MEET-DISTRIBUTIVE LATTICES HAVE THE INTERSECTION PROPERTY

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Abstract. This paper is an erratum of H. Mühle: Distributive lattices have the intersection property, Math. Bohem. (2021). Meet-distributive lattices form an intriguing class of lattices, because they are precisely the lattices obtainable from a closure operator with the so-called anti-exchange property. Moreover, meet-distributive lattices are join semidistributive. Therefore, they admit two natural secondary structures: the core label order is an alternative order on the lattice elements and the canonical join complex is the flag simplicial complex on the canonical join representations. In this article we present a characterization of finite meet-distributive lattices in terms of the core label order and the canonical join complex, and we show that the core label order of a finite meet-distributive lattice is always a meet-semilattice.

Keywords: meet-distributive lattice; congruence-uniform lattice; canonical join complex; core label order; intersection property

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1. Introduction

A lattice **L** is join semidistributive if every element admits a canonical expression as a join of join-irreducible elements, see [18], [19]. Consequently, the word problem can be solved efficiently in these lattices. The set of canonical join representations of a lattice forms a simplicial complex (see [16], Proposition 2.2), the *canonical join complex* of **L**. If **L** is join semidistributive, then the faces of the canonical join complex are naturally indexed by the elements of **L**.

Moreover, when **L** is join semidistributive, canonical join representations can be computed easily with the help of a certain edge-labeling which is determined by

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a perspectivity relation (see [4]). This labeling is essentially unique and can be used to define an alternative partial order on $\mathbf{L}$, the *core label order*. Ordering the elements of $\mathbf{L}$ by this order yields the core label poset $\mathrm{CLO}(\mathbf{L})$.

This order first appeared in Reading's research on congruence-uniform lattices of regions of real hyperplane arrangements. We have investigated this order abstractly for congruence-uniform lattices in [11]. For some special cases the core label order was studied combinatorially in [3], [5], [9], [10], [13], [14], [15].

An interesting subclass of join-semidistributive lattices are *meet-distributive lattices*, which have the property that every interval $[x, y]$—where x is the meet of the elements covered by y—is isomorphic to a Boolean lattice (see [6], [7]). It turns out that we can use the core label order and the canonical join complex to characterize meet-distributive lattices.

Theorem 1.1. *A finite join-semidistributive lattice* $\mathbf{L}$ *is meet-distributive if and only if the core label poset* $\mathrm{CLO}(\mathbf{L})$ *is the face poset of the canonical join complex of* $\mathbf{L}$.

We want to point out that we can also use the core label order to characterize finite Boolean lattices. They are precisely the join-semidistributive lattices that are isomorphic to their own core label order, see [11], Theorem 1.5. Consequently, the canonical join complex of a finite Boolean lattice is a simplex.

In [17], Problem 9.5, N. Reading asked under what conditions the core label order is again a lattice. In [11], Section 4.2 we found one such property, which we call the *intersection property*. This property can be used to characterize the join-semidistributive lattices whose core label orders are meet-semilattices (see [11], Theorem 4.8). We conclude this article with the observation that every meet-distributive lattice has the intersection property.

Theorem 1.2. *Every finite meet-distributive lattice* $\mathbf{L}$ *has the intersection property. Consequently, for a finite meet-distributive lattice* $\mathbf{L}$*, the core label poset* $\mathrm{CLO}(\mathbf{L})$ *is a meet-semilattice, and it is a lattice if and only if* $\mathbf{L}$ *is isomorphic to a Boolean lattice.*

We first recall the necessary basic notions in Section 2. After that we define the core label order of a lattice in Section 3.1 and the canonical join complex of a join-semidistributive lattice in Section 3.3, where we also prove Theorem 1.1. In Section 3.4 we define the intersection property and prove Theorem 1.2.

2.1. Basic notions. Let $\mathbf{P} = (P, \leqslant)$ be a partially ordered set (*poset* for short). The *dual* poset of $\mathbf{P}$ is $\mathbf{P}^* \overset{\text{def}}{=} (P, \geqslant)$.

An element $x \in P$ is *minimal* in $\mathbf{P}$ if $y \leqslant x$ implies $y = x$ for all $y \in P$. Dually, $x \in P$ is *maximal* in $\mathbf{P}$ if it is minimal in $\mathbf{P}^*$.

A *cover* of $\mathbf{P}$ is a pair (x, y) such that $x < y$ and there is no $z \in P$ such that $x < z < y$. We usually write $x \lessdot y$ for a cover, and we denote the set of all covers of $\mathbf{P}$ by $\mathcal{E}(\mathbf{P})$. Moreover, if $x \lessdot y$, then we call x a *lower cover* of y and y an *upper cover* of x.

A *chain* of $\mathbf{P}$ is a totally ordered subset of P and it is *saturated* if it can be written as a sequence of covers. A saturated chain is *maximal* if it contains a minimal and a maximal element of $\mathbf{P}$.

We say that $\mathbf{P}$ is a *lattice* if for every two elements $x, y \in P$ there exists a greatest lower bound $x \wedge y$ (the *meet*) and a least upper bound $x \vee y$ (the *join*). Every finite lattice has a unique minimal element (denoted by $\hat{0}$) and a unique maximal element (denoted by $\hat{1}$).

A lattice is *Boolean* if it is isomorphic to the family of subsets of some set M ordered by inclusion. If $|M| = n$, then we write $\mathrm{Bool}(n)$ for the Boolean lattice with 2^n elements.

2.2. Join-semidistributive lattices. Let $\mathbf{L} = (L, \leqslant)$ be a lattice. A *join representation* of $x \in L$ is a set $X \subseteq L$ with $x = \bigvee X$. A join representation X of x *join-refines* a join representation X' of x if for every $y \in X$ there exists some $y' \in X'$ such that $y \leqslant y'$. A join representation X of x is *irredundant* if no proper subset of X joins to x, and an irredundant join representation is *canonical* if it join-refines every other join representation of x. We denote the canonical join representation of $x \in L$ by $\Gamma(x)$ (if it exists).

It turns out that the finite lattices in which every element admits a canonical join representation can be characterized algebraically. A lattice $\mathbf{L} = (L, \leqslant)$ is *join semidistributive* if for all $x, y, z \in L$ the implication

$$x \vee y = x \vee z \Rightarrow x \vee y = x \vee (y \wedge z) \tag{JSD}$$

holds.

Theorem 2.1 ([8], Theorem 2.24)**.** *A finite lattice is join semidistributive if and only if every element admits a canonical join representation.*

2.3. Meet-distributive lattices. We now move to a subfamily of the join-semidistributive lattices. Let $\mathbf{L} = (L, \leqslant)$ be a lattice. For $x \in L$, we define its *nucleus* by

$$x_{\downarrow} \stackrel{\text{def}}{=} x \wedge \bigwedge_{y \in L\colon y \lessdot x} y.$$

We remark that the additional meet with x in the definition of $x_{\downarrow}$ is relevant, only when $x = \hat{0}$. Since $\hat{0}$ has no lower covers and the meet over the empty set is usually the greatest element, we need to add this correction term in order to ensure that $x_{\downarrow} \leqslant x$ for all $x \in L$. This has been overlooked in the analogous definition of [11].

We call the interval $[x_{\downarrow}, x]$ the *core* of x. Then, $\mathbf{L}$ is *meet distributive* if for every $x \in L$, the core $[x_{\downarrow}, x]$ is isomorphic to a Boolean lattice. Figure 1(a) shows a join-semidistributive lattice that is not meet distributive, and Figure 2(a) shows a meet-distributive lattice.

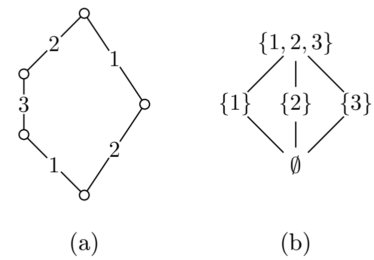

Figure 1. (a) A join-semidistributive lattice; (b) The core label order of the lattice from Figure 1(a).

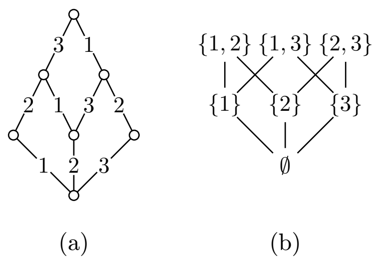

Figure 2. (a) A meet-distributive lattice; (b) The core label order of the lattice from Figure 2(a).

The following result characterizes meet-distributive lattices. Recall that $\mathbf{L}$ is *lower semimodular* if for all $x, y \in L$ it holds: whenever $x, y \lessdot x \vee y$, then $x \wedge y \lessdot x, y$.

Theorem 2.2 ([1], Theorem 1.9)**.** *A finite lattice is meet distributive if and only if it is join semidistributive and lower semimodular.*

Meet-distributive lattices are precisely the lattices that arise from a closure operator satisfying the so-called *anti-exchange property*, see [1], [2], [7].

3.1. The core label order. Motivated by the study of the poset of regions of real hyperplane arrangements, N. Reading introduced an alternate way to order the elements of a congruence-uniform lattice (see [17], Section 9–7.4). In fact, we may generalize this construction to arbitrary finite lattices.

Let $\mathbf{L} = (L, \leqslant)$ be a finite lattice, let M be a set and let $\lambda\colon \mathcal{E}(\mathbf{L}) \to M$ be an *edge labeling* of $\mathbf{L}$. The *core label set* of $x \in L$ (with respect to λ) is

$$\Psi_\lambda(x) \overset{\text{def}}{=} \{\lambda(u,v)\colon\ x_\downarrow \leqslant u \lessdot v \leqslant x\}.$$

We now put $x \leqslant_{\text{clo}} y$ if and only if $\Psi_\lambda(x) \subseteq \Psi_\lambda(y)$. In general, this results in a quasi-ordered set $\text{CLO}_\lambda(\mathbf{L}) \overset{\text{def}}{=} (L, \leqslant_{\text{clo}})$.

We say that λ is a *core labeling* if the assignment $x \mapsto \Psi_\lambda(x)$ is injective. If λ is a core labeling, then it is quickly checked that $\text{CLO}_\lambda(\mathbf{L})$ is in fact a partially ordered set; the *core label poset*. We call $\leqslant_{\text{clo}}$ the *core label order* of $\mathbf{L}$. Figures 1 and 2 illustrate this construction.

3.2. A perspectivity labeling. Two covers $(x_1, y_1), (x_2, y_2) \in \mathcal{E}(\mathbf{L})$ are *perspective* if either $y_1 \vee x_2 = y_2$ and $y_1 \wedge x_2 = x_1$ or $y_2 \vee x_1 = y_1$ and $y_2 \wedge x_1 = x_2$. We write $(x_1, y_1) \mathrel{\overline{\overline{\wedge}}} (x_2, y_2)$ in this case. This definition is illustrated in Figure 3.

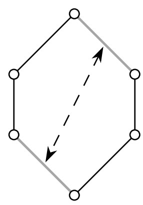

Figure 3. The grey edges represent perspective covers.

Recall another useful fact about join-semidistributive lattices. An element $j \in L \setminus \{\hat{0}\}$ is *join irreducible* if whenever $j = x \vee y$, then $j \in \{x, y\}$. The set of join-irreducible elements of $\mathbf{L}$ is denoted by $\text{J}(\mathbf{L})$. In particular, if $\mathbf{L}$ is finite and $j \in \text{J}(\mathbf{L})$, then there exists a unique lower cover j_* of j.

Lemma 3.1 ([1], Lemma 1.8)**.** *Let $\mathbf{L}$ be a finite join-semidistributive lattice. For every $(x, y) \in \mathcal{E}(\mathbf{L})$, the set $\{z \in L\colon\ z \leqslant y$ and $z \not\leqslant x\}$ has a unique minimal element j, and j is join irreducible.*

This gives rise to the following edge-labeling of a finite join-semidistributive lattice $\mathbf{L}$:

$$\lambda_{\mathrm{jsd}}\colon \mathcal{E}(\mathbf{L}) \to \mathrm{J}(\mathbf{L}), \quad (x, y) \mapsto \bigwedge\{z \in L\colon\ z \leqslant y \text{ and } z \nleqslant x\}. \tag{3.1}$$

This labeling is illustrated in Figures 1(a) and 2(a).

Lemma 3.2. *Let $(x, y) \in \mathcal{E}(\mathbf{L})$ and $j \in \mathrm{J}(\mathbf{L})$. If $(x, y) \mathrel{\overline{\overline{\barwedge}}} (j_*, j)$, then $j \leqslant y$.*

P r o o f. If $(x, y) \mathrel{\overline{\overline{\barwedge}}} (j_*, j)$, then either $j \leqslant y$ or $y \leqslant j$. The latter case, however, forces the existence of two lower covers of j, contradicting that j is join irreducible. □

We now show that the labeling λ_{jsd} is a *canonical labeling* of a finite join-semidistributive lattice, because it is determined by the perspectivity relation.

Lemma 3.3. *Let $(x, y) \in \mathcal{E}(\mathbf{L})$. Then $\lambda_{\mathrm{jsd}}(x, y) = j$ if and only if $(j_*, j) \mathrel{\overline{\overline{\barwedge}}} (x, y)$.*

P r o o f. Suppose that $\lambda_{\mathrm{jsd}}(x, y) = j$. By definition, $x \vee j = y$ and thus $x \wedge j < j$. Since j is minimal with the property that $j \leqslant y$ and $j \nleqslant x$, we see that $j_* \leqslant x$. This implies $x \wedge j = j_*$ and it follows that $(j_*, j) \mathrel{\overline{\overline{\barwedge}}} (x, y)$.

Conversely, suppose that $(j_*, j) \mathrel{\overline{\overline{\barwedge}}} (x, y)$. By Lemma 3.2, we get $j \vee x = y$ and $j \wedge x = j_*$. Thus, $\lambda_{\mathrm{jsd}}(x, y) \leqslant j$. But $j_* \leqslant x$, which means that $\lambda_{\mathrm{jsd}}(x, y) \nleqslant j_*$. Since $j \in \mathrm{J}(\mathbf{L})$, we must have $\lambda_{\mathrm{jsd}}(x, y) = j$. □

The labeling λ_{jsd} also allows for a simple computation of canonical join representations.

Proposition 3.4 ([4], Lemma 19)**.** *If $\mathbf{L} = (L, \leqslant)$ is a finite join-semidistributive lattice, then for every $x \in L$,*

$$\Gamma(x) = \{\lambda_{\mathrm{jsd}}(x', x)\colon\ x' \lessdot x\}.$$

Proposition 3.5. *The edge-labeling λ_{jsd} of a finite join-semidistributive lattice is a core labeling.*

P r o o f. Let $\mathbf{L} = (L, \leqslant)$ be a finite join-semidistributive lattice, and let $x \in L$. If $j \in \Psi_{\lambda_{\mathrm{jsd}}}(x)$, then there exist $x_1, x_2 \in L$ such that $x_\downarrow \leqslant x_1 \lessdot x_2 \leqslant x$ and $\lambda_{\mathrm{jsd}}(x_1, x_2) = j$. By Lemma 3.3, this means that $(j_*, j) \mathrel{\overline{\overline{\barwedge}}} (x_1, x_2)$ and by Lemma 3.2 it follows that $j \leqslant x_2 \leqslant x$. As a consequence, $\bigvee \Psi_{\lambda_{\mathrm{jsd}}}(x) \leqslant x$. Moreover, by Proposition 3.4, we have $\Gamma(x) \subseteq \Psi_{\lambda_{\mathrm{jsd}}}(x)$, and therefore $x = \bigvee \Gamma(x) \leqslant \bigvee \Psi_{\lambda_{\mathrm{jsd}}}(x)$. It follows that $\bigvee \Psi_{\lambda_{\mathrm{jsd}}}(x) = x$.

Now, if there exist $x, y \in L$ such that $\Psi_{\lambda_{\rm jsd}}(x) = \Psi_{\lambda_{\rm jsd}}(y)$, then

$$x = \bigvee \Psi_{\lambda_{\rm jsd}}(x) = \bigvee \Psi_{\lambda_{\rm jsd}}(y) = y.$$

Hence, the assignment $x \mapsto \Psi_{\lambda_{\rm jsd}}(x)$ is injective and $\lambda_{\rm jsd}$ is a core labeling. □

Theorem 3.6. *Let* $\mathbf{L} = (L, \leqslant)$ *be a finite join-semidistributive lattice. Then we have* $\Gamma(x) = \Psi_{\lambda_{\rm jsd}}(x)$ *for all* $x \in L$ *if and only if* $\mathbf{L}$ *is meet distributive.*

Proof. If $\mathbf{L}$ is meet distributive, then every core $[x_{\downarrow}, x]$ is isomorphic to a Boolean lattice. If $\mathrm{Bool}(k)$ is the Boolean lattice with the ground set $M = \{1, 2, \ldots, k\}$, then it is easy to verify that $\Gamma(M) = M = \Psi_{\lambda_{\rm jsd}}(M)$. This proves that $\Gamma(x) = \Psi_{\lambda_{\rm jsd}}(x)$ for all $x \in L$.

Conversely, suppose that $\mathbf{L}$ is not meet distributive. By Theorem 2.2, $\mathbf{L}$ is not lower semimodular, which means that there exist two elements $x, y \in L$ such that $x, y \lessdot x \vee y$ and—without loss of generality—$(x \wedge y, x) \notin \mathcal{E}(\mathbf{L})$. This means that there exists $z \in L$ with $x \wedge y < z \lessdot x$. Suppose that $\lambda_{\rm jsd}(z, x) = j$. By construction, $j \in \Psi_{\lambda_{\rm jsd}}(x \vee y)$. By perspectivity, $j \neq \lambda_{\rm jsd}(x, x \vee y)$.

Since $j \leqslant x$ and $j \not\leqslant z$, the assumption $x \wedge y < z$ implies that $j \not\leqslant y$. Moreover, $z \not\leqslant y$ because otherwise $z = x \wedge y$. This implies that $j \vee y = x \vee y = z \vee y$ and by (JSD) we get $y \neq x \vee y = y \vee (z \wedge j) = y \vee j_*$. Thus $j_* \not\leqslant y$ and since $j \in \mathrm{J}(\mathbf{L})$, we find $y \wedge j \neq j_*$. It follows that $\lambda_{\rm jsd}(y, x \vee y) \neq j$.

If x, y are the only lower covers of $x \vee y$, then we have just shown that $j \notin \Gamma(x \vee y)$, which yields $\Gamma(x \vee y) \subsetneq \Psi_{\lambda_{\rm jsd}}(x \vee y)$.

Suppose that there exists another lower cover u of $x \vee y$ (different from x and y). If $z \leqslant u$, then we get $x \vee y = x \vee u = y \vee u$ and therefore by (JSD) $x \vee y = u \vee (x \wedge y) \leqslant u \vee z = u$, which is a contradiction. If $j \leqslant u$, then $\lambda_{\rm jsd}(u, x \vee y) \neq j$ by Lemma 3.3. Otherwise, we get $j \vee u = x \vee y = z \vee u$ and therefore $u \neq x \vee y = u \vee (z \wedge j) = u \vee j_*$, and this implies $\lambda_{\rm jsd}(u, x \vee y) \neq j$. Since u was chosen arbitrarily, we conclude that $j \notin \Gamma(x \vee y)$ and we find that $\Gamma(x \vee y) \subsetneq \Psi_{\lambda_{\rm jsd}}(x \vee y)$. □

We may define the *Boolean defect* of a join-semidistributive lattice $\mathbf{L} = (L, \leqslant)$ by

$$\mathrm{bdef}(\mathbf{L}) \stackrel{\mathrm{def}}{=} \sum_{x \in L} |\Psi_{\lambda_{\rm jsd}}(x) \setminus \Gamma(x)|.$$

Theorem 3.6 has the following consequence, which strengthens [11], Proposition 5.2.

Corollary 3.7. *A finite join-semidistributive lattice* $\mathbf{L}$ *has* $\mathrm{bdef}(\mathbf{L}) = 0$ *if and only if* $\mathbf{L}$ *is meet distributive.*

3.3. The canonical join complex of a join-semidistributive lattice. Given a finite set M, a *simplicial complex* on M is a family $\Delta(M)$ of subsets of M such that for every $F \in \Delta(M)$ and every $F' \subseteq F$ we have $F' \in \Delta(M)$. The members of $\Delta(M)$ are *faces*. The *face poset* of $\Delta(M)$ is the poset $(\Delta(M), \subseteq)$.

N. Reading has observed in [16], Proposition 2.2 that the set of canonical join representations of a lattice is closed under taking subsets. In other words, it forms a simplicial complex; the *canonical join complex* of $\mathbf{L}$, denoted by $\mathrm{Can}(\mathbf{L})$.

We are now ready to prove Theorem 1.1.

P r o o f of Theorem 1.1. Let $\mathbf{L} = (L, \leqslant)$ be a finite join-semidistributive lattice. By definition, the face poset of $\mathrm{Can}(\mathbf{L})$ is precisely $(\{\Gamma(x) \colon x \in L\}, \subseteq)$ and $\mathrm{CLO}_{\lambda_{\mathrm{jsd}}}(\mathbf{L})$ is isomorphic to $(\{\Psi_{\lambda_{\mathrm{jsd}}}(x) \colon x \in L\}, \subseteq)$.

If $\mathbf{L}$ is meet distributive, then Theorem 3.6 states that these two posets are isomorphic.

If $\mathbf{L}$ is not meet distributive, then by Theorem 3.6, there exists some $x \in L$ such that $\Gamma(x) \subsetneq \Psi_{\lambda_{\mathrm{jsd}}}(x)$. In particular, there exists $j \in \Psi_{\lambda_{\mathrm{jsd}}}(x) \setminus \Gamma(x)$. It follows that $\{j\} \subseteq \Psi_{\lambda_{\mathrm{jsd}}}(x)$, but $\{j\} \not\subseteq \Gamma(x)$, so that the core label poset of $\mathbf{L}$ is not isomorphic to the face poset of $\mathrm{Can}(\mathbf{L})$. □

Figure 4 illustrates Theorem 1.1 on a bigger example.

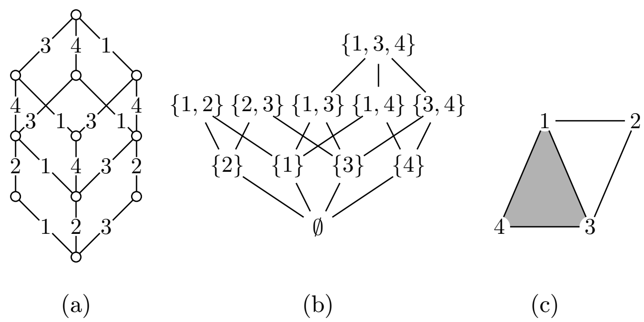

Figure 4. Illustration of Theorem 1.1: (a) A meet-distributive lattice; (b) The core label order of the lattice from Figure 4(a); (c) The canonical join complex of the lattice from Figure 4(a). The highlighted region indicates a two-dimensional face.

3.4. The intersection property. N. Reading asked in [17], Problem 9.5 for conditions on a congruence-uniform lattice $\mathbf{L}$ which would imply that $\mathrm{CLO}(\mathbf{L})$ is a lattice, too. We gave one such property in [11], Section 4.2, which extends to arbitrary lattices as follows. A finite lattice $\mathbf{L} = (L, \leqslant)$ with the edge labeling λ *has the intersection property* if for all $x, y \in L$ there exists $z \in L$ such that $\Psi_\lambda(x) \cap \Psi_\lambda(y) = \Psi_\lambda(z)$.

Provided that λ is a core labeling, the proof of [11], Theorems 1.3 and 4.7 carries over essentially verbatim to the more general case.

Theorem 3.8 ([11], Theorems 1.3 and 4.7)**.** *Let* $\mathbf{L}$ *be a finite lattice with core labeling* λ*. The core label poset* $\mathrm{CLO}_\lambda(\mathbf{L})$ *is a meet-semilattice if and only if* $\mathbf{L}$ *has the intersection property. It is a lattice if and only if* $\hat{1}_\downarrow = \hat{0}$*.*

We conclude this article with the proof of Theorem 1.2.

P r o o f of Theorem 1.2. Let $\mathbf{L} = (L, \leqslant)$ be a finite meet-distributive lattice. For $x, y \in L$ we conclude from Theorem 3.6 that $\Psi_{\lambda_{\rm jsd}}(x) = \Gamma(x)$ and $\Psi_{\lambda_{\rm jsd}}(y) = \Gamma(y)$. It follows that $Z = \Gamma(x) \cap \Gamma(y)$ is a face of $\mathrm{Can}(\mathbf{L})$, which means that there exists $z \in L$ with $Z = \Gamma(z) = \Psi_{\lambda_{\rm jsd}}(z)$. We have thus established that $\mathbf{L}$ has the intersection property.

Lemma 3.9 of [11] states that $\mathrm{CLO}(\mathbf{L})$ has a greatest element if and only if $\hat{1}_\downarrow = \hat{0}$. Now, if $\mathbf{L}$ is meet-distributive, then the interval $[\hat{1}_\downarrow, \hat{1}]$ is isomorphic to a Boolean lattice. Thus, $\hat{1}_\downarrow = \hat{0}$ if and only if $\mathbf{L}$ is Boolean. The claim then follows from Theorem 3.8. □

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