simply by $\|u\|_{p(x)}$, $\|u\|_{k,p(x)}$, respectively, whenever it is clear from the context. The properties of $L^{p(x)}$ and $W^{k,p(x)}$ can be found in the book [14]. Let us define $W_0^{k,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. We use the same notation for functional spaces and norms for both scalar and vector fields. For our purpose, we put

$$W_\nu^{1,p(x)}(\Omega) := \{u \in W^{1,p(x)}(\Omega)\colon\ u\cdot\nu|_{\partial\Omega} = 0\},$$
$$C_{\nu,\sigma}^\infty(\Omega) := \{u \in C^\infty(\Omega)\colon\ \operatorname{div} u = 0,\ u\cdot\nu|_{\partial\Omega} = 0\},$$
$$\mathcal{V}_\nu^{p(x)}(\Omega) := \{u \in W^{1,p(x)}(\Omega)\colon\ \operatorname{div} u = 0,\ u\cdot\nu|_{\partial\Omega} = 0\}.$$

By $p'(x)$ we denote the conjugate function of $p(x)$. As usual, we denote the Sobolev conjugate exponent by

$$p^*(x) := \begin{cases} \dfrac{np(x)}{n-p(x)}, & p(x) < n, \\ \infty, & p(x) \geqslant n. \end{cases}$$

Let us put $p_- := \operatorname{ess\,inf} p(x)$ and $p_+ := \operatorname{ess\,sup} p(x)$.

We say that a function $p\colon\ \overline{\Omega} \to \mathbb{R}$ is globally log-Hölder continuous on $\overline{\Omega}$ if there exists a constant $C_{\log} > 0$ such that

$$|p(x) - p(y)| \leqslant \frac{C_{\log}}{\ln(\mathrm{e} + |x-y|^{-1})} \quad \forall\, x, y \in \overline{\Omega},$$

and write $p \in \mathcal{P}^{\log}(\overline{\Omega})$.

For $(n \times n)$-matrices $F$ and $H$, let us put $F : H = \sum_{i,j=1}^{n} F_{ij} H_{ij}$, $|F| \equiv (F : F)^{1/2}$, and for vectors $a$ and $b$, $a \otimes b = (a_i b_j)_{n \times n}$. The constants can change even in a single string of estimates. The dependence of a constant on certain parameters is expressed, for example, by the notation $c = c(n, p)$.

**Definition 2.1.** Assume that $p_- \geqslant 2n/(n+2)$, $F \in L^{p'(x)}(\Omega)$. We say that a function $u$ is a weak solution to the problem (1.1)–(1.3) if $u \in \mathcal{V}_\nu^{p(x)}$ and it satisfies

$$\int_\Omega S(x, \mathcal{D}u) : \mathcal{D}\phi \,\mathrm{d}x + \int_\Omega (u \cdot \nabla) u \cdot \phi \,\mathrm{d}x = -\int_\Omega F : \nabla\phi \,\mathrm{d}x - \int_{\partial\Omega} \alpha u \cdot \phi \,\mathrm{d}\sigma \quad \forall\, \phi \in C^\infty_{\nu,\sigma}(\Omega).$$

The main result follows.

**Theorem 2.1.** *Let $p \in \mathcal{P}^{\log}(\overline{\Omega})$ and $\Omega$ be a bounded domain with $C^{1,1}$-boundary. Assume that $F \in L^{p'(x)}(\Omega)$, $p(x) > 2n/(n+2)$, $n \geqslant 2$, and the extra stress $S(x, \mathcal{D}u)$ satisfies* (1.4)–(1.6) *and, in addition, $\Omega$ is a non-axisymmetric if $\alpha = 0$.*

*Then there exists at least one weak solution $u \in \mathcal{V}_\nu^{p(x)}(\Omega)$ to the problem* (1.1)–(1.3) *which satisfies*

$$\|u\|_{1,p(x)} \leqslant K, \tag{2.1}$$

*where the constant $K$ depends only on $n$, $\Omega$, $p_-, p_+$, $C_{\log}$ and $\|F\|_{L^{p'(x)}(\Omega)}$.*

## 3. Auxiliary results

### 3.1. Poincaré-type inequality.

**Lemma 3.1.** *The set $W_\nu^{1,p(x)}(\Omega)$ is a closed subspace of $W^{1,p(x)}(\Omega)$.*

Proof. It is obvious that $W_\nu^{1,p(x)}(\Omega)$ is a linear subspace of $W^{1,p(x)}(\Omega)$. To complete the proof, we must prove closeness. We assume that a sequence $\{u^k\} \subset W_\nu^{1,p(x)}(\Omega)$ converges to $u$ in $W^{1,p(x)}(\Omega)$. Since $W^{1,p(x)}(\Omega) \hookrightarrow W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$ we have

$$\int_{\partial\Omega} |u \cdot \nu| \,\mathrm{d}\sigma = \int_{\partial\Omega} |(u^k - u) \cdot \nu| \,\mathrm{d}\sigma \leqslant \int_{\partial\Omega} |u^k - u| \,\mathrm{d}\sigma \leqslant c\|u^k - u\|_{1,p(x),\Omega} \to 0$$

as $k \to \infty$. Thus $\int_{\partial\Omega} |u \cdot \nu| \,\mathrm{d}\sigma = 0$ and, in turn, $u \cdot \nu|_{\partial\Omega} = 0$. □

**Theorem 3.1.** *Let $\Omega$ be a $C^{0,1}$ bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, and $p(x) \in \mathcal{P}^{\log}(\Omega)$, $1 \leqslant p(x) < \infty$. Then there exists a positive constant $c$, depending only on $\Omega$, $p_-$, $p_+$ and $C_{\log}$, such that for all $u \in W_\nu^{1,p(x)}(\Omega)$ there holds*

$$\|u\|_{p(x)} \leqslant c\|\nabla u\|_{p(x)}. \tag{3.1}$$

P r o o f. The inequality (3.1) is equivalent to the following one:

$$\|u\|_{1,p(x)} \leqslant c\|\nabla u\|_{p(x)}. \tag{3.2}$$

For contradiction, we assume that the inequality (3.2) does not hold. Hence for any $k \in \mathbb{N}$ there exists a function $u^k \in W_\nu^{1,p(x)}(\Omega)$ satisfying

$$\|u^k\|_{1,p(x)} > k\|\nabla u^k\|_{p(x)}.$$

Setting $v^k := u^k/\|u^k\|_{1,p(x)}$, we have $\|v^k\|_{1,p(x)} = 1$ and

$$1 > k\|\nabla v^k\|_{p(x)}.$$

Hence $\|\nabla v^k\|_{p(x)} \to 0$ as $k \to \infty$. On the other hand, by $\|v^k\|_{1,p(x)} = 1$ and Lemma 3.1, there exists a function $v \in W_\nu^{1,p(x)}(\Omega)$ such that $v^k \rightharpoonup v$ in $W_\nu^{1,p(x)}(\Omega)$. By [14], Theorem 8.2.4, we have

$$\|v^k - (v^k)_\Omega\|_{p(x)} \leqslant c\,\mathrm{diam}(\Omega)\|\nabla v^k\|_{p(x)}$$

and furthermore

$$\|v^k\|_{p(x)} + \|\nabla v^k\|_{p(x)} \leqslant c\|\nabla v^k\|_{p(x)} + c\|(v^k)_\Omega\|_{p(x)}.$$

Since $v^k \to v$ in $L^{p(x)}(\Omega)$ by compact embedding, it follows that $\{v^k\}$ is a Cauchy sequence in $L^{p(x)}(\Omega)$ and so is $\{(v^k)_\Omega\}$ in $\mathbb{R}$. These together with the previous inequality yield that $\{v^k\}$ is a Cauchy sequence in $W_\nu^{1,p(x)}(\Omega)$. By uniqueness of limit, we have $v^k \to v$ in $W_\nu^{1,p(x)}(\Omega)$ and thus $\nabla v \equiv 0$. Consequently we have $v = \mathrm{const} \neq 0$ since $\|v\|_{p(x)} = 1$.

On the other hand, by Poincaré's inequality in $W_\nu^{1,p}(\Omega)$ with a constant $p$ (for example [23], Exercise II 5.6) we obtain

$$\|v\|_1 \leqslant c\|\nabla v\|_1$$

since $v \in W_\nu^{1,p(x)}(\Omega) \subset W_\nu^{1,1}(\Omega)$. This implies that $\|v\|_1 = 0$, that is $v \equiv 0$ since $\nabla v \equiv 0$. This contradicts to $v = \mathrm{const} \neq 0$. $\square$

R e m a r k 3.1. It is not clear whether the constant $c$ in Theorem 3.1 is directly proportional to $\mathrm{diam}(\Omega)$ because its proof is based on the contradiction argument. If we use the identity as in [23], Exercise II 5.6,

$$\sum_{i,j=1}^{n} (\partial_i(u_i x_j u_j |u|^{p(x)-2}) - (\partial_i u_i) x_j u_j |u|^{p(x)-2} - |u|^{p(x)} - u_i x_j \partial_i(u_j |u|^{p(x)-2})) = 0$$

and follow the same argument as in the proof of [20], Theorem 2.1, then it can be possible to get a constant $c_p$ proportional to $\mathrm{diam}(\Omega)$. But in the case we need at least the Lipschitz continuity of $p(x)$.

### 3.2. Korn-type inequality.

**Theorem 3.2.** *Let $\Omega$ be a $C^{0,1}$ bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, and $p(x) \in \mathcal{P}^{\log}(\Omega)$, $p(x) > 1$. Then there exists a positive constant $c$, depending only on $\Omega$, $p_-$, $p_+$ and $C_{\log}$, such that for all $u \in W^{1,p(x)}(\Omega)$ with the trace $u \in L^r(\partial\Omega)$ for $r \in (1,\infty)$ the inequality*

$$\|u\|_{1,p(x)} \leqslant c(\|\mathcal{D}u\|_{p(x)} + \|u\|_{L^r(\partial\Omega)}) \tag{3.3}$$

*holds.*

Proof. We follow the argument from [9]. At first, we define the space

$$E(\Omega) := \{u \in L^{p(x)}(\Omega)\colon\ \mathcal{D}u \in L^{p(x)}(\Omega)\}$$

endowed with the norm $\|u\|_{E(\Omega)} = \|u\|_{p(x)} + \|\mathcal{D}u\|_{p(x)}$. Then $E(\Omega)$ is a Banach space.

Now we prove that $W^{1,p(x)}(\Omega)$ coincides with $E(\Omega)$. It is obvious that $W^{1,p(x)}(\Omega) \subset E(\Omega)$. Let us prove that $E(\Omega) \subset W^{1,p(x)}(\Omega)$. For $u \in E(\Omega)$, it is well-known that for all $i, j, l = 1, \dots, n$,

$$\frac{\partial^2 u_i}{\partial x_j \partial x_l} = \frac{\partial}{\partial x_j}(\mathcal{D}_{il}u) + \frac{\partial}{\partial x_l}(\mathcal{D}_{ij}u) - \frac{\partial}{\partial x_i}(\mathcal{D}_{jl}u) \tag{3.4}$$

in the sense of distributions, see [35], Appendix, (1.7). Since $\mathcal{D}u \in L^{p(x)}(\Omega)$, it follows from (3.4) that $\partial^2 u_i/(\partial x_j \partial x_l) \in W^{-1,p(x)}(\Omega)$. On the other hand, it follows from $u \in L^{p(x)}(\Omega)$ that $\partial u_i/\partial x_j \in W^{-1,p(x)}(\Omega)$. So by the negative norm theorem (for example [14], Theorem 14.3.18) we have $\nabla u \in L^{p(x)}(\Omega)$, thus $u \in W^{1,p(x)}(\Omega)$. Therefore $E(\Omega) \subset W^{1,p(x)}(\Omega)$ and finally, the space $W^{1,p(x)}(\Omega)$ coincides with $E(\Omega)$ and there holds

$$\|u\|_{1,p(x)} \leqslant c(p,\Omega)(\|u\|_{p(x)} + \|\mathcal{D}u\|_{p(x)}). \tag{3.5}$$

So in order to prove the inequality (3.3), it suffices to show that

$$\|u\|_{p(x)} \leqslant c(p,\Omega)(\|u\|_{L^r(\partial\Omega)} + \|\mathcal{D}u\|_{p(x)}).$$

For contradiction we take a sequence $\{u^k\}_{k=1}^{\infty}$ such that $\|u^k\|_{p(x)} = 1$ and

$$\|u^k\|_{L^r(\partial\Omega)} + \|\mathcal{D}u^k\|_{p(x)} < 1/k.$$

It is clear that

$$\|u^k\|_{L^r(\partial\Omega)} \to 0, \quad \|\mathcal{D}u^k\|_{p(x)} \to 0. \tag{3.6}$$

From (3.5) it follows that $\|u^k\|_{1,p(x)} \leqslant C' < \infty$. This implies the existence of $u \in W^{1,p(x)}(\Omega)$ and a subsequence, which is denoted again by $u^k$, such that

$$u^k \rightharpoonup u \quad \text{in } W^{1,p(x)}(\Omega).$$

By compact embedding we have

$$u^k \to u \quad \text{in } L^{p(x)}(\Omega)$$

and conclude that $\|u\|_{p(x)} = 1$.

On the other hand, we have $u \in W_0^{1,p(x)}(\Omega)$ and $\mathcal{D}u \equiv 0$ by (3.6). Thus from Korn's inequality in $W_0^{1,p(x)}(\Omega)$ (see [14], Theorem 14.3.21) it follows that $u \equiv 0$, which is a contradiction. $\square$

Remark 3.2. In order that $u \in W^{1,p(x)}(\Omega)$ has a trace $u \in L^r(\partial\Omega)$, it is needed the restriction $p_- > nr/(n+r-1)$ by the standard trace embedding.

If $\alpha = 0$ in (1.3), then in Definition 2.1, the term including integration on $\partial\Omega$ vanishes. So we want to show a Korn-type inequality not including any trace norm.

**Theorem 3.3.** *Let $\Omega$ be a $C^{0,1}$ bounded, non-axisymmetric domain of $\mathbb{R}^n$, $n \geqslant 2$, and $p(x) \in \mathcal{P}^{\log}(\Omega)$, $p(x) > 1$. Then there exists a positive constant $c$, depending only on $\Omega$, $p_-$, $p_+$ and $C_{\log}$, such that for all $u \in W_\nu^{1,p(x)}(\Omega)$ there holds*

$$\|u\|_{1,p(x)} \leqslant c\|\mathcal{D}u\|_{p(x)}. \tag{3.7}$$

Proof. For contradiction, we assume that the inequality (3.7) does not hold. Hence for any $k \in \mathbb{N}$ there exists a function $u^k \in W_\nu^{1,p(x)}(\Omega)$ satisfying

$$\|u^k\|_{1,p(x)} > k\|\mathcal{D}u^k\|_{p(x)}.$$

Setting $v^k := u^k/\|u^k\|_{1,p(x)}$, we have $\|v^k\|_{1,p(x)} = 1$ and

$$1 > k\|\mathcal{D}v^k\|_{p(x)}.$$

Hence $\|\mathcal{D}v^k\|_{p(x)} \to 0$ as $k \to \infty$. On the other hand, by $\|v^k\|_{1,p(x)} = 1$ and Lemma 3.1, there exists a function $v \in W_\nu^{1,p(x)}(\Omega)$ such that $v^k \rightharpoonup v$ in $W_\nu^{1,p(x)}(\Omega)$ and $\|\mathcal{D}v\|_{p(x)} = 0$. Thus $v = a + b \wedge x$ and $v \cdot \nu = 0$ on $\partial\Omega$. Then since $\Omega$ is axisymmetric if and only if there exists a nontrivial rigid motion $w$ which is tangent to $\partial\Omega$ ([12], Lemma 5), this implies that $\Omega$ is axisymmetric, which is a contradiction. $\square$

**3.3. Lipschitz truncation method.** In this subsection we show that a weakly convergent sequence of Sobolev functions can be approximated by a sequence of Lipschitz functions such that certain additional convergence properties hold and, in addition, it preserves the zero normal component. To begin with we recall the well-known Lipschitz extension theorem, see [15], [31].

**Proposition 3.1.** *Let $w\colon E \to \mathbb{R}^m$, defined on a nonempty set $E \subset \mathbb{R}^d$, be such that for certain $\lambda > 0$ and $\theta > 0$ and for all $x, y \in E$*

$$|w(x) - w(y)| \leqslant \lambda |x - y|, \quad |w| \leqslant \theta. \tag{3.8}$$

*Then there exists an extension $w_{\theta,\lambda}\colon \mathbb{R}^d \to \mathbb{R}^m$ satisfying (3.8) for all $x, y \in \mathbb{R}^d$, and $w_{\theta,\lambda} = w$ on $E$.*

Recall that the Hardy-Littlewood maximal function is defined as

$$Mf(x) := \sup_{r>0} |B_r(x)|^{-1} \int_{B_r(x)} |f(y)| \,\mathrm{d}y.$$

The following lemma is a keystone in proving Theorem 3.4 below.

**Lemma 3.2.** *Let $\Omega$ be a bounded domain with $C^{1,1}$-boundary and $w \in W^{1,1}_\nu(\Omega)$. Then for every $\theta, \lambda > 0$, there exist truncations $w_{\theta,\lambda} \in W^{1,\infty}_\nu(\Omega)$ such that*

$$\|w_{\theta,\lambda}\|_\infty \leqslant \theta, \tag{3.9}$$

$$\|\nabla w_{\theta,\lambda}\|_\infty \leqslant c_* \lambda, \tag{3.10}$$

*where $c_* > 0$ depends only on the dimension $n$ and the smoothness of $\partial\Omega$. Moreover, up to a null-set (i.e., a set of the Lebesgue measure zero)*

$$\{w_{\theta,\lambda} \neq w\} \subset \Omega \cap (\{Mw > \theta\} \cup \{M(\nabla w) > \lambda\}). \tag{3.11}$$

P r o o f. We follow the same argument as in the proof of [15], Theorem 2.3, with the important difference that it needs to construct Lipschitz truncations, normal components of which vanish on the boundary. By localization method we can assume that $\Omega = \mathbb{R}^n_+$. Then it is clear that $w_n = w \cdot \nu = 0$ on $\partial\mathbb{R}^n_+$. At first we extend $w$ from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ as

$$\widetilde{w}_i(x', x_n) := \begin{cases} w_i(x', x_n) & \text{if } x_n > 0, \\ w_i(x', -x_n) & \text{if } x_n < 0, \end{cases} \tag{3.12}$$

$$\widetilde{w}_n(x', x_n) := \begin{cases} w_n(x', x_n) & \text{if } x_n > 0, \\ -w_n(x', -x_n) & \text{if } x_n < 0, \end{cases}$$

where $i = 1, \dots, n-1$ and $x' = (x_1, \dots, x_{n-1})$. It is easy to see that a function $\widetilde{w} \in W^{1,1}(\mathbb{R}^n)$ and $\widetilde{w}_n|_{\partial\mathbb{R}^n_+} = 0$. Let $\mathcal{L}(\widetilde{w})$ be the set of its Lebesgue points. Then it is well-known that

$$|\mathbb{R}^n \setminus \mathcal{L}(\widetilde{w})| = 0$$

and there holds

$$|\widetilde{w}(x) - (\widetilde{w})_{B_r(x_0)}| \leqslant crM(\nabla\widetilde{w})(x) \tag{3.13}$$

for all balls $B_r(x_0) \subset \mathbb{R}^n$ and for all $x \in \mathcal{L}(\widetilde{w}) \cap B_r(x_0)$, see [15] or [31]. Then for any $x, y \in \mathcal{L}(\widetilde{w})$ we take $x = x_0$, $r = 2|y - x|$ in (3.13) and obtain

$$|\widetilde{w}(x) - \widetilde{w}(y)| \leqslant c|x - y|(M(\nabla\widetilde{w})(x) + M(\nabla\widetilde{w})(y)). \tag{3.14}$$

For $\lambda, \theta > 0$ we define

$$H_{\theta,\lambda} := \mathcal{L}(\widetilde{w}) \cap \{M\widetilde{w} \leqslant \theta\} \cap \{M(\nabla\widetilde{w}) \leqslant \lambda\}.$$

It follows from (3.14) that for all $x, y \in H_{\theta,\lambda}$

$$|\widetilde{w}(x) - \widetilde{w}(y)| \leqslant c\lambda|x - y|, \quad |\widetilde{w}(x)| \leqslant \theta. \tag{3.15}$$

Now we must construct a Lipschitz truncation such that its normal component vanishes on $\partial\mathbb{R}^n_+$. For $\widetilde{w}_i$ $(i = 1, \dots, n-1)$ assertions of the lemma follow from Proposition 3.1 applied to $E = H_{\theta,\lambda}$. Let $x \in H_{\theta,\lambda} \cap \Omega$ and $r := 2\,\mathrm{dist}(x, \Omega^C)$. Recalling the extension (3.12) and using Poincaré's inequality, we obtain that for $x \in H_{\theta,\lambda} \cap \Omega$

$$\begin{aligned}
|(\widetilde{w}_n)_{B_r(x)}| &\leqslant \frac{1}{|B_r|}\int_{B_r(x)\cap\Omega} |w_n(y)|\,\mathrm{d}y + \frac{1}{|B_r|}\int_{B_r(x)\cap\Omega^C} |\widetilde{w}_n(y)|\,\mathrm{d}y \\
&= \frac{1}{|B_r|}\int_{B_r(x)\cap\Omega} |w_n(y)|\,\mathrm{d}y + \frac{1}{|B_r|}\int_{(B_r(x)\cap\Omega^C)^{\mathrm{ref}}} |w_n(y)|\,\mathrm{d}y \\
&\leqslant \frac{cr}{|B_r|}\int_{B_r(x)\cap\Omega} |\nabla w_n(y)|dy + \frac{cr}{|B_r|}\int_{(B_r(x)\cap\Omega^C)^{\mathrm{ref}}} |\nabla w_n(y)|\,\mathrm{d}y \\
&\leqslant \frac{cr}{|B_r|}\int_{B_r(x)} |\nabla\widetilde{w}_n(y)|\,\mathrm{d}y \leqslant crM(\nabla\widetilde{w}_n)(x) \leqslant cr\lambda,
\end{aligned}$$

where $(B_r(x)\cap\Omega^C)^{\mathrm{ref}}$ is a reflection of $B_r(x)\cap\Omega^C$ with respect to the boundary $\partial\mathbb{R}^n_+$. This together with (3.13) implies that

$$|\widetilde{w}_n(x)| \leqslant |(\widetilde{w}_n)_{B_r(x)}| + crM(\nabla\widetilde{w}_n)(x) \leqslant cr\lambda.$$

Noting that $\widetilde{w}_n(y) = 0$ for all $y \in \partial\mathbb{R}^n_+$ we obtain that for $x \in H_{\theta,\lambda} \cap \Omega$ and $y \in \partial\mathbb{R}^n_+$

$$|\widetilde{w}_n(x) - \widetilde{w}_n(y)| \leqslant |\widetilde{w}_n(x)| \leqslant cr\lambda \leqslant c\lambda|x-y|,$$

since $r = 2\operatorname{dist}(x, \Omega^C) \leqslant c|x-y|$. It is obvious that

$$|\widetilde{w}_n(x) - \widetilde{w}_n(y)| = 0 \leqslant c\lambda|x-y|$$

for all $x, y \in \partial\mathbb{R}^n_+$. The two previous inequalities together with (3.15) yield that for all $x, y \in H_{\theta,\lambda} \cup \partial\mathbb{R}^n_+$

$$|\widetilde{w}_n(x) - \widetilde{w}_n(y)| \leqslant c\lambda|x-y|,$$

which, in other words, shows that $\widetilde{w}_n$ is Lipschitz continuous on $G_{\theta,\lambda} := H_{\theta,\lambda} \cup \partial\mathbb{R}^n_+$. Since $M\widetilde{w}_n \leqslant \theta$ on $H_{\theta,\lambda}$ and $\widetilde{w}_n = 0$ on $\partial\mathbb{R}^n_+$, we have $|\widetilde{w}_n| \leqslant \theta$ on $G_{\theta,\lambda}$. Applying Proposition 3.1 to $E = G_{\theta,\lambda}$ we can see that there exists an extension $(w_{\theta,\lambda})_n$ of $w_n$ with $(w_{\theta,\lambda})_n = w_n$ on $G_{\theta,\lambda}$ and satisfying (3.9) and (3.10). In particular, $(w_{\theta,\lambda})_n = 0$ on $\partial\mathbb{R}^n_+$ since it is contained in $G_{\theta,\lambda}$ and thus $w_{\theta,\lambda} \in W^{1,\infty}_\nu(\Omega)$. Using $w_{\theta,\lambda} = w$ on $G_{\theta,\lambda}$ and $|\mathcal{L}(w)^C| = 0$, and recalling that

$$G^C_{\theta,\lambda} = (\mathbb{R}^n_+ \cup \mathbb{R}^n_-) \cap \mathcal{L}(\widetilde{w})^C \cup \{M\widetilde{w} > \theta\} \cup \{M(\nabla\widetilde{w}) > \lambda\}$$

we have (3.11). □

The following theorem is the main result in this subsection.

**Theorem 3.4.** *Let $\Omega$ be a bounded domain with $C^{1,1}$-boundary and $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p_- \leqslant p_+ < \infty$. Let $w^k \in W^{1,p(x)}_\nu(\Omega)$ be such that $w^k \rightharpoonup 0$ in $W^{1,p(x)}_\nu(\Omega)$. Set*

$$K = \sup_k \|w^k\|_{1,p(x)} < \infty, \quad \tau_k = \|w^k\|_{p(x)} \to 0.$$

*Then there exists a null-sequence $\{\varepsilon^j\}$ and for every $j, k \in \mathbb{N}$ there exist a function $w^{k,j} \in W^{1,\infty}_\nu(\Omega)$ and a number $\lambda_{k,j} \in [2^{2^j}, 2^{2^{j+1}}]$ such that*

$$\lim_{k\to\infty}\Big(\sup_{j\in\mathbb{N}} \|w^{k,j}\|_{L^\infty(\Omega)}\Big) = 0, \quad \|\nabla w^{k,j}\|_{L^\infty(\Omega)} \leqslant cK\lambda_{k,j} \leqslant cK2^{2^{j+1}},$$

$$\limsup_{k\to\infty} \|\lambda_{k,j}\chi_{\{w^k \neq w^{k,j}\}}\|_{p(x)} \leqslant \varepsilon_j, \quad \limsup_{k\to\infty} \|\nabla w^{k,j}\chi_{\{w^k \neq w^{k,j}\}}\|_{p(x)} \leqslant \varepsilon_j,$$

*where the constant $c$ depends on $n$, $p_-$, $p_+$ and $C_{\log}$.*

*Moreover, for a fixed $j \in \mathbb{N}$, $\nabla w^{k,j} \rightharpoonup 0$ in $L^s(\Omega)$, $s < \infty$, and $\nabla w^{k,j} \overset{*}{\rightharpoonup} 0$ in $L^\infty(\Omega)$ as $k \to \infty$.*

Proof. By the same argument as in the proofs of [14], Theorem 9.5.2 and Corollary 9.5.4, we can prove the claims and so omit the details. □

## 4. Proof of the main result

In this section we prove the existence of weak solution of the problem (1.1)–(1.3), that is, Theorem 2.1.

Remark 4.1. The condition that $\Omega$ is a non-axisymmetric for $\alpha = 0$ in Theorem 2.1 is needed only due to the application of Korn's inequality (3.7).

We consider two cases: $2n/(n+2) < p_- \leqslant 3n/(n+2)$, and $p_- > 3n/(n+2)$.

*Case 1:* $p_- > 3n/(n+2)$. To begin with we define corresponding Galerkin approximation. Let $\{\phi^j\}$ be a Schauder basis of $\mathcal{V}_\nu^{p(x)}$. Let $\mathbf{X}^m$ be the span of $\{\phi^1, \phi^2, \dots, \phi^m\}$. Let us define the Galerkin approximation $u^m$ by $u^m := \sum\limits_{j=1}^{m} a^j \phi^j$ and an operator $A\colon \mathbb{R}^m \to \mathbb{R}^m$; $\mathbf{a} = \{a^1, \dots, a^m\} \to \mathbf{b} = \{b^1, \dots, b^m\}$, where $b^j$ is given by

$$
\begin{aligned}
b^j := \int_\Omega S(x, \mathcal{D}u^m) : \mathcal{D}\phi^j \,\mathrm{d}x + \int_\Omega (u^m \cdot \nabla) u^m \cdot \phi^j \,\mathrm{d}x \\
+ \int_\Omega F : \nabla \phi^j \,\mathrm{d}x + \int_{\partial\Omega} \alpha u^m \cdot \phi^j \,\mathrm{d}\sigma.
\end{aligned}
\tag{4.1}
$$

Note that $\|a\| := \|\mathcal{D}u^m\|_{p(x)}$ for $\alpha = 0$ or $\|a\| := \|\mathcal{D}u^m\|_{p(x)} + \|u^m\|_{L^2(\partial\Omega)}$ for $0 < \alpha < \infty$ is a norm in $\mathbb{R}^m$. Indeed, by Korn's and Poincaré's inequalities

$$
\|a\| = 0 \Rightarrow \left\| \mathcal{D}\Big( \sum_{j=1}^{m} a^j \phi^j \Big) \right\|_{p(x)} + \left\| \sum_{j=1}^{m} a^j \phi^j \right\|_{L^2(\partial\Omega)} = 0 \Rightarrow \sum_{j=1}^{m} a^j \phi^j = 0,
$$

which in turn, together with the fact that $\{\phi^j\}$ is linearly independent implies that

$$
\|a\| = 0 \Rightarrow a^j = 0 \quad \forall\, j,\ 1 \leqslant j \leqslant m.
$$

It is easy to verify the remaining axioms.

It is obvious that $A$ is continuous. Multiplying (4.1) by $a^j$ and summing up over $j$, we obtain that

$$
\begin{aligned}
A\mathbf{a} \cdot \mathbf{a} &= \int_\Omega S(x, \mathcal{D}u^m) : \mathcal{D}u^m \,\mathrm{d}x + \int_\Omega F : \nabla u^m \,\mathrm{d}x + \alpha \int_{\partial\Omega} |u^m|^2 \,\mathrm{d}\sigma \\
&= I_1 + I_2 + I_3,
\end{aligned}
\tag{4.2}
$$

where we use that

$$
\int_\Omega (u^m \cdot \nabla) u^m \cdot u^m \,\mathrm{d}x = 0.
$$

It follows from (1.4) that

$$
I_1 = \int_\Omega S(x, \mathcal{D}u^m) : \mathcal{D}u^m \,\mathrm{d}x \geqslant c_1 \int_\Omega |\mathcal{D}u^m|^{p(x)} \,\mathrm{d}x - c.
\tag{4.3}
$$

We use Korn's ((3.3) for $\alpha > 0$, (3.7) for $\alpha = 0$) and Young's inequalities to get

$$
\begin{aligned}
(4.4) \qquad I_2 \leqslant \|F\|_{p'(x)}\|\nabla u^m\|_{p(x)} &\leqslant \frac{c_0}{8}\int_\Omega |\mathcal{D}u^m|^{p(x)}\,\mathrm{d}x + \frac{c_0}{8}\int_{\partial\Omega}|u^m|^2\,\mathrm{d}\sigma \\
&\quad + c\max\{\|F\|^{(p_-)'}_{L^{p'(x)}(\Omega)}, \|F\|^{(p_+)'}_{p'(x)}\} + c\|F\|^2_{p'(x)} \\
&\leqslant \frac{c_0}{8}\int_\Omega |\mathcal{D}u^m|^{p(x)}\,\mathrm{d}x + \frac{c_0}{4}\int_{\partial\Omega}|u^m|^2\,\mathrm{d}\sigma + C,
\end{aligned}
$$

where $c_0 := \min\{c_1, \alpha\}$. Note that there is the trace $u \in L^2(\partial\Omega)$ for $p(x) > 3n/(n+2)$ by Remark 3.2.

The equation (4.2) together with (4.3)–(4.4) implies that

$$
A\mathbf{a}\cdot\mathbf{a} \geqslant c\int_\Omega |\mathcal{D}u^m|^{p(x)}\,\mathrm{d}x + c\alpha\int_{\partial\Omega}|u^m|^2\,\mathrm{d}\sigma - C.
$$

By [23], Lemma IX.3.1, this estimate shows the solvability of the Galerkin approximations $u^m \in \mathbf{X}^m$, that is,

$$
(4.5) \quad \int_\Omega S(x,\mathcal{D}u^m) : \mathcal{D}\phi\,\mathrm{d}x + \int_\Omega (u^m\cdot\nabla)u^m\cdot\phi\,\mathrm{d}x + \alpha\int_{\partial\Omega} u^m\cdot\phi\,\mathrm{d}\sigma = -\int_\Omega F : \nabla\phi\,\mathrm{d}x
$$

for all $\phi \in \mathbf{X}^m$. Moreover this provides the a priori estimate

$$
\int_\Omega |\mathcal{D}u^m|^{p(x)}\,\mathrm{d}x + \alpha\int_{\partial\Omega}|u^m|^2\,\mathrm{d}\sigma \leqslant K,
$$

where the constant $K$ depends on $n$, $\Omega$, $p_-, p_+$, $C_{\log}$ and $\|F\|_{L^{p'(x)}(\Omega)}$. Furthermore by Korn's inequality ((3.3) for $\alpha > 0$ and (3.7) for $\alpha = 0$), Poincaré's inequality (3.1), and (1.5) we obtain

$$
(4.6) \qquad \int_\Omega |u^m|^{p(x)}\,\mathrm{d}x + \int_\Omega |\nabla u^m|^{p(x)}\,\mathrm{d}x + \int_\Omega |S(x,\mathcal{D}u^m)|^{p'(x)}\,\mathrm{d}x \leqslant K.
$$

This shows that there exists a function $u \in \mathcal{V}^{p(x)}_\nu$ and a subsequence, which will be denoted again by $u^m$, such that

$$
(4.7) \qquad
\begin{aligned}
u^m &\rightharpoonup u \quad \text{in } W^{1,p(x)}(\Omega), \\
u^m &\to u \quad \text{in } L^2(\partial\Omega), \\
S(x,\mathcal{D}u^m) &\rightharpoonup \Xi \quad \text{in } L^{p'(x)}(\Omega)
\end{aligned}
$$

since $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^2(\partial\Omega)$ for $p_- > 3n/(n+2)$.

Note that $\frac{1}{2}np_-/(n-p_-) > p'_-$ for $p_- > 3n/(n+2)$. Taking into account this and using (4.7), the compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ ($q < np_-/(n-p_-)$) we have as $m \to \infty$

$$
(4.8) \qquad \int_\Omega (u^m\cdot\nabla)u^m\cdot\phi\,\mathrm{d}x \to \int_\Omega (u\cdot\nabla)u\cdot\phi\,\mathrm{d}x \quad \forall\,\phi \in \mathcal{V}^{p(x)}_\nu(\Omega).
$$

By passing to the limit $m \to \infty$ together with (4.8), (4.7), the equation (4.5) leads to

(4.9)
$$\int_\Omega \Xi : \mathcal{D}\phi \,\mathrm{d}x + \int_\Omega (u \cdot \nabla)u \cdot \phi \,\mathrm{d}x + \alpha \int_{\partial\Omega} u \cdot \phi \,\mathrm{d}\sigma = -\int_\Omega F : \nabla u \,\mathrm{d}x \quad \forall\, \phi \in \mathcal{V}_\nu^{p(x)}(\Omega).$$

On the other hand, substituting $\phi = u^m$ into (4.5) and letting $m \to \infty$, we have

$$\lim_{m\to\infty} \left( \int_\Omega S(x, \mathcal{D}u^m) : \mathcal{D}u^m \,\mathrm{d}x \right) + \alpha \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma = -\int_\Omega F : \nabla u \,\mathrm{d}x. \tag{4.10}$$

We use the monotonicity condition (1.6) to get

$$\lim_{m\to\infty} \int_\Omega (S(x, \mathcal{D}u^m) - S(x, \mathcal{D}\phi)) : \mathcal{D}(u^m - \phi) \,\mathrm{d}x \geqslant 0 \quad \forall\, \phi \in \mathcal{V}_\nu^{p(x)}(\Omega).$$

This together with (4.10) implies that

$$-\int_\Omega S(x, \mathcal{D}\phi) : \mathcal{D}(u - \phi) \,\mathrm{d}x - \int_\Omega \Xi : \mathcal{D}\phi \,\mathrm{d}x - \alpha \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma - \int_\Omega F : \nabla u \,\mathrm{d}x \geqslant 0.$$

Adding this to (4.9) with $\phi = u$, we have

$$\int_\Omega (\Xi - S(x, \mathcal{D}\phi)) : \mathcal{D}(u - \phi) \,\mathrm{d}x \geqslant 0. \tag{4.11}$$

Choosing $\phi = u \pm t\varphi$ in (4.11) and letting $t \to 0$, we conclude that

$$\int_\Omega (\Xi - S(x, \mathcal{D}u)) : \mathcal{D}\varphi \,\mathrm{d}x = 0 \quad \forall\, \varphi \in \mathcal{V}_\nu^{p(x)}(\Omega),$$

which implies that $\Xi = S(x, \mathcal{D}u)$. This shows that $u$ is a weak solution of the problem (1.1)–(1.3).

*Case 2:* $2n/(n+2) < p_- \leqslant 3n/(n+2)$. As in [15], [22] we use the Lipschitz truncation method in this case.

Remark 4.2. In the case $p = \text{const} > 2n/(n+2)$, it seems to be possible for us to prove the existence of a weak solution by the same method from [8]. Here we can give a shorter proof than that in [8] by relying on Theorem 3.4.

We first show the existence of a weak solution to the approximation problem

$$\begin{cases} -\operatorname{div} S(x, \mathcal{D}u^k) + (u^k \cdot \nabla)u^k + \nabla\pi + \dfrac{1}{k}|u^k|^{q-2}u^k = \operatorname{div} F & \text{in } \Omega, \\ \operatorname{div} u^k = 0 & \text{in } \Omega, \\ u^k \cdot \nu = 0,\ (S(x, \mathcal{D}u^k)\nu)_\tau + \alpha(u^k)_\tau = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.12}$$

where $k \in \mathbb{N}$ and $q = 2p'_-$.

Let $\{\phi^j\}$ be a Schauder basis of $\mathcal{V}_\nu^{p(x)}(\Omega) \cap L^q(\Omega)$. Let us define $\mathbf{X}^m$ as a span of $\{\phi^1, \phi^2, \dots, \phi^m\}$. We put $u^{k,m} := \sum_{j=1}^{m} a^j \phi^j$. Due to a suitable choice of the value for $q$ we have

$$\int_\Omega (u^{k,m} \cdot \nabla) u^{k,m} \cdot \phi^j \,\mathrm{d}x \leqslant \|u^{k,m}\|_q \|\nabla u\|_{p_-} \|\phi^j\|_q < \infty.$$

Following the argument as in Case 1, we have

$$A\mathbf{a} \cdot \mathbf{a} \geqslant c \int_\Omega |\mathcal{D}u^{k,m}|^{p(x)} \,\mathrm{d}x + \frac{1}{k}\|u^{k,m}\|_q^q + c\alpha \int_{\partial\Omega} |u^{k,m}|^2 \,\mathrm{d}\sigma - C.$$

It is worth noting that the term $\int_{\partial\Omega} |u^{k,m}|^2 \,\mathrm{d}\sigma$ has meaning since we can assume that $u^{k,m}$ are smooth by means of a possible choice of Schauder basis. Hence by [23], Lemma IX.3.1, there exist the Galerkin approximations $u^{k,m}$ satisfying

$$\|\mathcal{D}u^{k,m}\|_{p(x)} + \alpha \|u^{k,m}\|_{L^2(\partial\Omega)} + \frac{1}{k^{1/q}} \|u^{k,m}\|_q \leqslant C.$$

As in Case 1 we can prove the existence of weak solutions $u^k \in \mathcal{V}_\nu^{p(x)}(\Omega) \cap L^q(\Omega)$ to the problem (4.12) such that

$$\|\mathcal{D}u^k\|_{p(x)} + \alpha \|u^k\|_{L^{2(n-1)/n}(\partial\Omega)} + \frac{1}{k^{1/q}} \|u^k\|_q \leqslant C, \tag{4.13}$$

where we use that $\|u^k\|_{L^{2(n-1)/n}(\partial\Omega)} \leqslant c\|u^k\|_{L^2(\partial\Omega)}$ by the smoothness of $\partial\Omega$ and the trivial inequality $2(n-1)/n < 2$. We note that even though $u^k \in V_\nu^{p(x)} \cap L^q$, $\|u^k\|_{L^{2(n-1)/n}(\partial\Omega)}$ has meaning for $p_- \geqslant 2n/(n+2)$ by the standard trace embedding theorem. Hence there exist a function $u \in \mathcal{V}_\nu^{p(x)}$ and a (not relabeled) subsequence $u^k$ such that

$$\begin{aligned} \mathcal{D}u^k \rightharpoonup \mathcal{D}u, \nabla u^k \rightharpoonup \nabla u \quad & \text{in } L^{p(x)}(\Omega), \\ u^k \to u \quad & \text{a.e. in } \Omega, \\ S(x, \mathcal{D}u^k) \rightharpoonup \Xi_1 \quad & \text{in } L^{p'(x)}(\Omega), \\ u^k \to u \quad & \text{in } L^1(\partial\Omega), \end{aligned} \tag{4.14}$$

and satisfying

$$\|\mathcal{D}u\|_{p(x)} + \alpha \|u\|_{L^1(\partial\Omega)} \leqslant K. \tag{4.15}$$

It follows from (4.13) that as $k \to \infty$

$$\frac{1}{k} \int_\Omega |u^k|^{q-2} u^k \cdot \phi \,\mathrm{d}x \leqslant \frac{1}{k^{1/q}} \Big( \frac{1}{k^{1/q}} \|u^k\|_q \Big)^{q-1} \|\phi\|_q \to 0 \quad \forall\, \phi \in L^q(\Omega). \tag{4.16}$$

Since $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^1(\partial\Omega)$ for $p_- > 2n/(n+2)$ we get that as $k \to \infty$

$$\int_\Omega (u^k \otimes u^k) : \nabla\phi \,\mathrm{d}x \to \int_\Omega (u \otimes u) : \nabla\phi \,\mathrm{d}x \quad \forall\, \phi \in W^{1,\infty}_\nu(\Omega),$$
$$\int_{\partial\Omega} u^k \cdot \phi \,\mathrm{d}\sigma \to \int_{\partial\Omega} u \cdot \phi \,\mathrm{d}\sigma \quad \forall\, \phi \in W^{1,\infty}_\nu(\Omega).$$

This together with (4.14) and (4.16) implies that

$$\int_\Omega \Xi_1 : \mathcal{D}\phi \,\mathrm{d}x + \int_\Omega (u \otimes u) : \nabla\phi \,\mathrm{d}x + \alpha \int_{\partial\Omega} u \cdot \phi \,\mathrm{d}\sigma = -\int_\Omega F : \nabla\phi \,\mathrm{d}x \quad \forall\, \phi \in \mathcal{V}^\infty_\nu(\Omega).$$

To complete the proof, it suffices to show that $\Xi_1 = S(x, \mathcal{D}u)$. To this end, we use the Lipschitz truncation method in $W^{1,p(x)}_\nu(\Omega)$. Since $w^k := u^k - u \rightharpoonup 0$ in $W^{1,p(x)}_\nu(\Omega)$ by (4.14), we can apply Theorem 3.4 to the function. We denote by $w^{k,j}$ the Lipschitz approximations to the vector-valued functions $w^k$. Since functions $w^{k,j}$ are in general not divergence free, we cannot use them as test functions. By [14], Theorem 14.3.15 there exists a solution $\xi^{k,j} \in W^{1,p(x)}_0(\Omega)$ to the problem

$$\operatorname{div} \xi^{k,j} = \operatorname{div} w^{k,j} \quad \text{in } \Omega; \quad \xi^{k,j} = 0 \quad \text{on } \partial\Omega.$$

Furthermore it follows from Theorem 3.4 that for each $j \in \mathbb{N}$ and for all $s \in (1, \infty)$

$$\xi^{k,j} \rightharpoonup 0 \quad \text{in } W^{1,s}(\Omega), \quad \xi^{k,j} \to 0 \quad \text{in } L^s(\Omega). \tag{4.17}$$

Since $\operatorname{div} w^{k,j} = \operatorname{div} w^k = 0$ on $\{x\colon\ w^k(x) = w^{k,j}(x)\}$ we can easily see that by Theorem 3.4

$$\begin{aligned} \limsup_{k\to\infty} \|\xi^{k,j}\|_{1,p(x)} &\leqslant c \limsup_{k\to\infty} \|\operatorname{div} w^{k,j} \chi_{\{w^k \neq w^{k,j}\}}\|_{p(x)} \\ &\leqslant c \limsup_{k\to\infty} \|\nabla w^{k,j} \chi_{\{w^k \neq w^{k,j}\}}\|_{p(x)} \leqslant c\varepsilon_j. \end{aligned} \tag{4.18}$$

Setting $\eta^{k,j} = w^{k,j} - \xi^{k,j}$ and using Theorem 3.4 and (4.17) we obtain that for a fixed $j \in \mathbb{N}$

$$\eta^{k,j} \rightharpoonup 0 \quad \text{in } W^{1,s}(\Omega); \quad \eta^{k,j} \to 0 \quad \text{in } L^s(\Omega) \quad \text{as } k \to \infty. \tag{4.19}$$

We test (4.12) with $\eta^{k,j}$ to get

(4.20)

$$\begin{aligned} &\int_\Omega (S(x, \mathcal{D}u^k) - S(x, \mathcal{D}u)) : \mathcal{D}w^{k,j} \,\mathrm{d}x \\ &\quad = \int_\Omega S(x, \mathcal{D}u^k) : \mathcal{D}\xi^{k,j} \,\mathrm{d}x - \int_\Omega S(x, \mathcal{D}u) : \mathcal{D}w^{k,j} \,\mathrm{d}x - \frac{1}{k} \int_\Omega |u^k|^{q-2} u^k \cdot \eta^{k,j} \,\mathrm{d}x \\ &\qquad \pm \int_\Omega F : \nabla\eta^{k,j} \,\mathrm{d}x + \int_\Omega (u^k \otimes u^k) : \nabla\eta^{k,j} \,\mathrm{d}x - \alpha \int_{\partial\Omega} u^k \cdot \eta^{k,j} \,\mathrm{d}\sigma \\ &\quad =: J^1_{k,j} + J^2_{k,j} + J^3_{k,j} + J^4_{k,j} + J^5_{k,j} + J^6_{k,j}. \end{aligned}$$

By the same arguments as in [15] we can see that for all $j \in \mathbb{N}$

$$\lim_{k\to\infty} |J^2_{k,j} + J^3_{k,j} + J^4_{k,j} + J^5_{k,j}| = 0, \tag{4.21}$$

$$\lim_{k\to\infty} |J^1_{k,j}| \leqslant c(K)\varepsilon_j. \tag{4.22}$$

Since $\eta^{k,j} \to 0$ in $L^s(\partial\Omega)$ by (4.19) and the standard trace embedding, it follows that for all $j \in \mathbb{N}$

$$\lim_{k\to\infty} |J^6_{k,j}| = 0. \tag{4.23}$$

The equation (4.20) together with (4.21)–(4.23) implies that for all $j \in \mathbb{N}$

$$\limsup_{k\to\infty} \int_\Omega (S(x,\mathcal{D}u^k) - S(x,\mathcal{D}u)) : \mathcal{D}w^{k,j}\,\mathrm{d}x \leqslant c(K)\varepsilon_j.$$

Following the same lines as in [15] we can get that for a not relabeled subsequence

$$\int_\Omega (S(x,\mathcal{D}u^k) - S(x,\mathcal{D}u)) : \mathcal{D}\phi\,\mathrm{d}x \to 0 \quad \forall\,\phi \in W^{1,\infty}_\nu(\Omega),$$

which shows that $\Xi_1 = S(x,\mathcal{D}u)$.

Moreover the estimates (4.15), (4.6) together with Poincaré's inequality (3.1) and Korn's one ((3.3) for $\alpha > 0$, (3.7) for $\alpha = 0$) imply (2.1). $\square$

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