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OSCILLATORY PROPERTIES OF THIRD-ORDER SEMI-NONCANONICAL NONLINEAR DELAY DIFFERENCE EQUATIONS

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Abstract. We study the oscillatory properties of the solutions of the third-order nonlinear semi-noncanonical delay difference equation

$$D_3y(n) + f(n)y^{\beta}(\sigma(n)) = 0,$$

where $D_3y(n) = \Delta(b(n)\Delta(a(n)(\Delta y(n))^{\alpha}))$ is studied. The main idea is to transform the semi-noncanonical operator into canonical form. Then we obtain new oscillation theorems for the studied equation. Examples are provided to illustrate the importance of the main results.

Keywords: semi-noncanonical operator; third-order; delay difference equation; oscillation

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1. Introduction

Consider the third-order nonlinear delay difference equation

$$D_3y(n) + f(n)y^{\beta}(\sigma(n)) = 0, \quad n \in \mathbb{N}(n_0), \tag{E}$$

where $\mathbb{N}(n_0) = \{n_0, n_0+1, n_0+2, \ldots\}$, n_0 is a positive integer and D_3 denotes the difference operator

$$D_3y(n) = \Delta(b(n)\Delta(a(n)(\Delta y(n))^{\alpha})). \tag{1.1}$$

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Throughout the paper, we assume that

(C_1) $\{b(n)\}$, $\{a(n)\}$ and $\{f(n)\}$ are positive real sequences for all $n \geqslant n_0$;

(C_2) $\{\sigma(n)\}$ is a sequence of integers with $\sigma(n) \leqslant n-1$ and $\sigma(n) \to \infty$ as $n \to \infty$;

(C_3) α and β are ratios of odd positive integers;

(C_4) the operator D_3 is in semi-noncanonical form, that is,

$$\Pi(n_0) = \sum_{n=n_0}^{\infty} \frac{1}{b(n)} < \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} \Big(\frac{1}{a(n)}\Big)^{1/\alpha} = \infty. \tag{1.2}$$

By a solution of (E), we mean a nontrivial real sequence $\{y(n)\}$ that satisfies (E) for all $n \in \mathbb{N}(n_0)$. We only consider those solutions of (E) which exist for all $n \in \mathbb{N}(n_0)$ and satisfy the condition

$$\sup\{|y(n)|\colon\ N \leqslant n < \infty\} > 0 \quad \text{for any } N \in \mathbb{N}(n_0).$$

A solution of (E) is called *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, it is said to be *nonoscillatory*.

Several results have been reported in the literature on the oscillatory and asymptotic properties of the solutions of (E), see for example [2]–[19], [21], and [24]. Most papers are devoted to canonical type equations, that is,

$$\sum_{n=n_0}^{\infty} \frac{1}{b(n)} = \sum_{n=n_0}^{\infty} \frac{1}{a^{1/\alpha}(n)} = \infty.$$

This is due to the fact that the canonical type equation is relatively simpler to study. In this paper, we connect the semi-noncanonical equation (E) to the canonical type.

While considering the nonoscillatory solutions of (E), we can restrict our attention to the positive ones since the proof for the negative case is similar. It follows from a well-known result in [1], that the set of positive solutions of (E) has the property described by the following lemma:

Lemma 1.1. *Assume that* $\{y(n)\}$ *is a positive solution of* (E). *Then* $\{y(n)\}$ *satisfies one of the following conditions:*

(I) $(\Delta y(n))^\alpha > 0$, $\Delta(a(n)(\Delta y(n))^\alpha) > 0$, $\Delta(b(n)\Delta(a(n)(\Delta y(n))^\alpha)) < 0$,

(II) $(\Delta y(n))^\alpha < 0$, $\Delta(a(n)(\Delta y(n))^\alpha) > 0$, $\Delta(b(n)\Delta(a(n)(\Delta y(n))^\alpha)) < 0$,

(III) $(\Delta y(n))^\alpha > 0$, $\Delta(a(n)(\Delta y(n))^\alpha) < 0$, $\Delta(b(n)\Delta(a(n)(\Delta y(n))^\alpha)) < 0$

for all $n \in \mathbb{N}(n_1)$, $n_1 \geqslant n_0$.

From the above lemma it is clear that if we want to establish oscillation criteria for a semi-noncanonical equation, we have to eliminate the above mentioned classes. To overcome this, we present a simple condition that leads to a canonical representation of (E), which essentially simplifies the study of (E).

2. Main results

Throughout the paper, we use the notation

$$\Pi(n)=\sum_{s=n}^{\infty}\frac{1}{b(s)},\quad d(n)=b(n)\Pi(n)\Pi(n+1),\quad c(n)=\frac{a(n)}{\Pi(n)},$$

$$F(n)=\Pi(n+1)f(n),\quad G(n)=F(n)\biggl(\sum_{s=n_1}^{\sigma(n)-1}\Bigl(\frac{1}{c(s)}\sum_{t=n_1}^{s-1}\frac{1}{d(t)}\Bigr)^{1/\alpha}\biggr)^{\alpha},$$

$$\mu(n)=\sum_{s=n_1}^{n-1}\frac{1}{d(s)}\quad\text{and}\quad\eta(n)=\sum_{s=n_1}^{n-1}\Bigl(\frac{\mu(\sigma(s))}{c(\sigma(s))}\Bigr)^{1/\alpha}\quad\text{for all } n_1\geqslant n_0.$$

Theorem 2.1. *Assume that*

$$\sum_{n=n_0}^{\infty}\Bigl(\frac{\Pi(n)}{a(n)}\Bigr)^{1/\alpha}=\infty. \tag{2.1}$$

Then the semi-noncanonical equation (E) *can be written in the canonical form as*

$$\frac{1}{\Pi(n+1)}\Delta\Bigl(b(n)\Pi(n)\Pi(n+1)\Delta\Bigl(\frac{a(n)}{\Pi(n)}(\Delta y(n))^{\alpha}\Bigr)\Bigr)+f(n)y^{\beta}(\sigma(n))=0. \tag{E$_1$}$$

Proof. By a straightforward computation we can show that

$$b(n)\Pi(n)\Pi(n+1)\Delta\Bigl(\frac{a(n)}{\Pi(n)}(\Delta y(n))^{\alpha}\Bigr)=\Pi(n)b(n)\Delta(a(n)(\Delta y(n))^{\alpha})+a(n)(\Delta y(n))^{\alpha}.$$

So

$$\Delta\Bigl(b(n)\Pi(n)\Pi(n+1)\Delta\Bigl(\frac{a(n)}{\Pi(n)}(\Delta y(n))^{\alpha}\Bigr)\Bigr)=\Pi(n+1)\Delta(b(n)\Delta(a(n)(\Delta y(n))^{\alpha}))$$

or

$$\frac{1}{\Pi(n+1)}\Delta\Bigl(b(n)\Pi(n)\Pi(n+1)\Delta\Bigl(\frac{a(n)}{\Pi(n)}(\Delta y(n))^{\alpha}\Bigr)\Bigr)=\Delta(b(n)\Delta(a(n)(\Delta y(n))^{\alpha})),$$

which shows that equations (E) and (E$_1$) are equivalent.

Now, we show that (E_1) is canonical, that is,

$$\sum_{n=n_0}^{\infty} \frac{1}{b(n)\Pi(n)\Pi(n+1)} = \sum_{n=n_0}^{\infty} \Delta\Big(\frac{1}{\Pi(n)}\Big) = \lim_{n\to\infty} \frac{1}{\Pi(n)} - \frac{1}{\Pi(n_0)} = \infty$$

and

$$\sum_{n=n_0}^{\infty} \Big(\frac{\Pi(n)}{a(n)}\Big)^{1/\alpha} = \infty$$

by (2.1). The proof is complete. □

Corollary 2.2. *Let* (2.1) *hold. If* $\{y(n)\}$ *is a positive solution of* (E), *then* $\{y(n)\}$ *is a positive solution of*

$$(E_2) \qquad \Delta(d(n)\Delta(c(n)(\Delta y(n))^{\alpha})) + F(n)y^{\beta}(\sigma(n)) = 0$$

and $\{y(n)\}$ *satisfies either*

$$(\Delta y(n))^{\alpha} < 0, \quad \Delta(c(n)(\Delta y(n))^{\alpha}) > 0, \quad \Delta(d(n)\Delta(c(n)(\Delta y(n))^{\alpha})) < 0,$$

in which case we say $y(n) \in S_0$, *or*

$$(\Delta y(n))^{\alpha} > 0, \quad \Delta(c(n)(\Delta y(n))^{\alpha}) > 0, \quad \Delta(d(n)\Delta(c(n)(\Delta y(n))^{\alpha})) < 0,$$

in which we say $y(n) \in S_2$.

Corollary 2.2 simplifies the form of possible positive solutions of (E). Therefore, to obtain oscillation criteria for (E) it suffices to eliminate the classes S_0 and S_2 instead of the three classes postulated by Lemma 1.1.

Theorem 2.3. *Let* (2.1) *hold. Assume that* $\{y(n)\}$ *is a positive solution of* (E). *If* $\alpha = \beta$ *and*

$$(2.2) \qquad \limsup_{n\to\infty} \sum_{s=\sigma(n)}^{n-1} \Big(\frac{1}{c(s)} \sum_{t=s}^{n-1} \frac{1}{d(t)} \sum_{j=t}^{n-1} F(j)\Big)^{1/\alpha} > 1,$$

then $\{y(n)\}$ *does not satisfy* (S_0).

P r o o f. Assume to the contrary that $y(n) \in S_0$. Summing up (E_2) from s to $n-1$ yields

$$d(s)\Delta(c(s)(\Delta y(s))^{\alpha}) \geqslant \sum_{t=s}^{n-1} F(t)y^{\beta}(\sigma(t)) \geqslant y^{\beta}(\sigma(n)) \sum_{t=s}^{n-1} F(t).$$

Summing it up twice from s to $n-1$, we obtain

$$y(s) \geqslant y^{\beta/\alpha}(\sigma(n)) \sum_{t=s}^{n-1} \left(\frac{1}{c(t)} \sum_{j=t}^{n-1} \frac{1}{d(j)} \sum_{i=j}^{n-1} F(i) \right)^{1/\alpha}.$$

By setting $s = \sigma(n)$ and $\alpha = \beta$ we reach a contradiction to (2.2). This completes the proof. □

Theorem 2.4. *Let* (2.1) *hold. Assume that* $\{y(n)\}$ *is a positive solution of* (E). *If* $\alpha = \beta$ *and*

$$\liminf_{n\to\infty} \sum_{s=\sigma(n)}^{n-1} G(s) > \frac{1}{\mathrm{e}}, \tag{2.3}$$

then $\{y(n)\}$ *does not satisfy* (S_2).

P r o o f. Assume to the contrary that $y(n) \in S_2$. Since $d(n)\Delta(c(n)(\Delta y(n))^{\alpha})$ is positive and decreasing, we can verify that

$$c(n)(\Delta y(n))^{\alpha} \geqslant \sum_{s=n_1}^{n-1} \frac{d(s)\Delta(c(s)(\Delta y(s))^{\alpha})}{d(s)} \geqslant d(n)\Delta(c(n)(\Delta y(n))^{\alpha}) \sum_{s=n_1}^{n-1} \frac{1}{d(s)}.$$

Summing it up again from n_1 to $n-1$, we get

$$y(n) \geqslant (d(n)\Delta(c(n)(\Delta y(n))^{\alpha}))^{1/\alpha} \sum_{s=n_1}^{n-1} \left(\frac{1}{c(s)} \sum_{t=n_1}^{s-1} \frac{1}{d(t)} \right)^{1/\alpha}.$$

Substituting this into (E_2), we see that $w(n) = d(n)\Delta(c(n)(\Delta y(n))^{\alpha})$ is a positive solution of the difference inequality

$$\Delta w(n) + F(n) \left(\sum_{s=n_1}^{\sigma(n)-1} \left(\frac{1}{c(s)} \sum_{t=n_1}^{s-1} \frac{1}{d(t)} \right)^{1/\alpha} \right)^{\alpha} w(\sigma(n)) \leqslant 0.$$

On the other hand, by Lemma 2.7 in [23], the corresponding equation

$$\Delta w(n) + G(n)w(\sigma(n)) = 0 \tag{2.4}$$

also has a positive solution. But by (2.3) and Theorem 2.1 in [20], equation (2.4) has no positive solution, which is a contradiction. This completes the proof. □

Combining Theorems 2.3 and 2.4, we immediately obtain the following theorem.

Theorem 2.5. *Let* $\alpha = \beta$ *and* (2.1), (2.2), (2.3) *hold. Then* (E) *is oscillatory.*

Next, we consider the case $\alpha > \beta$.

Theorem 2.6. *Let* $\alpha > \beta$ *and* (2.1) *hold. If*

$$\limsup_{n\to\infty} \sum_{s=\sigma(n)}^{n-1} \left(\frac{1}{c(s)} \sum_{t=s}^{n-1} \frac{1}{d(t)} \sum_{j=t}^{n-1} F(j)\right)^{1/\alpha} = \infty \tag{2.5}$$

and

$$\sum_{n=n_1}^{\infty} F(n) = \infty \tag{2.6}$$

for all $n_1 \in \mathbb{N}(n_0)$, *then equation* (E) *is oscillatory.*

P r o o f. Assume that $\{y(n)\}$ is a nonoscillatory solution of (E), say $y(n) > 0$ and $y(\sigma(n)) > 0$ for all $n \geqslant n_1 \in \mathbb{N}(n_0)$. From Corollary 2.2, $y(n) \in S_0$ or $y(n) \in S_2$ for all $n \geqslant n_1$.

First assume $y(n) \in S_0$. Then proceeding as in the proof of Theorem 2.3, we obtain

$$y(\sigma(n)) \geqslant y^{\beta/\alpha}(\sigma(n)) \sum_{t=\sigma(n)}^{n-1} \left(\frac{1}{c(t)} \sum_{j=t}^{n-1} \frac{1}{d(j)} \sum_{i=j}^{n-1} F(i)\right)^{1/\alpha},$$

or

$$y^{1-\beta/\alpha}(\sigma(n)) \geqslant \sum_{t=\sigma(n)}^{n-1} \left(\frac{1}{c(t)} \sum_{j=t}^{n-1} \frac{1}{d(j)} \sum_{i=j}^{n-1} F(i)\right)^{1/\alpha}.$$

Since $\{y(n)\}$ is decreasing and $\alpha > \beta$, we have that $y^{1-\beta/\alpha}(\sigma(n)) < M$ for all $n \geqslant n_2 \geqslant n_1$. Using this, we obtain

$$M \geqslant \sum_{t=\sigma(n)}^{n-1} \left(\frac{1}{c(t)} \sum_{j=t}^{n-1} \frac{1}{d(j)} \sum_{i=j}^{n-1} F(i)\right)^{1/\alpha},$$

which contradicts (2.5) as $n \to \infty$.

Next assume $y(n) \in S_2$. Then, since $y(n)$ is increasing, there exists a constant $M_1 > 0$ for all $n \geqslant n_1$. Then from (E_2) we obtain

$$\Delta(d(n)\Delta(c(n)(\Delta y(n))^{\alpha})) + M_1^{\beta} F(n) \leqslant 0.$$

Summing the above inequality from n_2 to $n-1$, we have

$$M_1^{\beta} \sum_{s=n_2}^{n-1} F(s) \leqslant d(n_2)\Delta(c(n_2)(\Delta y(n_2))^{\alpha}) < \infty,$$

which contradicts (2.6). The proof is complete. □

Lemma 2.7. *Assume that* (E_2) *possesses an eventually positive solution* $y(n)\in S_2$. *Then*

$$\frac{c(n)(\Delta y(n))^{\alpha}}{\mu(n)} \quad \textit{is eventually decreasing.}$$

P r o o f. Assume that $y(n)\in S_2$. Since $d(n)\Delta(c(n)(\Delta y(n))^{\alpha})$ is positive and decreasing, we have

$$c(n)(\Delta y(n))^{\alpha} \geqslant \sum_{s=n_1}^{n-1} d(s)\Delta(c(s)(\Delta y(s))^{\alpha})\frac{1}{d(s)} \geqslant d(n)\Delta(c(n)(\Delta y(n))^{\alpha})\mu(n),$$

which implies that

$$\Delta\Big(\frac{c(n)(\Delta y(n))^{\alpha}}{\mu(n)}\Big) \leqslant 0.$$

This proof is complete. □

Lemma 2.8. *Assume that* (E_2) *possesses an eventually positive solution* $y(n)\in S_2$. *Then*

$$y(\sigma(n)) \geqslant \eta(n)(d(n)\Delta(c(n)(\Delta y(n))^{\alpha}))^{1/\alpha}$$

for all $n\geqslant n_1\in\mathbb{N}(n_0)$.

P r o o f. The proof is similar to Lemma 2.2 in [22]. Thus, the details are omitted. □

Theorem 2.9. *Let* (2.1) *hold. Assume that* $\alpha=\beta$ *and both*

$$\sum_{n=n_1}^{\infty}\Big(\frac{1}{c(n)}\sum_{s=n}^{\infty}\frac{1}{d(s)}\sum_{t=s}^{\infty}F(t)\Big)^{1/\alpha}=\infty \tag{2.7}$$

and

$$\limsup_{n\to\infty}\Big\{\frac{1}{\mu(\sigma(n))}\sum_{s=n_1}^{\sigma(n)-1}\mu(s+1)F(s)\varrho^{\alpha}(\sigma(s)) + \sum_{s=\sigma(n)}^{n-1}F(s)\varrho^{\alpha}(\sigma(s))+\mu(\sigma(n))\sum_{s=n}^{\infty}F(s)\frac{\varrho^{\alpha}(\sigma(s))}{\mu(\sigma(s))}\Big\}>1. \tag{2.8}$$

Then every nonoscillatory solution $\{y(n)\}$ *of* (E) *satisfies* $\lim\limits_{n\to\infty}y(n)=0$.

P r o o f. Let $\{y(n)\}$ be an eventually positive solution of (E). Then by Corollary 2.2, $\{y(n)\}$ is also a positive solution of (E_2) and so either $y(n) \in S_0$ or $y(n) \in S_2$ for all $n \geqslant n_1 \in \mathbb{N}(n_0)$.

First assume that $y(n) \in S_2$. By Lemma 2.7, the function $c(n)(\Delta y(n))^\alpha/\mu(n)$ is decreasing and thus

$$y(n) \geqslant \sum_{s=n_1}^{n-1} \frac{c^{1/\alpha}(s)\Delta y(s)}{\mu^{1/\alpha}(s)} \frac{\mu^{1/\alpha}(s)}{c^{1/\alpha}(s)} \geqslant \frac{c^{1/\alpha}(n)\Delta y(n)}{\mu^{1/\alpha}(n)} \varrho(n),$$

where $\varrho(n) = \sum\limits_{s=n_1}^{n-1} (\mu(s)/c(s))^{1/\alpha}$. Substituting the last inequality into (E_2), we see that $x(n) = c(n)(\Delta y(n))^\alpha$ is a positive increasing solution of the difference inequality

$$\Delta(d(n)\Delta x(n)) + F(n)\frac{\varrho^\alpha(\sigma(n))}{\mu(\sigma(n))} x(\sigma(n)) \leqslant 0. \tag{2.9}$$

Moreover, the sequence $\{x(n)/\mu(n)\}$ is decreasing. Now, summing up (2.9) from n to ∞ yields

$$\Delta x(n) \geqslant \frac{1}{d(n)} \sum_{s=n}^{\infty} F(s)\frac{\varrho^\alpha(\sigma(s))}{\mu(\sigma(s))} x(\sigma(s)).$$

Summing it up again from n_1 to $n-1$ and then changing the order of the summations, we obtain

$$\begin{aligned} x(n) &\geqslant \sum_{s=n_1}^{n-1} \frac{1}{d(s)} \sum_{t=s}^{\infty} \frac{F(t)\varrho^\alpha(\sigma(t))}{\mu(\sigma(t))} x(\sigma(t)) \\ &\geqslant \sum_{s=n_1}^{n-1} \frac{1}{d(s)} \sum_{t=s}^{n-1} \frac{F(t)\varrho^\alpha(\sigma(t))}{\mu(\sigma(t))} x(\sigma(t)) + \sum_{s=n_1}^{n-1} \frac{1}{d(s)} \sum_{t=n}^{\infty} \frac{F(t)\varrho^\alpha(\sigma(t))}{\mu(\sigma(t))} x(\sigma(t)) \\ &= \sum_{s=n_1}^{n-1} \mu(s+1) \frac{F(s)\varrho^\alpha(\sigma(s))}{\mu(\sigma(s))} x(\sigma(s)) + \mu(n) \sum_{t=n}^{\infty} \frac{F(t)\varrho^\alpha(\sigma(t))}{\mu(\sigma(t))} x(\sigma(t)). \end{aligned}$$

Consequently,

$$\begin{aligned} x(\sigma(n)) \geqslant \sum_{s=n_1}^{\sigma(n)-1} \mu(s+1)F(s)\varrho^\alpha(\sigma(s)) \frac{x(\sigma(s))}{\mu(\sigma(s))} + \mu(\sigma(n)) \sum_{t=\sigma(n)}^{n-1} \frac{F(t)\varrho^\alpha(\sigma(t))}{\mu(\sigma(t))} x(\sigma(t)) \\ + \mu(\sigma(n)) \sum_{t=n}^{\infty} \frac{F(t)\varrho^\alpha(\sigma(t))}{\mu(\sigma(t))} x(\sigma(t)). \end{aligned}$$

Since $\{x(n)\}$ and $\{x(n)/\mu(n)\}$ are monotonically decreasing, we have that

$$x(\sigma(n)) \geqslant \frac{x(\sigma(n))}{\mu(\sigma(n))} \sum_{s=n_1}^{\sigma(n)-1} \mu(s+1)F(s)\varrho^{\alpha}(\sigma(s)) + x(\sigma(n)) \sum_{t=\sigma(n)}^{n-1} F(t)\varrho^{\alpha}(\sigma(t)) + \mu(\sigma(n))x(\sigma(n)) \sum_{t=n}^{\infty} \frac{F(t)\varrho^{\alpha}(\sigma(t))}{\mu(\sigma(t))}.$$

That is,

$$1 \geqslant \frac{1}{\mu(\sigma(n))} \sum_{s=n_1}^{\sigma(n)-1} \mu(s+1)F(s)\varrho^{\alpha}(\sigma(s)) + \sum_{s=\sigma(n)}^{n-1} F(s)\varrho^{\alpha}(\sigma(s)) + \mu(\sigma(n)) \sum_{s=n}^{\infty} \frac{F(s)\varrho^{\alpha}(\sigma(s))}{\mu(\sigma(s))},$$

which is a contradiction and we conclude that $y(n) \notin S_2$.

Next, we assume that $y(n) \in S_0$. It follows from the monotonicity of $y(n)$ that there exists $\lim_{n\to\infty} y(n) = l \geqslant 0$. We claim that $l = 0$. If not, then $y(n) \geqslant l > 0$. Summing up (E_2) from n to ∞ yields

$$d(n)\Delta(c(n)(\Delta y(n))^{\alpha}) \geqslant \sum_{s=n}^{\infty} F(s)y^{\alpha}(\sigma(s)) \geqslant l^{\alpha} \sum_{s=n}^{\infty} F(s).$$

Summing it up once more from n to ∞, we obtain

$$-c(n)(\Delta y(n))^{\alpha} \geqslant l^{\alpha} \sum_{s=n}^{\infty} \frac{1}{d(s)} \sum_{t=s}^{\infty} F(t), \quad \text{or} \quad -\Delta y(n) \geqslant l \left(\frac{1}{c(n)} \sum_{s=n}^{\infty} \frac{1}{d(s)} \sum_{t=s}^{\infty} F(t) \right)^{1/\alpha}.$$

Now, by summing this up from n_1 to ∞, one can see that

$$y(n_1) \geqslant l \sum_{n=n_1}^{\infty} \left(\frac{1}{c(n)} \sum_{s=n}^{\infty} \frac{1}{d(s)} \sum_{t=s}^{\infty} F(t) \right)^{1/\alpha},$$

which contradicts (2.7). Thus, we conclude that $\lim_{n\to\infty} y(n) = 0$. □

Our final result concerns the case when

$$\sum_{n=n_0}^{\infty} F(n) < \infty. \tag{2.10}$$

Theorem 2.10. *Let* (2.1), $\beta = \alpha$ *and* (2.2) *hold. If* (2.10) *and*

$$\lim_{n\to\infty} \sup \eta^{\alpha}(n) \sum_{s=n}^{\infty} F(s) > 1, \tag{2.11}$$

then equation (E) *is oscillatory.*

Proof. Let $\{y(n)\}$ be an eventually positive solution of (E). Then by Corollary 2.2, $\{y(n)\}$ is also a positive solution of (E_2) and so either $y(n) \in S_0$ or $y(n) \in S_2$ for all $n \geqslant n_1 \in \mathbb{N}(n_0)$.

First assume that $y(n) \in S_2$. Define

$$w(n) = \frac{d(n)\Delta(c(n)(\Delta y(n))^\alpha)}{y^\alpha(\sigma(n))}, \quad n \geqslant n_1.$$

Then $w(n) > 0$. Using (E_2), we have

$$\Delta w(n) = -F(n) - \frac{d(n+1)\Delta(c(n+1)(\Delta y(n+1))^\alpha)}{y^\alpha(\sigma(n))y^\alpha(\sigma(n+1))}\Delta y^\alpha(\sigma(n)) \leqslant -F(n).$$

Summing the last inequality from n to ∞, we obtain

$$\sum_{s=n}^{\infty} F(s) \leqslant \frac{d(n)\Delta(c(n)(\Delta y(n))^\alpha)}{y^\alpha(\sigma(n))}.$$

Using Lemma 2.8 in the above inequality, we get

$$\eta^\alpha(n)\sum_{s=n}^{\infty} F(s) \leqslant 1,$$

which contradicts (2.11) as $n \to \infty$. The proof for the case S_0 is similar to that of Theorem 2.3. This completes the proof. □

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. Consider the semi-noncanonical third-order delay difference equation

$$\Delta\Big(n(n+1)\Delta\Big(\frac{1}{n}(\Delta y(n))^{1/3}\Big)\Big) + 2^{7/3}(n+1)y^{1/3}(n-2) = 0, \quad n \geqslant 1. \tag{3.1}$$

Here $b(n) = n(n+1)$, $a(n) = 1/n$, $\alpha = \beta = \frac{1}{3}$, $f(n) = 2^{7/3}(n+1)$, $\sigma(n) = n-2$. Clearly (3.1) is semi-noncanonical. A straightforward calculation shows that $\Pi(n) = 1/n$, $d(n) = 1$, $c(n) = 1$, $F(n) = 2^{7/3}$. The transformed canonical equation is

$$\Delta^2((\Delta y(n))^{1/3}) + 2^{7/3}y^{1/3}(n-2) = 0, \quad n \geqslant 1. \tag{3.2}$$

It is easy to see that condition (2.1) is satisfied. Condition (2.2) becomes

$$\limsup_{n\to\infty}\sum_{s=n-2}^{n-1}\left(\sum_{t=s}^{n-1}\sum_{j=t}^{n-1}2^{7/3}\right)^3=3584>1,$$

while condition (2.3) becomes

$$\liminf_{n\to\infty}\sum_{s=n-2}^{n-1}2^{7/3}\left(\sum_{t=1}^{s-3}\left(\sum_{j=1}^{t-1}1\right)^3\right)^{1/3}=2^{5/3}\liminf_{n\to\infty}(n-5)^{2/3}((n-6)^{2/3}+(n-4)^{2/3})$$
$$=\infty.$$

Hence, all the conditions of Theorem 2.5 are satisfied and therefore equation (3.1) is oscillatory. Using (3.2) one can easily verify that $y(n)=\{(-1)^n\}$ is one such oscillatory solution of (3.1).

Example 3.2. Consider the semi-noncanonical third-order delay difference equation

$$\Delta\Big(2^{n+1}\Delta\Big(\frac{1}{2^n}(\Delta y(n))\Big)\Big)+\lambda y(n-2)=0,\quad n\geqslant 1. \tag{3.3}$$

Here $b(n)=2^{n+1}$, $a(n)=1/2^n$, $\alpha=\beta=1$, $f(n)=\lambda>0$, $\sigma(n)=n-2$. Clearly (3.3) is semi-noncanonical. A straightforward calculation shows that $\Pi(n)=1/2^n$, $d(n)=1/2^n$, $c(n)=1$, $F(n)=\lambda/2^{n+1}$, $\eta(n)=2^{n-2}-n+\frac{3}{4}$.

It is easy to see that condition (2.1) is satisfied. Condition (2.2) becomes

$$\limsup_{n\to\infty}\sum_{s=n-2}^{n-1}\left(\sum_{t=s}^{n-1}2^t\sum_{j=t}^{n-1}\frac{\lambda}{2^{j+1}}\right)=\frac{9\lambda}{4}>1,$$

while condition (2.11) becomes

$$\lim_{n\to\infty}\Big(2^{n-2}-n+\frac{3}{4}\Big)\sum_{s=n}^{\infty}\frac{\lambda}{2^{s+1}}=\frac{\lambda}{4}>1.$$

Hence, Theorem 2.10 is satisfied if $\lambda>4$, and so equation (3.3) is oscillatory.

4. Conclusion

In this paper, we have obtained new oscillation criteria for equation (E) by transforming (E) into canonical form. By this technique, we get a new kind of oscillation criteria for equation (E). We have shown the application of the results through some illustrative examples.

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