ON THE AVERAGING OF DIFFERENTIAL INCLUSIONS WITH MAXIMA

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Abstract. We apply the averaging method to ordinary differential inclusions with maxima perturbed by a small parameter and illustrate the method by some examples.

Keywords: differential inclusion; maxima; averaging

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1. Introduction and notations

The theory of differential equations and inclusions with maxima has attracted a lot of interest in the recent years. The justification of the averaging for the case of differential equations with maxima was considered in, e.g., [4], [10], [13], [14], [15], and the averaging method of set valued differential equations with maxima is considered in [9]. The method of averaging of differential inclusions with maxima was also considered recently in [5].

We consider the following initial value problem associated to a differential inclusion with maxima;

$$\begin{cases} \dot{x} \in \varepsilon F\Big(t, x(t), \max\limits_{s\in S(t)} x(s)\Big), \quad t \geqslant 0, \\ x(t) = x_0, \end{cases} \tag{1.1}$$

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where $F\colon \mathbb{R}_+ \times \mathbb{R}^p \times \mathbb{R}^p \to \mathcal{P}(\mathbb{R}^p)$ and $S\colon \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}_+)$, with $S(t) \subset [0,t]$ for $t \geqslant 0$, are multifunctions and

$$\max_{s\in S(t)} x(s) := \Big(\max_{s\in S(t)} x_1(s), \max_{s\in S(t)} x_2(s), \dots, \max_{s\in S(t)} x_p(s)\Big).$$

In the case where the multifunctions $\overline{F}\colon \mathbb{R}^p \times \mathbb{R}^p \to \mathcal{P}(\mathbb{R}^p)$ and $\overline{S}\colon \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}_+)$, given by

$$\overline{F}(x,z) = \lim_{T\to\infty} \frac{1}{T} \int_0^T F(t,x,z)\,\mathrm{d}t \tag{1.2}$$

and

$$\lim_{\varepsilon\to 0} \varepsilon S\Big(\frac{\tau}{\varepsilon}\Big) = \overline{S}(\tau) \tag{1.3}$$

exist, where the integral is understood in Aumann-Hukuhara sense and the convergence in sense of the Hausdorff metric (see [3]).

We can consider the following initial value problem:

$$\begin{cases} z' \in \overline{F}\Big(z(\tau), \max\limits_{\tau\in\overline{S}(\tau)} z(s)\Big), & \tau \geqslant 0,\\ z(0) = x_0 \end{cases} \tag{1.4}$$

such that $z' = \mathrm{d}z/\mathrm{d}\tau$.

The structure of the paper is as follows. In Section 2 we present our main results: Theorems 2.1 and Corollary 2.1. We state and prove some preliminary results in Section 3 and then give the proofs of Theorem 2.1.

We finish this section with some definitions and notations. Let $\mathbb{R}^p$ denot the p-dimensional space with the Euclidean norm $|\cdot|$. $\mathrm{Comp}(\mathbb{R}^p)$ ($\mathrm{Conv}(\mathbb{R}^p)$, respectively) stands for the class of all nonempty compact (nonempty compact and convex, respectively) subsets of $\mathbb{R}^p$. In $\mathrm{Comp}(\mathbb{R}^p)$ the so-called Hausdorff metric is defined by

$$H(A,B) = \max\Big(\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\Big) \quad \forall\, A,B \in \mathrm{Comp}(\mathbb{R}^p),$$

where $d(\alpha, C) = \inf\{|\alpha - c|, c \in C\}$ for any $\alpha \in \mathbb{R}^p$ and any $C \in \mathrm{Comp}(\mathbb{R}^p)$.

Definition 1.1 ([7]). A multifunction $G\colon \Omega \subset \mathbb{R}^m \to \mathrm{Comp}(\mathbb{R}^n)$ is said to be *continuous at a point* $x_0 \in \Omega$ if for all $\varepsilon > 0$, exists $\delta > 0$ such that for all $x \in \Omega$ where $\|x - x_0\| < \delta$ then $H(G(x), G(x_0)) \leqslant \varepsilon$. G is said to be continuous if it is continuous at every point of Ω.

Definition 1.2 ([2], [3]). The *integral of a multifunction* $G\colon I \subset \mathbb{R} \to \mathrm{Comp}(\mathbb{R}^n)$ on the interval I is defined by

$$\int_I G := \Big\{ \int_I g,\ g \in \Gamma \Big\},$$

where Γ is the set of functions g which are integrable on I and which verify $g(t) \in G(t)$ for all $t \in I$.

Let $F\colon \mathbb{R}_+ \times \mathbb{R}^p \times \mathbb{R}^p \to \mathrm{Comp}(\mathbb{R}^p)$ be a multifunction. By a solution of the differential inclusion with maxima $\dot{x}(t) \in F\Big(t, x, \max_{s\in S(t)} x(s)\Big)$ we mean an absolutely continuous function x defined on some interval and satisfying $\dot{x}(t) \in F\Big(t, x, \max_{s\in S(t)} x(s)\Big)$ almost everywhere.

Let $\alpha, A > 0$. We call $K(\alpha, A)$ the class of multifunctions $S\colon \mathbb{R}_+ \to \mathrm{Comp}(\mathbb{R}_+)$ which verify: for all $t_1, t_2 \in \mathbb{R}_+$, $|t_1 - t_2| \leqslant \alpha \Rightarrow H(S(t_1), S(t_2)) \leqslant A$. This class is always nonempty, and for every S uniformly continuous and $\alpha > 0$ there is $A = A(\alpha) > 0$ such that $S \in K(\alpha, A)$.

For the basic theory of differential inclusions we refer to the books of Deimling (see [7]), Aubin and Frankowska (see [2]), Aubin and Cellina (see [1]) and Smirnov (see [16]).

2. Averaging results

First, let us formulate the assumptions on F and S that we need to prove our averaging results.

(H1) $F\colon \mathbb{R}_+ \times \mathbb{U} \times \mathbb{U} \to \mathrm{Conv}(\mathbb{R}^p)$, where $\mathbb{U}$ is an open subset of $\mathbb{R}^p$, is measurable in t, continuous in (x, y) uniformly in t, and

$$H(F(t, x, y), \{0\}) \leqslant m(t) \quad \forall\, (t, x, y) \in \mathbb{R}_+ \times \mathbb{U} \times \mathbb{U}$$

with

$$\int_{t_1}^{t_2} m(t)\,\mathrm{d}t \leqslant M(t_2 - t_1) \quad \forall\, t_1, t_2 \in \mathbb{R}_+.$$

(H2) For all $x, y \in \mathbb{U}$, the limit

$$\overline{F}(x, y) := \lim_{L\to\infty} \frac{1}{L} \int_0^L F(\tau, x, y)\,\mathrm{d}\tau$$

exists uniformly with respect to (x, y), where the integral is meant in Aumann-Hukuhara's sense.

(H3) There is $\overline{S}:\mathbb{R}_+\to \mathrm{Comp}(\mathbb{R}^p)$ such that, for every $L>0$, the quantity

$$\xi_\varepsilon(L)=\sup\Big\{H\Big(\varepsilon S\Big(\frac{\tau}{\varepsilon}\Big),\overline{S}(\tau)\Big),\tau\in[0,L]\Big\}\text{ is such that }\lim_{\varepsilon\to 0}\xi_\varepsilon(L)=0.$$

We have the following theorem:

Theorem 2.1. *Assume that the assumptions* (H1)–(H3) *are fulfilled. Then, for any $T>0$, where $S\in K(T,A_T)$ for a certain fixed A_T and is measurable, and for any $\eta>0$, there exists $\varepsilon_0=\varepsilon(\eta,T)>0$ such that, for any $\varepsilon\in(0,\varepsilon_0]$ and for every solution x_ε of problem* (1.1) *which is defined on $[0,T/\varepsilon]$, there is a solution z of problem* (1.4) *such that z is defined on $[0,T]$ and satisfies*

$$|x_\varepsilon(t)-z(\varepsilon t)|\leqslant\eta\quad\forall\, t\in[0,T/\varepsilon].$$

In the case where problem (1.4) has a unique solution, we have the following result which is a corollary of Theorem 2.1.

Corollary 2.1. *Assume that the assumptions* (H1)–(H3) *are fulfilled and let $T>0$ such that problem* (1.4) *has a unique solution $y_\varepsilon(\cdot)$ defined on $[0,T]$, and $S\in K(T,A_T)$ for a certain fixed A_T and is measurable, then, for any $\eta>0$, there exists $\varepsilon_0=\varepsilon_0(\eta,T)>0$ such that for any $\varepsilon\in(0,\varepsilon_0]$ and for every $x_\varepsilon(\cdot)$ solution of problem* (1.1)

$$|x_\varepsilon(t)-y_\varepsilon(\varepsilon t)|\leqslant\eta\quad\forall\, t\in[0,T/\varepsilon].$$

Remark 2.1.

(1) If we take the particular case $S(t)=\{t\}$, then we obtain an ordinary differential inclusion (i.e., without maximum), and the results above are similar to those found in [12], Theorem 2.1, and to those in [11], Theorem 8.
(2) When $S(t)=[t-r,t]$ and F is single valued, we deduce from the above corollary the result in [15].
(3) In [5], [10], the multifunction S is supposed to be uniformly continuous, and the function F verifies a certain Lipschitz condition. In Theorem 2.1 above, S is a general multifunction which is not necessarily continuous, and F is only continuous.

Example 2.1. As an example for Theorem 2.1, we consider the system

$$\begin{cases}\dot{x}(t)\in\varepsilon|\sin(t)|^2\Big([0,1]+\Big(\max\limits_{s\in[g_1(t),g_2(t)]}x(s)\Big)^{1/2}\Big), & t\geqslant 0,\\ x(0)=1,\end{cases}$$

where $g_1(t) = \max\left\{0, \frac{1}{2}\left(t - \sqrt{t}\right)\right\}$, and $g_2(t) = \min\{t, \frac{1}{2}\lceil t \rceil\}$, where $\lceil \cdot \rceil$ is the ceiling function. Notice that $S(\cdot) \equiv [g_1(\cdot), g_2(\cdot)]$ is not continuous, but $S \in K(1,1)$ and $\overline{S}$ exists and is given by $\overline{S}(\tau) = \frac{1}{2}\tau$, for $\tau \geqslant 0$, where $\tau = t/\varepsilon$.

$$\begin{cases} z'(\tau) \in \dfrac{1}{2}\left([0,1] + \sqrt{z\left(\dfrac{\tau}{2}\right)}\right), \\ z(0) = 1. \end{cases}$$

Theorem 2.1 gives us that every solution of the original problem could be approximated by a solution of the averaged one; notice also that from a numerical point of view the averaged problem is less expensive, because the evaluation needed is only in a single point (i.e., $\frac{1}{2}\tau$), whereas in the original one we must find the maximum on an interval.

E x a m p l e 2.2. Let us take the following example considered in [9]

$$\begin{cases} \dot{x}_1(\tau) = \varepsilon\Big[-2\lambda x_1 \sin\Big(\tau + \max\limits_{s\in[g_1(\tau),g_2(\tau)]} x_2(s)\Big) + \mu x_1^3 \cos^3(\tau + x_2)\Big] \sin(\tau + x_2), \\ \dot{x}_2(\tau) = -\dfrac{\varepsilon}{x_1}\Big[-2\lambda x_1 \sin\Big(\tau + \max\limits_{s\in[g_1(\tau),g_2(\tau)]} x_2(s)\Big) + \mu x_1^3 \cos^3(\tau + x_2)\Big] \cos(\tau + x_2) \end{cases}$$

with $x_1(0) = 2$, $x_2(0) = \frac{1}{2}\pi$, $g_1(\tau) = \max\{0, \tau - \frac{1}{2}\}$, and $g_2(\tau) = \max\{0, \tau - \frac{1}{4}\}$, $\lambda = 0.7$, $\mu = 0.2$. The averaged system is given by:

$$\begin{cases} y_1'(t) = -\lambda y_1(t), & y_1(0) = 2, \\ y_2'(t) = -\dfrac{3\mu}{8} y_1^2(t), & y_2(0) = \dfrac{\pi}{2}. \end{cases}$$

This means that $y_1(t) = 2\exp(-\lambda t)$, and $y_2(t) = \frac{1}{2}\pi + (\exp(-2\lambda t) - 1)3\mu/(16\lambda)$

ε	0.5	0.1	0.01	0.001
$\max\lvert x_1(\tau) - y_1(\varepsilon\tau)\rvert$	0.2470	0.0740	0.0083	0.0008
$\max\lvert x_2(\tau) - y_2(\varepsilon\tau)\rvert$	0.2890	0.0692	0.0076	0.0007

Although the solution of the problem can be computed analytically (contrary to that in [10]), the error bounds are sharper than those in the aforementioned work, but this is not true in general.

E x a m p l e 2.3. The need for one-sided Lipschitz condition (see [5]) for approximating solutions of problem (1.4) by those of (1.1), is justified by the following system:

$$\begin{cases} \dot{x}(t) = \varepsilon\left(\sqrt{\left|\max\limits_{s\in[0,t]} x(s)\right|} + \sin(t)\right), \\ x(0) = 0. \end{cases} \tag{2.1}$$

Theorem 2.1 still holds, and the averaged system is:

$$\text{(2.2)} \qquad \begin{cases} \dot{y}(t) = \varepsilon\sqrt{\left|\max\limits_{s\in[0,t]} y(s)\right|}, \\ y(0) = 0. \end{cases}$$

There are no solutions of problem (2.1) approximating the trivial solution $y(t) \equiv 0$ of the averaged problem (2.2) (see [8]).

3. Proofs of the results

To prove Theorems 2.1–2.2 we need to establish the following preliminary lemma:

Lemma 3.1. *Let $F : \mathbb{R}_+ \times \mathbb{U} \times \mathbb{U} \to \mathrm{Conv}(\mathbb{R}^p)$ be a multifunction.*

(1) *If F satisfies Assumption* (H1), *then its average $\overline{F}\colon \mathbb{U} \times \mathbb{U} \to \mathrm{Conv}(\mathbb{R}^p)$ is uniformly bounded by the constant M and is continuous.*
(2) *Let $S\colon \mathbb{R}_+ \to \mathrm{Comp}(\mathbb{R}_+)$ be in a class $K(T, A_T)$. Then $\overline{S}\colon \mathbb{R}_+ \to \mathrm{Comp}(\mathbb{R}^p)$ defined in* (H4) *is continuous.*

Proof. For (1), see [6].

Now suppose that $\tau_1, \tau \in \mathbb{R}_+$, and $0 < \varepsilon \ll 1$ such that $|\tau_1/\varepsilon - \tau/\varepsilon| \leqslant T$, which implies $H(S(\tau_1/\varepsilon), S(\tau/\varepsilon)) \leqslant A_T$. Thus

$$\begin{aligned} H(\overline{S}(\tau_1), \overline{S}(\tau)) &\leqslant H\Big(\overline{S}(\tau_1), \varepsilon S\Big(\frac{\tau_1}{\varepsilon}\Big)\Big) + \varepsilon H\Big(S\Big(\frac{\tau_1}{\varepsilon}\Big), S\Big(\frac{\tau}{\varepsilon}\Big)\Big) + H\Big(\varepsilon S\Big(\frac{\tau}{\varepsilon}\Big), \overline{S}(\tau)\Big) \\ &\leqslant 2\xi_\varepsilon(T) + \varepsilon A_T. \end{aligned}$$

Piecing it all together gives

$$|\tau_1 - \tau| \leqslant T\varepsilon \to 0 \Rightarrow H(\overline{S}(\tau_1), \overline{S}(\tau)) \leqslant 2\xi_\varepsilon(T) + \varepsilon A_T \to 0.$$

This finishes the proof. □

We need the following lemma, which is a generalization of Lemma 1 (see [1], page 99); the proof is similar to the one mentioned.

Lemma 3.2 (Integral representation)**.** *Let $G\colon [0, L]\times\mathbb{U}\times\mathbb{U} \to \mathrm{Conv}(\mathbb{R}^p)$, where $\mathbb{U}$ is an open subset of $\mathbb{R}^p$, be an $\varepsilon - \delta$ upper semicontinuous multifunction (see* [7]*), and $H(G(t, x, y), 0) \leqslant m(t)$ for all $(t, x, y) \in I \times \mathbb{U} \times \mathbb{U}$, where $m(\cdot)$ as in* (H1), *and* $\widetilde{S}\colon \mathbb{R}_+ \to \mathrm{Comp}(\mathbb{R}^p)$ *be continuous (w.r.t the metric* H*). Then the continuous function* $x(\cdot)$ *is a solution on* $I = [0, L]$ *to the inclusion*

$$\dot{x}(t) \in G\Big(t, x(t), \max_{s\in\widetilde{S}(t)} x(s)\Big)$$

if and only if for every pair $t_1, t_2 \in I$

$$x(t_2) - x(t_1) \in \int_{t_1}^{t_2} G\Big(t, x(t), \max_{s\in\widetilde{S}(t)} x(s)\Big)\,\mathrm{d}t.$$

P r o o f. The necessity of the statement is obvious, so we prove only its sufficiency.

First, notice that $|x(t_2)-x(t_1)| \leqslant \int_{t_1}^{t_2} m(t)\,\mathrm{d}t \leqslant M(t_2-t_1)$. Thus x is differentiable a.e.; also the fact that $\widetilde{S}$ and x are continuous means that the function $\max\limits_{s\in\widetilde{S}(\cdot)} x(s)$ is continuous as well. Hence $\varphi(\cdot) = G\Big(\cdot, x(\cdot), \max\limits_{s\in\widetilde{S}(\cdot)} x(s)\Big)$ is $\varepsilon-\delta$ upper semicontinuous. Fix t and let $\delta > 0$ be such that for $t' \in I$, we have that $|t-t'| \leqslant \delta$ implies that $\varphi(t') \subset \varphi(t) + \varepsilon B$, where B is the unit ball in $\mathbb{R}^p$. Then

$$x(t_1) - x(t) \in \int_{t}^{t_1} G\Big(l, x(l), \max_{s\in\widetilde{S}(l)} x(s)\Big)\,\mathrm{d}l \in \Big(G\Big(t, x(t), \max_{s\in\widetilde{S}(t)} x(s)\Big) + \varepsilon B\Big)(t_1 - t),$$

The last inclusion means that $\dot{x}(t) \in G\Big(t, x(t), \max\limits_{s\in\widetilde{S}(t)} x(s)\Big) + \varepsilon B$, ε is arbitrary and G is closed valued, i.e.,

$$\dot{x}(t) \in G\Big(t, x(t), \max_{s\in\widetilde{S}(t)} x(s)\Big).$$

This finishes the proof. □

P r o o f of Theorem 2.1. First, the fact that F is continuous in (x, y) uniformly in t means that there exists some function ω such that

$$\omega(F,\gamma) = \sup\{H(F(t,x_1,y_1), F(t,x_2,y_2))\colon\ |x_1-x_2|+|y_1-y_2| \leqslant \gamma,\ t \in \mathbb{R}_+,\ x_i, y_i \in \mathbb{U}\}$$

and $\lim\limits_{\gamma\to 0} \omega(F,\gamma) = 0$.

Let us make the following change of variable: $\tau = \varepsilon t$, and let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a non-increasing sequence converging to 0, and let x_n be a solution of (1.1) for $\varepsilon = \varepsilon_n$; thus x_n is a solution of the inclusion:

$$\begin{cases} x_n' \in F\Big(\dfrac{\tau}{\varepsilon_n}, x_n(\tau), \max\limits_{s\in\varepsilon_n S(\tau/\varepsilon_n)} x_n(s)\Big), & t \geqslant 0, \\ x_n(0) = x_0. \end{cases} \tag{3.1}$$

It is easy to prove that the set $\{x_n\}$ is uniformly bounded and equicontinuous; thus by Ascoli-Arzelà's theorem there is a subsequence that converges to a function z, i.e., $\lim_n \|x_n - z\|_{C[0,T]} = 0$. For $\alpha, \beta \in [0,T]$, let us divide the interval $[\alpha,\beta]$ into intervals $[\tau_i, \tau_{i+1}]$, such that $\tau_i = \alpha + i(\beta-\alpha)/m$ where $i \leqslant m-1$, and define $\overline{z}$ as a step function defined by $\overline{z}(\tau) = z(\tau_i)$, for $\tau \in [\tau_i, \tau_{i+1}[$ and $i \leqslant m-1$.

Let us take $n > n_0$ and $m > m_0$ such that $\|x_n - z\| \leqslant \delta$ and $\|\overline{z} - z\| \leqslant \delta$, we have

(3.2)

$$\begin{aligned}
H\Big(&\int_\alpha^\beta F\Big(\frac{\tau}{\varepsilon_n}, x_n(\tau), \max_{s\in\varepsilon_n S(\tau/\varepsilon_n)} x_n(s)\Big)\,\mathrm{d}\tau, \int_\alpha^\beta \overline{F}\Big(z(\tau), \max_{s\in\overline{S}(\tau)} z(s)\Big)\,\mathrm{d}\tau\Big)\\
&\leqslant H\Big(\int_\alpha^\beta F\Big(\frac{\tau}{\varepsilon_n}, x_n(\tau), \max_{s\in\varepsilon_n S(\tau/\varepsilon_n)} x_n(s)\Big)\,\mathrm{d}\tau, \int_\alpha^\beta F\Big(\frac{\tau}{\varepsilon_n}, z(\tau), \max_{s\in\varepsilon_n S(\tau/\varepsilon_n)} z(s)\Big)\,\mathrm{d}\tau\Big)\\
&\quad + H\Big(\int_\alpha^\beta F\Big(\frac{\tau}{\varepsilon_n}, z(\tau), \max_{s\in\varepsilon_n S(\tau/\varepsilon_n)} z(s)\Big)\,\mathrm{d}\tau, \int_\alpha^\beta F\Big(\frac{\tau}{\varepsilon_n}, \overline{z}(\tau), \max_{s\in\overline{S}(\tau)} \overline{z}(s)\Big)\,\mathrm{d}\tau\Big)\\
&\quad + H\Big(\int_\alpha^\beta F\Big(\frac{\tau}{\varepsilon_n}, \overline{z}(\tau), \max_{s\in\overline{S}(\tau)} \overline{z}(s)\Big)\,\mathrm{d}\tau, \int_\alpha^\beta \overline{F}\Big(\overline{z}(\tau), \max_{s\in\overline{S}(\tau)} \overline{z}(s)\Big)\,\mathrm{d}\tau\Big)\\
&\quad + H\Big(\int_\alpha^\beta \overline{F}\Big(\overline{z}(\tau), \max_{s\in\overline{S}(\tau)} \overline{z}(s)\Big)\,\mathrm{d}\tau, \int_\alpha^\beta \overline{F}\Big(z(\tau), \max_{s\in\overline{S}(\tau)} z(s)\Big)\,\mathrm{d}\tau\Big)
\end{aligned}$$

and

(3.3)

$$\begin{aligned}
H\Big(&\int_\alpha^\beta F\Big(\frac{\tau}{\varepsilon_n}, x_n(\tau), \max_{s\in\varepsilon_n S(\tau/\varepsilon_n)} x_n(s)\Big)\,\mathrm{d}\tau, \int_\alpha^\beta F\Big(\frac{\tau}{\varepsilon_n}, z(\tau), \max_{s\in\varepsilon_n S(\tau/\varepsilon_n)} z(s)\Big)\,\mathrm{d}\tau\Big)\\
&\leqslant \int_\alpha^\beta H\Big(F\Big(\frac{\tau}{\varepsilon_n}, x_n(\tau), \max_{s\in\varepsilon_n S(\tau/\varepsilon_n)} x_n(s)\Big), F\Big(\frac{\tau}{\varepsilon_n}, z(\tau), \max_{s\in\varepsilon_n S(\tau/\varepsilon_n)} z(s)\Big)\Big)\,\mathrm{d}\tau\\
&\leqslant T\omega(F, 2\delta).
\end{aligned}$$

We have also

$$\begin{aligned}
\Big|\max_{s\in\varepsilon_n S(\tau/\varepsilon_n)} z(s) - \max_{s\in\overline{S}(\tau)} \overline{z}(s)\Big| &\leqslant \Big|\max_{s\in\varepsilon_n S(\tau/\varepsilon_n)} z(s) - \max_{s\in\overline{S}(\tau)} z(s)\Big|\\
&\quad + \Big|\max_{s\in\overline{S}(\tau)} z(s) - \max_{s\in\overline{S}(\tau)} \overline{z}(s)\Big|\\
&\leqslant M\xi_{\varepsilon_n}(T) + \delta,
\end{aligned}$$

where $\lim_{n\to\infty} \xi_{\varepsilon_n}(T) = 0$. By virtue of the last inequality, we obtain

$$\begin{aligned}
H\Big(&\int_\alpha^\beta F\Big(\frac{\tau}{\varepsilon_n}, z(\tau), \max_{s\in\varepsilon_n S(\tau/\varepsilon_n)} z(s)\Big)\,\mathrm{d}\tau, \int_\alpha^\beta F\Big(\frac{\tau}{\varepsilon_n}, \overline{z}(\tau), \max_{s\in\overline{S}(\tau)} \overline{z}(s)\Big)\,\mathrm{d}\tau\Big)\\
&\leqslant \int_\alpha^\beta H\Big(F\Big(\frac{\tau}{\varepsilon_n}, z(\tau), \max_{s\in\varepsilon_n S(\tau/\varepsilon_n)} z(s)\Big), F\Big(\frac{\tau}{\varepsilon_n}, \overline{z}(\tau), \max_{s\in\overline{S}(\tau)} \overline{z}(s)\Big)\Big)\,\mathrm{d}\tau\\
&\leqslant T\omega(F, M\xi_{\varepsilon_n}(T) + 2\delta).
\end{aligned} \tag{3.4}$$

It is also easy to prove (see [11]) that for every $\mu > 0$ we have

$$H\Big(\int_{\tau_i}^{\tau_{i+1}} F\Big(\frac{\tau}{\varepsilon_n}, \overline{z}(\tau), \max_{s\in\overline{S}(\tau)} \overline{z}(s)\Big)\,\mathrm{d}\tau, \int_{\tau_i}^{\tau_{i+1}} \overline{F}\Big(\overline{z}(\tau), \max_{s\in\overline{S}(\tau)} \overline{z}(s)\Big)\,\mathrm{d}\tau\Big) \leqslant (\tau_{i+1}-\tau_i)\mu.$$

Hence,

$$H\Big(\int_{\alpha}^{\beta} F\Big(\frac{\tau}{\varepsilon_n}, \overline{z}(\tau), \max_{s\in\overline{S}(\tau)} \overline{z}(s)\Big)\,\mathrm{d}\tau, \int_{\alpha}^{\beta} \overline{F}\Big(\overline{z}(\tau), \max_{s\in\overline{S}(\tau)} \overline{z}(s)\Big)\,\mathrm{d}\tau\Big) \leqslant T\mu, \tag{3.5}$$

$$H\Big(\int_{\alpha}^{\beta} \overline{F}\Big(\overline{z}(\tau), \max_{s\in\overline{S}(\tau)} \overline{z}(s)\Big)\,\mathrm{d}\tau, \int_{\alpha}^{\beta} \overline{F}\Big(z(\tau), \max_{s\in\overline{S}(\tau)} z(s)\Big)\,\mathrm{d}\tau\Big) \leqslant T\omega(F, 2\delta). \tag{3.6}$$

By virtue of (3.2), (3.3), (3.4), (3.5), and (3.6), we obtain

$$\begin{aligned} H\Big(\int_{\alpha}^{\beta} F\Big(\frac{\tau}{\varepsilon_n}, x_n(\tau), \max_{s\in\varepsilon_n S(\tau/\varepsilon_n)} x_n(s)\Big)\,\mathrm{d}\tau, \int_{\alpha}^{\beta} \overline{F}\Big(z(\tau), \max_{s\in\overline{S}(\tau)} z(s)\Big)\,\mathrm{d}\tau\Big) \\ \leqslant 2T\omega(F, 2\delta) + T\omega(F, M\xi_{\varepsilon_n}(T) + 2\delta) + T\mu. \end{aligned}$$

The last quantity could be made as small as we want, and thus z verifies

$$z(\beta) - z(\alpha) \in \int_{\alpha}^{\beta} \overline{F}\Big(z(\tau), \max_{s\in\overline{S}(\tau)} z(s)\Big)\,\mathrm{d}\tau.$$

Taking into account Lemma 3.1 ($\overline{F}$ and $\overline{S}$ are continuous), and applying Lemma 3.2 to the last inclusion means that z is solution of (1.4). This finishes the proof of Theorem 2.1. □

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