ON THE INCLUSIONS OF X^Φ SPACES

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Abstract. We give some equivalent conditions (independent from the Young functions) for inclusions between some classes of X^Φ spaces, where Φ is a Young function and X is a quasi-Banach function space on a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$.

Keywords: Young function; Orlicz space; quasi-Banach function space; inclusion

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1. Introduction

In [4] an improvement of the following interesting result was given for generalized Orlicz spaces.

Theorem 1.1 ([6]). *Let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space and* $1 \leqslant p, q \leqslant \infty$ *such that* $p < q$. *Then*

(i) $L^p(\mu) \subset L^q(\mu)$ *if and only if* $\inf\{\mu(A)\colon A \in \mathcal{A},\ \mu(A) > 0\} > 0$;
(ii) $L^q(\mu) \subset L^p(\mu)$ *if and only if* $\sup\{\mu(A)\colon A \in \mathcal{A},\ \mu(A) < \infty\} < \infty$.

See also [5], [3]. In this paper, by some methods similar to [4] and with different details, we give a new version of the above theorem for Orlicz spaces X^Φ which are associated to a quasi-Banach function space X. The obtained results are novel for Lebesgue spaces associated to a Banach function space and for weighted Orlicz spaces too. These new structures which contain usual (weighted) Orlicz spaces were recently studied in [1]. In fact, $(L^1)^\Phi = L^\Phi$, where Φ is a Young function.

Throughout this paper, $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space in which μ is a non-negative measure, and the set of all $\mathcal{A}$-measurable complex-valued functions on Ω is denoted by $\mathcal{M}_0(\Omega)$. Two functions in $\mathcal{M}_0(\Omega)$ which are equal almost everywhere are considered the same.

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Definition 1.2. A continuous convex function $\Phi\colon [0,\infty) \to [0,\infty)$ is called a *Young function* if $\Phi(0) = \lim_{x\to 0} \Phi(x) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$. We denote the set of all strictly increasing Young functions by $\mathit{\Phi}$.

Definition 1.3. Let X be a linear subspace of $\mathcal{M}_0(\Omega)$. If X equipped with a given quasi-norm $\|\cdot\|_X$ is a quasi-Banach space, we say that X is a *quasi-Banach function space on* Ω. In this situation, X is called *solid* if for each $f \in X$ and $g \in \mathcal{M}_0(\Omega)$ satisfying $|g| \leqslant |f|$ a.e. we have $g \in X$ and $\|g\|_X \leqslant \|f\|_X$.

Definition 1.4. Let X be a quasi-Banach function space on Ω. For each function $f \in \mathcal{M}_0(\Omega)$ we put

$$\|f\|_\Phi := \inf\Big\{\lambda > 0\colon\ \Phi\Big(\frac{|f|}{\lambda}\Big) \in X,\ \Big\|\Phi\Big(\frac{|f|}{\lambda}\Big)\Big\|_X \leqslant 1\Big\}. \tag{1.1}$$

Then, the set of all $f \in \mathcal{M}_0(\Omega)$ with $\|f\|_\Phi < \infty$ is denoted by X^Φ.

As in [1], Theorem 4.11, $(X^\Phi, \|\cdot\|_\Phi)$ is a quasi-Banach function space on Ω. If $p > 0$ and the function $\Phi_{(p)}$ is defined by $\Phi_{(p)}(x) := x^p$ for all $x \geqslant 0$, then we denote $X^p := X^{\Phi_{(p)}}$. In particular, if $X := L^1(\Omega, \mathcal{A}, \mu)$, then $X^\Phi = L^\Phi(\Omega)$ and $X^p = L^p(\Omega)$, the usual Orlicz and Lebesgue spaces.

Notation. *For each Young function* Φ *and* $a > 0$ *we denote*

$$\Phi_a(t) := \Phi(t^{1/a}), \quad t \in [0,\infty).$$

In general, Φ_a *is not a convex function even while* $\Phi \in \mathit{\Phi}$. *For each* $\Phi \in \mathit{\Phi}$ *we set*

$$D_\Phi := \{a \in (0,1)\colon\ \Phi_{1/a} \in \mathit{\Phi}\}.$$

Remark 1.5.

(1) Let $\Phi \in \mathit{\Phi}$ and $0 < a < \infty$ with $\Phi_a \in \mathit{\Phi}$. Then for each $f \in \mathcal{M}_0(\Omega)$ we have

$$\begin{aligned}
\|f\|_{\Phi_a} &= \inf\Big\{\lambda > 0\colon\ \Phi_a\Big(\frac{|f|}{\lambda}\Big) \in X \text{ and } \Big\|\Phi_a\Big(\frac{|f|}{\lambda}\Big)\Big\|_X \leqslant 1\Big\}\\
&= \inf\Big\{\lambda > 0\colon\ \Phi\Big(\frac{|f|^{1/a}}{\lambda^{1/a}}\Big) \in X \text{ and } \Big\|\Phi\Big(\frac{|f|^{1/a}}{\lambda^{1/a}}\Big)\Big\|_X \leqslant 1\Big\}\\
&= \inf\Big\{t^a\colon\ t > 0,\ \Phi\Big(\frac{|f|^{1/a}}{t}\Big) \in X \text{ and } \Big\|\Phi\Big(\frac{|f|^{1/a}}{t}\Big)\Big\|_X \leqslant 1\Big\}\\
&= \Big(\inf\Big\{t\colon\ t > 0,\ \Phi\Big(\frac{|f|^{1/a}}{t}\Big) \in X \text{ and } \Big\|\Phi\Big(\frac{|f|^{1/a}}{t}\Big)\Big\|_X \leqslant 1\Big\}\Big)^a\\
&= (\||f|^{1/a}\|_\Phi)^a.
\end{aligned}$$

(2) For each $\Phi \in \mathit{\Phi}$ and $a \in (0,1)$ we have $X^\Phi \cap L^\infty(\Omega) \subseteq X^{\Phi_a}$. Indeed, if $f \in X^\Phi \cap L^\infty(\Omega)$, then for some $\lambda > 1$ we have $\Phi(|f|/\lambda) \in X$ and $|f| \leqslant \lambda$ a.e. This implies that

$$\Phi_a\Bigl(\frac{|f|}{\lambda}\Bigr) = \Phi\Bigl(\frac{|f|^{1/a}}{\lambda^{1/a}}\Bigr) \leqslant \Phi\Bigl(\frac{|f|}{\lambda}\Bigr) \in X,$$

and so by solidity of X, $\Phi_a(|f|/\lambda) \in X$, i.e., $f \in X^{\Phi_a}$.

(3) Let $\Phi \in \mathit{\Phi}$. If X is a solid quasi-Banach function space on Ω, then X^Φ is also a solid space. Indeed, if $f, g \in \mathcal{M}_0(\Omega)$, $|f| \leqslant |g|$ a.e. and $g \in X^\Phi$, then there exists $\lambda > 0$ such that $\Phi(|g|/\lambda) \in X$. Now, since Φ is an increasing function, we have

$$\Phi\Bigl(\frac{|f|}{\lambda}\Bigr) \leqslant \Phi\Bigl(\frac{|g|}{\lambda}\Bigr),$$

and this implies that $\Phi(|f|/\lambda) \in X$ because X is solid, and the proof is complete.

In this paper, Φ is always a Young function, and X is a *solid* quasi-Banach function space on Ω such that for each $A \in \mathcal{A}$ with $\mu(A) < \infty$, $\chi_A \in X$.

2. Main results

Denote

$$\mathcal{A}_0 := \{E \in \mathcal{A}\colon\ 0 < \mu(E) \text{ and } \chi_E \in X\}.$$

Trivially, for each $E \in \mathcal{A}$ with $\chi_E \in X$, we have $\|\chi_E\|_X = 0$ if and only if $\mu(E) = 0$.

The following result would be an improvement of [4], Theorem 2.4 and [6], Theorem 1, and it is novel for Lebesgue spaces associated to the space X.

Theorem 2.1. *The following conditions are equivalent.*

(i) *For* $0 < p, q < \infty$ *with* $p < q$, $X^p \subset X^q$.

(ii) *For each* $0 < p, q < \infty$ *with* $p < q$, $X^p \subset X^q$.

(iii) *For* $\Phi \in \mathit{\Phi}$, $X^\Phi \subset L^\infty(\mu)$.

(iv) *For each* $\Phi \in \mathit{\Phi}$, $X^\Phi \subset L^\infty(\mu)$.

(v) *For* $\Phi \in \mathit{\Phi}$ *and* $a \in (0,1)$, $X^\Phi \subset X^{\Phi_a}$.

(vi) *For each* $\Phi \in \mathit{\Phi}$ *and* $a \in (0,1)$, $X^\Phi \subset X^{\Phi_a}$.

(vii) $\inf\{\|\chi_E\|_X\colon\ E \in \mathcal{A}_0\} > 0$.

Proof. It would be enough to prove (iii) $\Rightarrow$ (vii) $\Rightarrow$ (iv) and (v) $\Rightarrow$ (vii) $\Rightarrow$ (vi).

(iii) $\Rightarrow$ (vii): By [4], Lemma 2.3, there exists $K > 0$ such that for all $f \in X^\Phi$,

$$\|f\|_\infty \leqslant K\|f\|_\Phi. \tag{2.1}$$

We can assume that K is large enough, and hence without losing the generality we let $\Phi(2K) > 0$ since $\lim\limits_{x\to\infty} \Phi(x) = \infty$. By (2.1), for each $E \in \mathcal{A}_0$ with $\mu(E) < \infty$ we have $1/(2K) < \|\chi_E\|_\Phi$ because $\chi_E \in X^\Phi$. On the other hand, for each $\lambda > 0$ we have

$$\Phi\Big(\frac{\chi_E}{\lambda}\Big) = \Phi\Big(\frac{1}{\lambda}\Big)\chi_E,$$

and so

$$\|\chi_E\|_\Phi = \inf\Big\{\lambda > 0\colon\ \Phi\Big(\frac{1}{\lambda}\Big)\|\chi_E\|_X \leqslant 1\Big\}.$$

Therefore, $\Phi(2K)\|\chi_E\|_X > 1$ and the proof is complete.

(vii) $\Rightarrow$ (iv): Let $\Phi \in \mathit{\Phi}$ and $f \in X^\Phi$. For each $N \in \mathbb{N}$ put

$$A_N := \{x \in \Omega\colon\ |f(x)| > N\}.$$

Then $N\chi_{A_N} \leqslant |f|$ and so by solidity of X^Φ (see Remark 1.5) we have $N\|\chi_{A_N}\|_\Phi \leqslant \|f\|_\Phi$ for all $N \in \mathbb{N}$. Now, the assumption $\inf\{\|\chi_E\|_X\colon\ E \in \mathcal{A}_0\} > 0$ implies that for some $N \in \mathbb{N}$, $\|\chi_{A_N}\|_\Phi = 0$, i.e., $\mu(A_N) = 0$, and this implies that $f \in L^\infty(\Omega)$.

(v) $\Rightarrow$ (vii): By Remark 1.5 and [4], Lemma 2.3, there exists a constant $k > 0$ such that

$$\||f|^{1/a}\|_\Phi^a = \|f\|_{\Phi_a} \leqslant k\|f\|_\Phi \tag{2.2}$$

for all $f \in X^\Phi$. Let $E \in \mathcal{A}_0$. Then $\chi_E \neq 0$ in X. By (2.2), $0 < k^{1/(a-1)} \leqslant \|\chi_E\|_\Phi$. Now, setting $l^{-1} := \frac{1}{2}k^{1/(a-1)}$ we have

$$\|\chi_E\|_\Phi = \inf\Big\{\lambda > 0\colon\ \Phi\Big(\frac{\chi_E}{\lambda}\Big) \in X,\ \Big\|\Phi\Big(\frac{\chi_E}{\lambda}\Big)\Big\|_X \leqslant 1\Big\} \geqslant k^{1/(a-1)} > \frac{1}{l} > 0.$$

This implies that $\Phi(l)\|\chi_E\|_X > 1$ and therefore

$$\inf\{\|\chi_E\|_X\colon\ E \in \mathcal{A}_0\} > \frac{1}{\Phi(l)} > 0.$$

(vii) $\Rightarrow$(vi): Let $\inf\{\|\chi_E\|_X\colon\ E \in \mathcal{A}_0\} > 0$. Let $\Phi \in \mathit{\Phi}$ and $a \in (0,1)$. Then by the implication (vii) $\Rightarrow$ (iv) above we have $X^\Phi \subseteq L^\infty(\Omega)$. Now, by Remark 1.5,

$$X^\Phi = X^\Phi \cap L^\infty(\Omega) \subseteq X^{\Phi_a}.$$

□

R e m a r k 2.2. The condition $\Phi \in \mathit{\Phi}$ implies that "$\Phi(x) > 0$ for all $x > 0$" and this fact is used just in the proof of (v) $\Rightarrow$ (vii) in the above theorem.

Denote $\mathcal{A}_\infty := \{E \in \mathcal{A}\colon\ \chi_E \in X\}$. We say that X satisfies the MC (Monotone Convergence) property if for each increasing sequence $\{E_n\}_{n=1}^\infty \subseteq \mathcal{A}$ with χ_{E_n}, $\chi_E \in X$, $n = 1, 2, \ldots$, we have $\|\chi_{E_n}\|_X \to \|\chi_E\|_X$, where $E := \bigcup\limits_{n=1}^\infty E_n$.

The next lemma, which is similar to [1], Lemma 4.8 (i) with some minor changes, will be useful in the proof of part (vii) $\Rightarrow$ (v) of Theorem 2.4.

Lemma 2.3. *If $\Phi \in \mathit{\Phi}$, $A \in \mathcal{A}$ and $0 \neq \chi_A \in X^{\Phi}$, then we have*

$$\|\chi_A\|_{\Phi} = \frac{1}{\Phi^{-1}(\|\chi_A\|_X^{-1})}. \tag{2.3}$$

P r o o f. Let $A \in \mathcal{A}$ and $\chi_A \in X^{\Phi}$. Then by Definition 1.4 there exists some $\lambda_0 > 0$ such that

$$\Phi\Big(\frac{1}{\lambda_0}\Big)\chi_A = \Phi\Big(\frac{\chi_A}{\lambda_0}\Big) \in X,$$

and so $\chi_A \in X$ (note that $\Phi(1/\lambda_0) > 0$ since Φ is strictly increasing). Now,

$$\begin{aligned}
\|\chi_A\|_{\Phi} &= \inf\Big\{\lambda > 0\colon\ \Big\|\Phi\Big(\frac{\chi_A}{\lambda}\Big)\Big\|_X \leqslant 1\Big\} = \inf\Big\{\lambda > 0\colon\ \Phi\Big(\frac{1}{\lambda}\Big)\|\chi_A\|_X \leqslant 1\Big\} \\
&= \inf\Big\{\lambda > 0\colon\ \Phi\Big(\frac{1}{\lambda}\Big) \leqslant \frac{1}{\|\chi_A\|_X}\Big\} = \inf\Big\{\lambda > 0\colon\ \frac{1}{\lambda} \leqslant \Phi^{-1}\Big(\frac{1}{\|\chi_A\|_X}\Big)\Big\} \\
&= \inf\Big\{\lambda > 0\colon\ \lambda \geqslant \frac{1}{\Phi^{-1}(\|\chi_A\|_X^{-1})}\Big\},
\end{aligned}$$

and this completes the proof. □

The following result is an improvement of [4], Theorem 2.7; [4], Theorem 2.8 and [6], Theorem 2.

For each $f \in X^{\Phi}$ we denote $E_f := \{x \in \Omega\colon\ 0 < |f(x)|\}$.

Theorem 2.4. *Let X be a solid quasi-Banach function space satisfying the MC property. Then the following conditions are equivalent.*

(i) *For $0 < p, q < \infty$ with $p < q$, $X^q \subset X^p$.*
(ii) *For each $0 < p, q < \infty$ with $p < q$, $X^q \subset X^p$.*
(iii) *For $\Phi \in \mathit{\Phi}$, $\chi_{E_f} \in X$ for all $f \in X^{\Phi}$.*
(iv) *For each $\Phi \in \mathit{\Phi}$, $\chi_{E_f} \in X$ for all $f \in X^{\Phi}$.*
(v) *For $\Phi \in \mathit{\Phi}$, $\chi_{E_f} \in X$ for all $f \in X^{\Phi}$, and $\sup\limits_{f\in X^{\Phi}} \|\chi_{E_f}\|_X < \infty$.*
(vi) *For each $\Phi \in \mathit{\Phi}$, $\chi_{E_f} \in X$ for all $f \in X^{\Phi}$, and $\sup\limits_{f\in X^{\Phi}} \|\chi_{E_f}\|_X < \infty$.*
(vii) *For $\Phi \in \mathit{\Phi}$ and $a \in D_{\Phi}$, $X^{\Phi} \subset X^{\Phi_{1/a}}$.*
(viii) *For each $\Phi \in \mathit{\Phi}$ and $a \in D_{\Phi}$, $X^{\Phi} \subset X^{\Phi_{1/a}}$.*
(ix) $\sup\{\|\chi_E\|_X\colon\ E \in \mathcal{A}_{\infty}\} < \infty$.

P r o o f. We prove the nontrivial implications.

(v) $\Rightarrow$ (ix): Let $\Phi \in \mathit{\Phi}$ and $\sup_{f\in X^\Phi} \|\chi_{E_f}\|_X < \infty$. If $E \in \mathcal{A}$ and $\chi_E \in X^\Phi$, then

$$\|\chi_E\|_X \leqslant \sup_{f\in X^\Phi} \|\chi_{E_f}\|_X < \infty,$$

and so (ix) holds.

(ix) $\Rightarrow$ (vi): Let $\sup\{\|\chi_E\|_X : E \in \mathcal{A}_\infty\} < \infty$, and $\Phi \in \mathit{\Phi}$. Since X^Φ is solid (see Remark 1.5), for each $f \in X^\Phi \setminus \{0\}$ and $N \in \mathbb{N}$ we have $\chi_{A_{N,f}} \in X^\Phi$ and

$$\frac{1}{N}\|\chi_{A_{N,f}}\|_\Phi \leqslant \|f\|_\Phi,$$

where $A_{N,f} := \{x \in \Omega : 1/N < |f(x)|\}$. So, for some $\lambda > 0$,

$$\Phi\Big(\frac{1}{\lambda}\Big)\chi_{A_{N,f}} = \Phi\Big(\frac{\chi_{A_{N,f}}}{\lambda}\Big) \in X,$$

which shows that $\chi_{A_{N,f}} \in X$ because $\Phi(1/\lambda) \neq 0$. Hence, by assumption (ix), for each $N \in \mathbb{N}$ we have

$$\|\chi_{A_{N,f}}\|_X \leqslant K,$$

where $K := \sup\{\|\chi_E\|_X : E \in \mathcal{A}_\infty\} < \infty$. Finally, since X satisfies the MC property, we have

$$\|\chi_{E_f}\|_X = \lim_{N\to\infty} \|\chi_{A_{N,f}}\|_X \leqslant K,$$

and this completes the proof.

(vii) $\Rightarrow$ (v): Let $\Phi \in \mathit{\Phi}$ and $a \in D_\Phi$ such that $X^\Phi \subset X^{\Phi_{1/a}}$. By [4], Lemma 2.3 and Remark 1.5 there exists $K > 0$ such that for each $f \in X^\Phi$,

$$\||f|^a\|_\Phi^{1/a} = \|f\|_{\Phi_{1/a}} \leqslant K\|f\|_\Phi. \tag{2.4}$$

For each $0 \neq f \in X^\Phi$ we have $\chi_{\{x:\ N^{-1}<|f(x)|<N\}} \leqslant |Nf|$, and so

$$\chi_{\{x:\ N^{-1}<|f(x)|<N\}} \in X^\Phi$$

for all $N \in \mathbb{N}$.

Therefore, by the assumption we have $\chi_{\{x:\ N^{-1}<|f(x)|<N\}} \in X^{\Phi_{1/a}}$ for all $N \in \mathbb{N}$. By relation (2.4) and Lemma 2.3,

$$\begin{aligned}\frac{1}{\Phi^{-1}(\|\chi_{E_f}\|_X^{-1})} &= \lim_{N\to\infty} \frac{1}{\Phi^{-1}(\|\chi_{\{N^{-1}<|f|<N\}}\|_X^{-1})} \\ &= \lim_{N\to\infty} \|\chi_{\{N^{-1}<|f|<N\}}\|_\Phi \leqslant K^{a/(1-a)}.\end{aligned}$$

Hence,

$$\|\chi_{E_f}\|_X \leqslant \frac{1}{\Phi\big(K^{a/(a-1)}\big)},$$

and this completes the proof.

(iv) $\Rightarrow$ (viii): Let $\Phi \in \varPhi$ and $a \in D_\Phi$. By assumption (iv), for each $f \in X^\Phi$ we have $\chi_{E_f} \in X$. Let $f \in X^\Phi$. Then there is $\lambda > 0$ such that $\Phi(|f|/\lambda) \in X$. Note that

$$\Phi_{1/a}\Big(\frac{|f|}{\lambda^{1/a}}\Big) = \Phi\Big(\frac{|f|^a}{\lambda}\Big) = \Phi\Big(\frac{|f|^a}{\lambda}\Big)\,\chi_{\{|f|\leqslant 1\}} + \Phi\Big(\frac{|f|^a}{\lambda}\Big)\,\chi_{\{|f|>1\}}.$$

We have

$$\Phi\Big(\frac{|f|^a}{\lambda}\Big)\,\chi_{\{|f|>1\}} \leqslant \Phi\Big(\frac{|f|}{\lambda}\Big) \in X \quad \text{and} \quad \Phi\Big(\frac{|f|^a}{\lambda}\Big)\,\chi_{\{|f|\leqslant 1\}} \leqslant \Phi\Big(\frac{1}{\lambda}\Big)\,\chi_{E_f} \in X.$$

Thus, $f \in X^{\Phi_{1/a}}$. □

In the sequel, we intend to give a new version of [2], Theorem 3, page 155 for X^Φ spaces, where X is a Banach function space on a measure space $(\Omega, \mathcal{A}, \mu)$ and $\Phi \in \varPhi$. For this, we give the next definition from [2], page 15.

Definition 2.5. Let Φ_1 and Φ_2 be two Young functions. We say that Φ_2 is *stronger* than Φ_1, and write $\Phi_1 \prec \Phi_2$ if there exist $a > 0$ and $x_0 \geqslant 0$ such that $\Phi_1(x) \leqslant \Phi_2(ax)$ for all $x \geqslant x_0$. While $x_0 = 0$, we say that Φ_2 is *stronger (globally)* than Φ_1.

Theorem 2.6. *Suppose that Φ_1 and Φ_1 are two Young functions, and for each $A \in \mathcal{A}$ with $\mu(A) < \infty$, $\chi_A \in X$. If $\Phi_1 \prec \Phi_2$ (globally if $\mu(\Omega) = \infty$), then $X^{\Phi_2} \subseteq X^{\Phi_1}$.*

P r o o f. Let $\Phi_1 \prec \Phi_2$ and $f \in X^{\Phi_2}$. Then there exists $\lambda > 0$ such that $\Phi_2(|f|/\lambda) \in X$. In the case $\mu(\Omega) = \infty$ and $\Phi_1 \prec \Phi_2$ (globally), for some $b > 0$ we have $\Phi_1(|f|/(b\lambda)) \leqslant \Phi_2(|f|/\lambda) \in X$. Hence, $\Phi_1(f/(b\lambda)) \in X$ by solidity of X, and so $f \in X^{\Phi_1}$. In the case $\Phi_1 \prec \Phi_2$ (not necessarily globally) and $\mu(\Omega) < \infty$, there exist real numbers $b > 0$ and $x_0 \geqslant 0$ such that $\Phi_1(x) \leqslant \Phi_2(bx)$ for all $x \geqslant x_0$. Setting $B := \{x \in \Omega\colon\ f(x) < x_0\}$ we have

$$\begin{aligned}\Phi_1\Big(\frac{f}{\lambda}\Big) &= \Phi_1\Big(\frac{f\chi_B}{b\lambda}\Big) + \Phi_1\Big(\frac{f\chi_{\Omega-B}}{b\lambda}\Big)\\ &\leqslant \Phi_1\Big(\frac{x_0}{b\lambda}\Big)\chi_B + \Phi_2\Big(\frac{f\chi_{\Omega-B}}{\lambda}\Big)\\ &\leqslant \Phi_1\Big(\frac{x_0}{b\lambda}\Big)\chi_\Omega + \Phi_2\Big(\frac{f}{\lambda}\Big) \in X,\end{aligned}$$

and this completes the proof. □

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