148 (2023) MATHEMATICA BOHEMICA No. 2, 181–195

ON BHARGAVA RINGS

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Received September 6, 2021. Published online May 4, 2022.
Communicated by Dagmar Medková

Abstract. Let D be an integral domain with the quotient field K, X an indeterminate over K and x an element of D. The Bhargava ring over D at x is defined to be $\mathbb{B}_x(D) := \{f \in K[X]\colon \text{ for all } a \in D,\ f(xX+a) \in D[X]\}$. In fact, $\mathbb{B}_x(D)$ is a subring of the ring of integer-valued polynomials over D. In this paper, we aim to investigate the behavior of $\mathbb{B}_x(D)$ under localization. In particular, we prove that $\mathbb{B}_x(D)$ behaves well under localization at prime ideals of D, when D is a locally finite intersection of localizations. We also attempt a classification of integral domains D such that $\mathbb{B}_x(D)$ is locally free, or at least faithfully flat (or flat) as a D-module (or $D[X]$-module, respectively). Particularly, we are interested in domains that are (locally) essential. A particular attention is devoted to provide conditions under which $\mathbb{B}_x(D)$ is trivial when dealing with essential domains. Finally, we calculate the Krull dimension of Bhargava rings over MZ-Jaffard domains. Interesting results are established with illustrating examples.

Keywords: Bhargava ring; localization; (locally) essential domain; locally free module; (faithfully) flat module; Krull dimension

MSC 2020: 13C11, 13F20, 13F05, 13B30, 13C15

Introduction

Let D be an integral domain with the quotient field K and X an indeterminate over K. The ring of *integer-valued polynomials over* D is defined by $\operatorname{Int}(D) := \{f \in K[X]\colon f(D) \subseteq D\}$. Clearly, $\operatorname{Int}(D)$ is a ring between $D[X]$ and $K[X]$. For any element x of D, the *Bhargava ring* over D at x is defined as

$$\mathbb{B}_x(D) := \{f \in K[X]\colon \text{ for all } a \in D,\ f(xX+a) \in D[X]\}.$$

Investigation on Bhargava rings, relatively to commutative algebra, is still in the beginning and it is a promising research topic. Originally, Manjual Bhargava introduced the notion of integer-valued polynomials of modulus x in June 2000 during the

DOI: 10.21136/MB.2022.0137-21

second meeting on the integer-valued polynomials held at C.I.R.M. (Centre International de Rencontres Mathématiques) in Marseille. In 2004, under the supervision of Paul-Jean Cahen, Yeramian devoted her Ph.D. thesis (see [27]) to the study of that remarkable subring of the ring of integer-valued polynomials which she named the "Bhargava ring" in honor to Bhargava. She studied Bhargava rings over an arbitary integral domain and investigated localization properties, trivial cases, the module structure and the spectrum of Bhargava rings over some particular integral domains.

Further, in 2009, Elliott in [10] pointed out that the question of flatness of the D-module $\mathrm{Int}(D)$ can be reduced to the question of flatness of Bhargava rings. In other words, if $\mathbb{B}_x(D)$ is flat as a D-module for every nonzero element x of D then $\mathrm{Int}(D)$ is also flat as a D-module (cf. [10], Proposition 6.4). It is worth noting that in [5], Bhargava et al. refer to $\mathbb{B}_x(D)$ by $\mathrm{Int}_x(D)$ and use the term "the ring of integer-valued polynomials of modulus x" instead of "the Bhargava ring". Later in [1], Alrasasi and Izelgue continued the previous investigations and focused on the description of the prime spectrum and the evaluation of the Krull and valuative dimensions of Bhargava rings over general integral domains. Recently in 2020, Park and Tartarone investigated the PvMD property of Bhargava rings over some particular classes of PvMDs, namely valuation domains, Krull-type domains and almost Dedekind domains (see [22]).

Note that $\mathbb{B}_0(D) = \mathrm{Int}(D)$ and $\mathbb{B}_x(D) = D[X]$ for any unit element x in D. Also, we have the containments $D[X] \subseteq \mathbb{B}_x(D) \subseteq \mathrm{Int}(D)$. Moreover, as proved in [27], we have $\mathrm{Int}(D) = \bigcup\limits_{0 \neq x \in D} \mathbb{B}_x(D)$ and, for each nonzero element x of D, $\mathbb{B}_x(D) = \bigcap\limits_{a \in D} D[(X-a)/x]$. For a detailed review on Bhargava rings see [1], [27], [28], [29].

The purpose of this paper is to investigate some properties of Bhargava rings such as localization, local freeness, faithful flatness and calculation of the Krull dimension over various classes of integral domains. Thus, in Section 1 we collect some useful facts concerning the localization of $\mathbb{B}_x(D)$. Our main result in this section states that if D is an integral domain such that $D = \bigcap\limits_{\mathfrak{p} \in \mathcal{P}} D_{\mathfrak{p}}$ for some subset $\mathcal{P}$ of $\mathrm{Spec}(D)$ and the intersection $\bigcap\limits_{\mathfrak{p} \in \mathcal{P}} D_{\mathfrak{p}}$ is locally finite, then $\mathbb{B}_x(D)_{\mathfrak{p}} = \mathbb{B}_x(D_{\mathfrak{p}})$ for each prime ideal $\mathfrak{p}$ of D and for each element x of D (Proposition 1.3). In Section 2, we attempt a classification of integral domains D such that $\mathbb{B}_x(D)$ is locally free, or at least faithfully flat (or flat) as a D-module (or as an overring of $D[X]$, respectively). Among other results, we show that, for any locally essential domain D, the D-module $\mathbb{B}_x(D)$ is faithfully flat and it is locally free under an additional condition on the associated primes of D (Theorem 2.1). This last result includes the case when the integral domain D is either almost Krull, generalized Krull or t-almost Dedekind (Corollary 2.2). Also, we derive an interesting corollary about the D-module $\mathrm{Int}(D)$

(Corollary 2.4). In the case of essential domains, we show that $\mathbb{B}_x(D)$ is flat over $D[X]$ if and only if $\mathbb{B}_x(D)$ is trivial, that is, $\mathbb{B}_x(D) = D[X]$ (Theorem 2.7). Then, we give some conditions under which $\mathbb{B}_x(D)$ is trivial (Theorem 2.9). Further, we calculate the Krull dimension of $\mathbb{B}_x(D)$ when D is an MZ-Jaffard domain. We end with a list of examples illustrating some of our results.

In order to avoid trivialities, we always assume that D is not a field.

1. Preliminary results

In this section we provide many preliminary results that will be used at some further point in the article to prove our main results. First, we begin with the following useful lemmas.

Lemma 1.1. *Let D be an integral domain such that $D = \bigcap_i S_i^{-1}D$, where S_i is a multiplicative subset of D for each i. Then $\mathbb{B}_x(D) = \bigcap_i \mathbb{B}_x(S_i^{-1}D) = \bigcap_i S_i^{-1}\mathbb{B}_x(D)$ for each element x of D.*

Proof. By [28], Lemma 2.1, $\mathbb{B}_x(D) \subseteq S_i^{-1}\mathbb{B}_x(D) \subseteq \mathbb{B}_x(S_i^{-1}D)$ for each i, so we only need to show that $\bigcap_i \mathbb{B}_x(S_i^{-1}D) \subseteq \mathbb{B}_x(D)$. To do this, let $f \in \bigcap_i \mathbb{B}_x(S_i^{-1}D)$. Then $f(xX + a) \in S_i^{-1}D[X]$ for each i and each $a \in S_i^{-1}D$, and hence $f(xX + a) \in S_i^{-1}D[X]$ for each i and each $a \in D$ because $D = \bigcap_i S_i^{-1}D$. Thus $f(xX + a) \in \bigcap_i S_i^{-1}D[X] = D[X]$ for each $a \in D$, that is, $f \in \mathbb{B}_x(D)$, and this completes the proof. $\square$

The previous result can be viewed as a generalization of the first statement of [22], Proposition 2.5.

Lemma 1.2. *Let D be an integral domain. Then $\mathbb{B}_x(D_\mathfrak{p})_\mathfrak{q} = (D_\mathfrak{p})_\mathfrak{q}[X]$ for each pair of prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of D with $\mathfrak{p} \neq \mathfrak{q}$ and each element x of D, and thus $\mathbb{B}_x(D_\mathfrak{p})_\mathfrak{q} = \mathbb{B}_x(D_\mathfrak{q})_\mathfrak{p}$.*

Proof. It follows immediately from [11], Lemma 2.5 and the inclusions $(D_\mathfrak{p})_\mathfrak{q}[X] \subseteq \mathbb{B}_x(D_\mathfrak{p})_\mathfrak{q} \subseteq \mathrm{Int}(D_\mathfrak{p})_\mathfrak{q}$. $\square$

For polynomial rings, the equality $S^{-1}(D[X]) = (S^{-1}D)[X]$ always holds for any integral domain D and any multiplicative subset S of D. However, in the case of $\mathbb{B}_x(D)$, we always have $S^{-1}\mathbb{B}_x(D) \subseteq \mathbb{B}_x(S^{-1}D)$ (see [28], Lemma 2.1) and when the reverse inclusion holds we say that $\mathbb{B}_x(D)$ has the property of *the good behavior under localization*, i.e., $S^{-1}\mathbb{B}_x(D) = \mathbb{B}_x(S^{-1}D)$ for each multiplicative subset S of D. This last property plays a pivotal role in studying the ring or the module structure of $\mathbb{B}_x(D)$. In fact this condition allows us to reduce our studies to the local case; for example when dealing with domains that are either t-locally rank-one discrete valuation or t-locally valuation ones (such as Krull domains, generalized Krull domains and Krull-type domains). Thus, with the good behavior under localization of $\mathbb{B}_x(D)$ we only need to investigate the case $\mathbb{B}_x(V)$, where V is a rank-one discrete valuation domain or a valuation domain. From now on, for brevity's sake, we will use DVR to refer rank-one discrete valuation domain.

In [29], Proposition 1.1, Yeramian proved that $\mathbb{B}_x(D)$ behaves well under localization when D is a Noetherian domain, and if D is a Krull domain and $x \in D\setminus\{0\}$ then $\mathbb{B}_x(D)_{\mathfrak{p}} = \mathbb{B}_x(D_{\mathfrak{p}})$ for each height-one prime ideal $\mathfrak{p}$ of D. Recently in [22], the authors have generalized the previous result as follows: if D is an integral domain such that $D = \bigcap\limits_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$ for some subset $\mathcal{P}$ of $\operatorname{Spec}(D)$ and the intersection $\bigcap\limits_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$ is locally finite, then $\mathbb{B}_x(D)_{\mathfrak{p}} = \mathbb{B}_x(D_{\mathfrak{p}})$ for each $\mathfrak{p} \in \mathcal{P}$ and for each $x \in D$. So, in the following, we improve that result by showing that the equality holds for all prime ideals.

Let $\{D_\alpha\}_{\alpha\in\Lambda}$ be a family of integral domains contained in the same field. The intersection $\bigcap\limits_{\alpha\in\Lambda} D_\alpha$ is said to be *locally finite* if every nonzero element of this intersection is a unit in D_α for all but finitely many $\alpha \in \Lambda$.

Proposition 1.3. *Let* $D = \bigcap\limits_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$, *where* $\mathcal{P} \subseteq \operatorname{Spec}(D)$, *be an integral domain and let* x *be an element of* D. *If the intersection* $\bigcap\limits_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$ *is locally finite, then* $S^{-1}\mathbb{B}_x(D) = \mathbb{B}_x(S^{-1}D)$ *for each multiplicative subset* S *of* D. *Moreover, we have:* $\mathbb{B}_x(D)_{\mathfrak{p}} = \mathbb{B}_x(D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$ *for each* $\mathfrak{p} \in \operatorname{Spec}(D) \setminus \mathcal{P}$.

Proof. It is sufficient to prove that $\mathbb{B}_x(D)_{\mathfrak{p}} = \mathbb{B}_x(D_{\mathfrak{p}})$ for each prime ideal $\mathfrak{p}$ of D. So, let $\mathfrak{p} \in \operatorname{Spec}(D)$. Two cases are possible:

Case 1: $\mathfrak{p} \in \mathcal{P}$. Then the desired equality follows from the second statement of [22], Proposition 2.5.

Case 2: $\mathfrak{p} \notin \mathcal{P}$. Then the finite character and [26], Lemma 1.5 assert that $D_{\mathfrak{p}} = \bigcap\limits_{\mathfrak{q}\in\mathcal{P}} (D_{\mathfrak{q}})_{\mathfrak{p}}$. On the other hand, for each $\mathfrak{q} \in \mathcal{P}$, we have $\mathfrak{p} \neq \mathfrak{q}$ and then, by Lemma 1.2, $\mathbb{B}_x((D_{\mathfrak{q}})_{\mathfrak{p}}) = (D_{\mathfrak{q}})_{\mathfrak{p}}[X]$. Thus it follows from Lemma 1.1 that

$$\mathbb{B}_x(D_{\mathfrak{p}}) = \bigcap_{\mathfrak{q}\in\mathcal{P}} \mathbb{B}_x((D_{\mathfrak{q}})_{\mathfrak{p}}) = \bigcap_{\mathfrak{q}\in\mathcal{P}} (D_{\mathfrak{q}})_{\mathfrak{p}}[X] = D_{\mathfrak{p}}[X].$$

Therefore $\mathbb{B}_x(D)_{\mathfrak{p}} = \mathbb{B}_x(D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$, and the proof is complete. □

Remark 1.4. With a slight modification of the proof of [22], Proposition 2.5, we can adapt it to prove Proposition 1.3 as follows:

Let $\mathfrak{q} \in \mathrm{Spec}(D)$, we always have $\mathbb{B}_x(D)_{\mathfrak{q}} \subseteq \mathbb{B}_x(D_{\mathfrak{q}})$ (see [28], Lemma 2.1). For the other inclusion, let $f \in \mathbb{B}_x(D_{\mathfrak{q}})$. Clearly, there exists $d \in D \setminus \{0\}$, such that $df \in D[X]$. Set $\mathcal{B} := \{\mathfrak{p} \in \mathcal{P},\ d \in \mathfrak{p}\}$. The finite character of D implies that $\mathcal{B}$ is a finite set. Then, by Lemma 1.2, we have $\mathbb{B}_x(D_{\mathfrak{q}}) \subseteq \mathbb{B}_x(D_{\mathfrak{p}})_{\mathfrak{q}}$ for each $\mathfrak{p} \in \mathcal{B}$. So, for each $\mathfrak{p} \in \mathcal{B}$ there exists $s_{\mathfrak{p}} \in D \setminus \mathfrak{q}$ such that $s_{\mathfrak{p}} f \in \mathbb{B}_x(D_{\mathfrak{p}})$. Now, we set $s =: \prod_{\mathfrak{p} \in \mathcal{B}} s_{\mathfrak{p}}$. Then, since $\mathcal{B}$ is finite, $s \in D \setminus \mathfrak{q}$ and $sf \in \mathbb{B}_x(D_{\mathfrak{p}})$ for each $\mathfrak{p} \in \mathcal{B}$. On the other hand, since d is a unit in $D_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathcal{P} \setminus \mathcal{B}$ and $df \in D[X]$, one has $f \in D_{\mathfrak{p}}[X]$ and then $sf \in D_{\mathfrak{p}}[X] \subseteq \mathbb{B}_x(D_{\mathfrak{p}})$. Hence $sf \in \mathbb{B}_x(D_{\mathfrak{p}})$ for each $\mathfrak{p} \in \mathcal{P}$, and then $sf \in \bigcap_{\mathfrak{p} \in \mathcal{P}} \mathbb{B}_x(D_{\mathfrak{p}})$. Thus, by Lemma 1.1, $sf \in \mathbb{B}_x(D)$. Therefore $f \in \mathbb{B}_x(D)_{\mathfrak{q}}$ since $s \in D \setminus \mathfrak{q}$, and the equality is proved.

By taking $x = 0$ in Proposition 1.3, we recover the following well-known result.

Corollary 1.5 ([11], Theorem 1.1). *Let $D = \bigcap_{\mathfrak{p} \in \mathcal{P}} D_{\mathfrak{p}}$, where $\mathcal{P} \subseteq \mathrm{Spec}(D)$, be an integral domain. If the intersection $\bigcap_{\mathfrak{p} \in \mathcal{P}} D_{\mathfrak{p}}$ is locally finite, then $S^{-1}\mathrm{Int}(D) = \mathrm{Int}(S^{-1}D)$ for each multiplicative subset S of D.*

Following [7], a prime ideal $\mathfrak{p}$ of D is called an *associated prime of a principal ideal aD* of D if $\mathfrak{p}$ is minimal over $(aD : bD)$ for some $b \in D \setminus aD$. For brevity, we call $\mathfrak{p}$ an *associated prime* of D and we denote by $\mathrm{Ass}(D)$ the set of all associated prime ideals of D.

Proposition 1.6. *Let D be an integral domain and let x be an element of D. Then $\mathbb{B}_x(D)_{\mathfrak{m}} = \mathbb{B}_x(D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$ for each maximal ideal $\mathfrak{m}$ of D which is not an associated prime.*

P r o o f. Let $\mathfrak{m}$ be a maximal ideal of D that is not associated prime. Then, we can write $D_{\mathfrak{m}} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(D), \mathfrak{p} \subsetneq \mathfrak{m}} D_{\mathfrak{p}}$ [7], Proposition 4. Since $D_{\mathfrak{p}}$ has an infinite residue field for each $\mathfrak{p} \in \mathrm{Ass}(D)$ with $\mathfrak{p} \subsetneq \mathfrak{m}$, then $\mathbb{B}_x(D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$ and hence, by Lemma 1.1,

$$\mathbb{B}_x(D_{\mathfrak{m}}) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(D), \mathfrak{p} \subsetneq \mathfrak{m}} \mathbb{B}_x(D_{\mathfrak{p}}) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(D), \mathfrak{p} \subsetneq \mathfrak{m}} D_{\mathfrak{p}}[X] = D_{\mathfrak{m}}[X].$$

Therefore, the thesis follows from the inclusions $D_{\mathfrak{m}}[X] \subseteq \mathbb{B}_x(D)_{\mathfrak{m}} \subseteq \mathbb{B}_x(D_{\mathfrak{m}})$. □

As a first application, we characterize the good behavior under localization of Bhargava rings in terms of associated primes.

Corollary 1.7. *Let D be an integral domain and let x be an element of D. Then, $\mathbb{B}_x(D)$ has a good behavior under localization if and only if $\mathbb{B}_x(D)_{\mathfrak{m}} = \mathbb{B}_x(D_{\mathfrak{m}})$ for each $\mathfrak{m} \in \mathrm{Max}(D) \cap \mathrm{Ass}(D)$ with finite residue field.*

On an integral domain D with the quotient field K the *t-operation* is defined by $I_t := \bigcup (J^{-1})^{-1}$, where J ranges over the set of all nonzero finitely generated ideals contained in I and $J^{-1} := \{x \in K, xJ \subseteq D\}$. A nonzero ideal I of D is a *t-ideal* if $I_t = I$, and a *t-maximal* ideal is an ideal that is maximal among the proper t-ideals (and hence it is a prime ideal). As any associated prime ideal with finite residue field is t-maximal (cf. [10], Proposition 3.3), we deduce:

Corollary 1.8. *Let D be an integral domain and let x be an element of D. Then $\mathbb{B}_x(D)_{\mathfrak{m}} = \mathbb{B}_x(D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$ for each maximal ideal $\mathfrak{m}$ of D which is not t-maximal.*

We say that $\mathbb{B}_x(D)$ is a *locally free* D-module if $\mathbb{B}_x(D)_{\mathfrak{m}}$ is a free $D_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m}$ of D. Next, as a corollary, we give a key tool result that characterizes the local freeness (or faithful flatness) of the D-module $\mathbb{B}_x(D)$ in terms of maximal ideals that are associated primes.

Corollary 1.9. *For any integral domain D and any element x of D, the following statements are equivalent:*

(1) *$\mathbb{B}_x(D)$ is locally free (or faithfully flat, respectively) as a D-module;*
(2) *$\mathbb{B}_x(D)_{\mathfrak{m}}$ is free (or faithfully flat, respectively) as a $D_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m}$ of D which is an associated prime.*

Proof. It follows from Proposition 1.6 and the fact that faithful flatness is a local property, see [6], Chapitre II, §3, n°4, Corollaire de la Proposition 15. □

2. Main results and examples

In this section, we give various results on the ring $\mathbb{B}_x(D)$ concerning local freeness, (faithful) flatness and its Krull dimension.

First, we start by recalling that an *essential domain* is an integral domain D such that $D = \bigcap_{\mathfrak{p} \in \mathcal{P}} D_{\mathfrak{p}}$, where $\mathcal{P}$ is a subset of $\operatorname{Spec}(D)$ and $D_{\mathfrak{p}}$ is a valuation domain for each $\mathfrak{p} \in \mathcal{P}$, and then the subset $\mathcal{P}$ is called a *defining family* of D. As this notion does not carry up to localizations (cf. [15]), D is said to be a *locally essential domain* if $D_{\mathfrak{q}}$ is an essential domain for each $\mathfrak{q} \in \operatorname{Spec}(D)$; or equivalently, $D_{\mathfrak{p}}$ is a valuation domain for each $\mathfrak{p} \in \operatorname{Ass}(D)$ (cf. [21]).

Thus, we state our first main result for $\mathbb{B}_x(D)$ to be locally free, or at least faithfully flat, as a D-module when D is a locally essential domain.

Theorem 2.1. *Let D be a locally essential domain with the quotient field K and let x be an element of D. We have:*

(1) $\mathbb{B}_x(D)$ *is a faithfully flat* D*-module.*
(2) *If* $\mathrm{Ass}(D) = X^1(D)$, *where* $X^1(D)$ *denotes the set of all height-one prime ideals of* D, *then* $\mathbb{B}_x(D)$ *is locally free as a* D*-module.*

Proof. (1) By Corollary 1.9, we only need to prove that $\mathbb{B}_x(D)_{\mathfrak{m}}$ is a faithfully flat $D_{\mathfrak{m}}$-module for each $\mathfrak{m} \in \mathrm{Max}(D) \cap \mathrm{Ass}(D)$. So, let $\mathfrak{m} \in \mathrm{Max}(D) \cap \mathrm{Ass}(D)$. We have $D_{\mathfrak{m}} = \mathbb{B}_x(D)_{\mathfrak{m}} \cap K$ and $D_{\mathfrak{m}}$ is a valuation domain (because D is locally essential). Then, by [17], Remark 3.4, $\mathbb{B}_x(D)_{\mathfrak{m}}$ is a faithfully flat $D_{\mathfrak{m}}$-module. Thus $\mathbb{B}_x(D)$ is a faithfully flat D-module, as desired.

(2) Let $\mathfrak{m} \in \mathrm{Max}(D) \cap \mathrm{Ass}(D)$. Since $\mathrm{Ass}(D) = X^1(D)$, $D_{\mathfrak{m}}$ is a one-dimensional valuation domain and so two cases are possible:

Case 1: $\mathfrak{m}D_{\mathfrak{m}}$ is principal. Then $D_{\mathfrak{m}}$ is a DVR and hence it follows from [8], Corollary II.1.6 that $\mathbb{B}_x(D)_{\mathfrak{m}}$ is a free $D_{\mathfrak{m}}$-module because $D_{\mathfrak{m}}[X] \subseteq \mathbb{B}_x(D)_{\mathfrak{m}} \subseteq \mathbb{B}_x(D_{\mathfrak{m}}) \subseteq \mathrm{Int}(D_{\mathfrak{m}})$ and $D_{\mathfrak{m}}$ is a PID.

Case 2: $\mathfrak{m}D_{\mathfrak{m}}$ is not principal. Then $D_{\mathfrak{m}}$ is a valuation domain with nonprincipal ideal and hence, by [8], Proposition I.3.16, $\mathrm{Int}(D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$. Thus $\mathbb{B}_x(D)_{\mathfrak{m}} = D_{\mathfrak{m}}[X]$ because $D_{\mathfrak{m}}[X] \subseteq \mathbb{B}_x(D)_{\mathfrak{m}} \subseteq \mathbb{B}_x(D_{\mathfrak{m}}) \subseteq \mathrm{Int}(D_{\mathfrak{m}})$. Therefore $\mathbb{B}_x(D)_{\mathfrak{m}}$ is a free $D_{\mathfrak{m}}$-module.

Consequently, by Corollary 1.9, the D-module $\mathbb{B}_x(D)$ is locally free. □

An integral domain D is called a *generalized Krull domain* (in the sense of Gilmer, see [13], Section 43) if the intersection $D = \bigcap_{\mathfrak{p} \in X^1(D)} D_{\mathfrak{p}}$ is locally finite and $D_{\mathfrak{p}}$ is a valuation domain for each $\mathfrak{p} \in X^1(D)$. According to [23], an integral domain D is said to be an *almost Krull domain* if $D_{\mathfrak{m}}$ is Krull for each maximal ideal $\mathfrak{m}$ of D. An integral domain D is said to be a t*-almost Dedekind domain* if $D_{\mathfrak{m}}$ is a DVR for each t-maximal ideal $\mathfrak{m}$ of D. Notice that Krull domains form a proper subclass of generalized Krull domains, almost Krull domains and t-almost Dedekind domains.

Corollary 2.2. *Let* D *be an integral domain and let* x *be an element of* D. *If* D *is either* t*-almost Dedekind, almost Krull or generalized Krull, then* $\mathbb{B}_x(D)$ *is a locally free* D*-module.*

We say that an integral domain D has t*-dimension one* if it is not a field and each t-maximal ideal of D has height one. Notice that generalized Krull domains and t-almost Dedekind domains are locally essential domains of t-dimension one, and if D has t-dimension one then $\mathrm{Ass}(D) = X^1(D)$.

Corollary 2.3. *For any locally essential domain* D *of* t*-dimension one, the* D*-module* $\mathbb{B}_x(D)$ *is locally free.*

If we take $x = 0$ in the previous results, we derive the following corollary.

Corollary 2.4. *Let D be an integral domain. We have:*

(1) *If D is locally essential, then* $\mathrm{Int}(D)$ *is a faithfully flat D-module.*

(2) *If D is locally essential with* $\mathrm{Ass}(D) = X^1(D)$ *(this holds, for example, if either D is t-almost Dedekind, almost Krull, generalized Krull, or locally essential of t-dimension one), then* $\mathrm{Int}(D)$ *is locally free as a D-module.*

It is well-known that T is a localization of an integral domain R if and only if $T = R_S$, where $S = \{r \in R,\ r \text{ is a unit in } T\}$ (cf. [16], Introduction). Then, from the fact that the set $\{f \in D[X],\ f \text{ is a unit in } \mathbb{B}_x(D)\}$ is exactly $\mathcal{U}(D)$, the multiplicative group of units of D, we deduce the following:

Proposition 2.5. *Let D be an integral domain and let x be an element of D. Then $\mathbb{B}_x(D)$ is never a localization of $D[X]$ unless in the trivial case.*

For an overring R of an integral domain D, we recall that R is said to be *t-linked over D* if, for each nonzero finitely generated fractional ideal I of D, $I^{-1} = D$ implies that $(IR)^{-1} = R$. Notice that any flat overring is t-linked. An integral domain D is called a *GCD domain* if the intersection of two principal ideals of D is principal. Notice that valuation domains are GCD domains.

Corollary 2.6. *Let D be a GCD domain and let x be an element of D. Then $\mathbb{B}_x(D)$ is t-linked over $D[X]$ if and only if $\mathbb{B}_x(D)$ is trivial.*

P r o o f. This is an immediate application of Proposition 2.5 since GCD domains have the property that every t-linked overring is a localization (cf. [9], Corollary 3.8). □

In what follows we deal with the flatness of $D[X] \hookrightarrow \mathbb{B}_x(D)$ when D is an essential domain.

Theorem 2.7. *Let D be an essential domain and let x be an element of D. Then $\mathbb{B}_x(D)$ is flat over $D[X]$ if and only if $\mathbb{B}_x(D)$ is trivial.*

P r o o f. The direct implication is obvious. For the converse, assume that $\mathbb{B}_x(D)$ is flat over $D[X]$. Since D is essential, there exists a subset $\mathcal{P}$ of $\mathrm{Spec}(D)$ such that $D = \bigcap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$ and $D_{\mathfrak{p}}$ is a valuation domain for each $\mathfrak{p} \in \mathcal{P}$. So, let $\mathfrak{p} \in \mathcal{P}$. Then $D_{\mathfrak{p}}$ is a valuation domain and hence $D_{\mathfrak{p}}[X]$ is GCD [13], Theorem 34.10. Thus the flatness of $\mathbb{B}_x(D)_{\mathfrak{p}}$ as an overring of $D_{\mathfrak{p}}[X]$ and [3], Corollary 2 ensure that $\mathbb{B}_x(D)_{\mathfrak{p}}$ is a localization of $D_{\mathfrak{p}}[X]$. Therefore, as cited before Proposition 2.5, $\mathbb{B}_x(D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X]_S$, where $S = \{f \in D_{\mathfrak{p}}[X],\ f \text{ is a unit in } \mathbb{B}_x(D)_{\mathfrak{p}}\}$. Consequently, $\mathbb{B}_x(D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$ and hence $\bigcap_{\mathfrak{p}\in\mathcal{P}} \mathbb{B}_x(D)_{\mathfrak{p}} = \bigcap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}[X] = D[X]$. Therefore, by Lemma 1.1, $\mathbb{B}_x(D) = D[X]$, as desired. □

We next recover the case of the ring of integer-valued polynomials.

Corollary 2.8 ([19], Theorem 3)**.** *Let D be an essential domain. Then* $\operatorname{Int}(D)$ *is flat over $D[X]$ if and only if* $\operatorname{Int}(D) = D[X]$.

P r o o f. Just take $x = 0$ in Theorem 2.7. □

We next state some conditions under which the Bhargava ring $\mathbb{B}_x(D)$ is trivial.

Theorem 2.9. *Let D be an essential domain with the defining family $\mathcal{P}$ and let x be an element of D. If x is a nonunit in $D_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathcal{P}$, then the following statements are equivalent.*

(1) $\mathbb{B}_x(D) = D[X]$ *and* $\mathbb{B}_x(D_{\mathfrak{p}}) = \mathbb{B}_x(D)_{\mathfrak{p}}$ *for each* $\mathfrak{p} \in \mathcal{P}$;

(2) $\mathbb{B}_x(D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$ *for each* $\mathfrak{p} \in \mathcal{P}$;

(3) $D_{\mathfrak{p}}$ *has an infinite residue field or nonprincipal maximal ideal for each* $\mathfrak{p} \in \mathcal{P}$.

To prove this result we need the following well-known result.

Lemma 2.10. *Let V be a valuation domain and let x be a nonunit element of V. Then $\mathbb{B}_x(V) = V[X]$ if and only if the maximal ideal of V is not principal or its residue field is infinite.*

P r o o f. This is a consequence of [22], Proposition 1.3 (1). □

P r o o f of Theorem 2.9. (1) $\Rightarrow$ (2). This is clear.

(2) $\Rightarrow$ (1). Since $D = \bigcap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$ and $\mathbb{B}_x(D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$ for each $\mathfrak{p} \in \mathcal{P}$, then it follows from Lemma 1.1 that

$$\mathbb{B}_x(D) = \bigcap_{\mathfrak{p}\in\mathcal{P}} \mathbb{B}_x(D_{\mathfrak{p}}) = \bigcap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}[X] = D[X].$$

The second statement follows from the inclusions $D_{\mathfrak{p}}[X] \subseteq \mathbb{B}_x(D)_{\mathfrak{p}} \subseteq \mathbb{B}_x(D_{\mathfrak{p}})$.

(2) $\Leftrightarrow$ (3). This follows from Lemma 2.10 because $D_{\mathfrak{p}}$ is a valuation domain for each $\mathfrak{p} \in \mathcal{P}$. □

Following [20], an integral domain D is called a *Mott-Zafrullah DVR* (in short, an MZ-DVR) if $D_{\mathfrak{p}}$ is a DVR for each $\mathfrak{p} \in \operatorname{Ass}(D)$. Notice that (almost) Krull domains and (t-)almost Dedekind domains are MZ-DVRs, and any MZ-DVR D is an essential domain with the defining family $X^1(D)$.

Corollary 2.11. *Let D be an MZ-DVR and let x be an element of D. If x is a nonunit in $D_{\mathfrak{p}}$ for each $\mathfrak{p} \in X^1(D)$, then the following statements are equivalent.*

(1) $\mathbb{B}_x(D) = D[X]$ *and* $\mathbb{B}_x(D_{\mathfrak{p}}) = \mathbb{B}_x(D)_{\mathfrak{p}}$ *for each* $\mathfrak{p} \in X^1(D)$;

(2) $\mathbb{B}_x(D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$ *for each* $\mathfrak{p} \in X^1(D)$;

(3) *Each height-one prime ideal of D has an infinite residue field.*

An integral domain D is called a *Krull-type domain* if it is an essential domain with the defining family $\mathcal{P}$ such that the intersection $\bigcap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$ is locally finite. So, it follows from Proposition 1.3 and Theorem 2.9:

Corollary 2.12. *Let D be a Krull-type domain with the defining family $\mathcal{P}$ and let x be an element of D. If x is a nonunit in $D_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathcal{P}$, then the following statements are equivalent.*
(1) $\mathbb{B}_x(D) = D[X]$;
(2) $\mathbb{B}_x(D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$ *for each* $\mathfrak{p} \in \mathcal{P}$;
(3) $D_{\mathfrak{p}}$ *has an infinite residue field or nonprincipal maximal ideal for each* $\mathfrak{p} \in \mathcal{P}$.

Corollary 2.13. *Let D be a Krull domain and let x be an element of D. If x is a nonunit in $D_{\mathfrak{p}}$ for each $\mathfrak{p} \in X^1(D)$, then $\mathbb{B}_x(D) = D[X]$ if and only if each height-one prime ideal of D has an infinite residue field.*

Remark 2.14. Notice that the condition "$\mathbb{B}_x(D_{\mathfrak{p}}) = \mathbb{B}_x(D)_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathcal{P}$" is always true for any Krull-type domain D as asserted in Proposition 1.3. However, this is not the case for almost Dedekind domains. For instance, [22], Example 2.13 provides an almost Dedekind domain, and hence an essential domain, D such that $\mathbb{B}_x(D) = D[X]$ for all $x \in D$, but $\mathbb{B}_x(D_{\mathfrak{p}}) \neq \mathbb{B}_x(D)_{\mathfrak{p}}$ for some $\mathfrak{p} \in X^1(D)$. Therefore, the condition "$\mathbb{B}_x(D_{\mathfrak{p}}) = \mathbb{B}_x(D)_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathcal{P}$" is not superfluous in Theorem 2.9.

To prove our next main result, we shall need the following preliminary lemmas. In particular, the first lemma can be viewed as a generalization of [25], Lemma 1.2.

Lemma 2.15. *Let R be a commutative ring. For any R-algebra A, one has*

$$\dim(A) = \sup\{\dim(A_{\mathfrak{m}}), \mathfrak{m} \in \operatorname{Max}(R)\},$$

where $A_{\mathfrak{m}} := A_{R\setminus\mathfrak{m}}$ denotes the localization of A at $R \setminus \mathfrak{m}$.

Proof. It is well-known that $\dim(A) = \sup\{\dim(A_{\mathfrak{M}}), \mathfrak{M} \in \operatorname{Max}(A)\}$. The inequality "$\geqslant$" follows from the fact that $\dim(A) \geqslant \dim(A_{\mathfrak{m}})$ for each maximal ideal $\mathfrak{m}$ of R. For the reverse inequality, let $\mathfrak{M}$ be a maximal ideal of A and set $\mathfrak{p} := \mathfrak{M} \cap R$. Clearly, $\mathfrak{p}$ is a prime ideal of R and so it is contained in a maximal ideal $\mathfrak{m}$ of R. Then $A_{\mathfrak{M}} = (A_{\mathfrak{p}})_{\mathfrak{M}}$ and $A_{\mathfrak{p}} = (A_{\mathfrak{m}})_{\mathfrak{p}}$, and hence

$$\dim(A_{\mathfrak{M}}) \leqslant \dim(A_{\mathfrak{p}}) \leqslant \dim(A_{\mathfrak{m}}).$$

Thus, $\dim(A_{\mathfrak{M}}) \leqslant \sup\{\dim(A_{\mathfrak{m}}), \mathfrak{m} \in \operatorname{Max}(R)\}$ and therefore

$$\dim(A) \leqslant \sup\{\dim(A_{\mathfrak{m}}), \mathfrak{m} \in \operatorname{Max}(R)\}.$$

Consequently, $\dim(A) = \sup\{\dim(A_{\mathfrak{m}}), \mathfrak{m} \in \operatorname{Max}(R)\}$. □

Recall that the *valuative dimension* of a domain D, denoted by $\dim_v(D)$, is defined to be the supremum of Krull dimensions of the valuation overrings of D. A finite Krull dimensional domain D is said to be *Jaffard* if $\dim_v(D) = \dim(D)$, where $\dim(D)$ denotes the Krull dimension of D. For instance, in the finite Krull dimensional case, Noetherian domains and Prüfer domains are examples of Jaffard domains.

Lemma 2.16. *Let D be an integral domain with the quotient field K and let R be a ring between $D[X]$ and $D + XK[X]$. Then D is Jaffard if and only if R is Jaffard and* $\dim(R) = 1 + \dim(D)$.

Proof. This is a particular case of [2], Theorem 2 (3). □

An integral domain D is called an *MZ-Jaffard domain* if $D_{\mathfrak{p}}$ is Jaffard for each $\mathfrak{p} \in \operatorname{Ass}(D)$. In the finite Krull dimensional setting, it is clear that any locally essential domain is MZ-Jaffard.

Now, we are ready to calculate the Krull dimension of Bhargava rings over MZ-Jaffard domains.

Theorem 2.17. *Let D be an MZ-Jaffard domain and let x be an element of D. Then,* $\dim(\mathbb{B}_x(D)) = \dim(D[X])$.

Proof. Let $\mathfrak{m}$ be a maximal ideal of D and we examine the following two possible cases:

Case 1: $\mathfrak{m} \notin \operatorname{Ass}(D)$. Then, by Proposition 1.6, $\dim(\mathbb{B}_x(D)_{\mathfrak{m}}) = \dim(D_{\mathfrak{m}}[X])$.

Case 2: $\mathfrak{m} \in \operatorname{Ass}(D)$. As D is an MZ-Jaffard domain, $D_{\mathfrak{m}}$ is a Jaffard domain and then it follows from Lemma 2.16 that $\dim(\mathbb{B}_x(D)_{\mathfrak{m}}) = 1 + \dim(D_{\mathfrak{m}}) = \dim(D_{\mathfrak{m}}[X])$ because $D_{\mathfrak{m}}[X] \subseteq \mathbb{B}_x(D)_{\mathfrak{m}} \subseteq D_{\mathfrak{m}} + XK[X]$.

Consequently, $\dim(\mathbb{B}_x(D)_{\mathfrak{m}}) = \dim(D_{\mathfrak{m}}[X])$ for each maximal ideal $\mathfrak{m}$ of D, and therefore the desired equality follows by applying Lemma 2.15. □

As a consequence of Theorem 2.17, we deduce [12], Theorem 2.1 as a corollary.

Corollary 2.18 ([12], Theorem 2.1)**.** *For any locally essential domain D, we have* $\dim(\operatorname{Int}(D)) = \dim(D[X])$.

An integral domain D with the quotient field K is said to be *seminormal* if, for each $\alpha \in K$, whenever $\alpha^2, \alpha^3 \in D$, then $\alpha \in D$. Note that Prüfer domains form a subclass of seminormal domains.

It is well-known that $\operatorname{Int}(D)$ is a seminormal domain if and only if so is D (cf. [4], Proposition 2.7).

Proposition 2.19. *Let D be an integral domain with the quotient field K and let x be an element of D. Then $\mathbb{B}_x(D)$ is a seminormal domain if and only if so is D.*

P r o o f. Assume that $\mathbb{B}_x(D)$ is a seminormal domain and let $\alpha \in K$ be such that $\alpha^2, \alpha^3 \in D$. Since $D \subseteq \mathbb{B}_x(D)$, $\alpha^2, \alpha^3 \in \mathbb{B}_x(D)$ and then, by the seminormality of $\mathbb{B}_x(D)$, $\alpha \in \mathbb{B}_x(D)$. Thus $\alpha \in \mathbb{B}_x(D) \cap K = D$ and therefore D is a seminormal domain. For the converse, assume that D is a seminormal domain. Let $f \in K(X)$ such that $f^2, f^3 \in \mathbb{B}_x(D)$ and let $a \in D$. Since $K[X]$ is seminormal, $f \in K[X]$. We let c_i denote the coefficient of X^i in $f(xX + a)$ (which is an element of K). Then c_i^2 and c_i^3 appear in $f^2(xX+a)$ and $f^3(xX+a)$, respectively. Hence, as $f^2(xX+a)$ and $f^3(xX + a)$ are polynomials of $D[X]$, $c_i^2, c_i^3 \in D$ and thus, by seminormality of D, $c_i \in D$, i.e., $f(xX + a) \in D[X]$. Therefore, $f \in \mathbb{B}_x(D)$ and consequently, $\mathbb{B}_x(D)$ is a seminormal domain. □

By taking $x = 0$ in Proposition 2.19, we deduce the following corollary.

Corollary 2.20. *Let D be an integral domain. Then $\mathrm{Int}(D)$ is a seminormal domain if and only if so is D.*

An integral domain D is said to be a *Prüfer v-multiplication domain* (in short, a PvMD) if $D_{\mathfrak{m}}$ is a valuation domain for each t-maximal ideal $\mathfrak{m}$ of D. Notice that generalized Krull domains, Krull-type domains and t-almost Dedekind domains are PvMDs.

Proposition 2.21. *Let D be a generalized Krull domain and let x be an element of D. Then, $\mathbb{B}_x(D)$ is a PvMD. If, in addition, $x \neq 0$ then $\mathbb{B}_x(D)$ is of t-dimension one.*

P r o o f. Since D is a generalized Krull domain, $D_{\mathfrak{m}}$ is a one-dimensional valuation domain for each t-maximal ideal $\mathfrak{m}$ of D, and then, by [22], Corollary 1.9, $\mathbb{B}_x(D_{\mathfrak{m}})$ is a PvMD. Therefore the thesis follows from [22], Theorem 2.6. Moreover, if $x \neq 0$ then the t-dimension of $\mathbb{B}_x(D)$ is one as asserted in [22], Corollary 2.10. □

For the sake of illustration, we list the following examples.

E x a m p l e 2.22. Let x be an element of $\mathbb{Z}$. It is well-known that $\mathbb{Z}$ is a PID (principal ideal domain) with finite residue fields. By [8], Corollary II.1.6, $\mathbb{B}_x(\mathbb{Z})$ is a free $\mathbb{Z}$-module. If, moreover, x is nonzero, it follows from [27], Corollaire 4.5 and Lemma 2.16 that $\mathbb{B}_x(\mathbb{Z})$ is a two-dimensional Noetherian integrally closed domain, and hence it is a Krull domain that is not Prüfer. Now, if x is a nonzero nonunit in $\mathbb{Z}$, the factor ring $\mathbb{Z}/(x)$ is finite and then it follows from [1], Proposition 2.4 that $\mathbb{B}_x(\mathbb{Z})$ is not trivial. Hence, by Theorem 2.7, $\mathbb{B}_x(\mathbb{Z})$ is not flat over $\mathbb{Z}[X]$. We note that $\mathbb{B}_0(\mathbb{Z})$ is also not flat over $\mathbb{Z}[X]$ because $\mathbb{B}_0(\mathbb{Z}) = \mathrm{Int}(\mathbb{Z})$ and $\mathrm{Int}(\mathbb{Z}) \neq \mathbb{Z}[X]$.

R e m a r k 2.23. From this last example, we can see that $\mathbb{B}_x(\mathbb{Z})$ is not a Prüfer domain for any nonzero element x of $\mathbb{Z}$, but this is not the case for $\mathrm{Int}(\mathbb{Z}) = \mathbb{B}_0(\mathbb{Z})$. In fact, as pointed out by the referee, if D is an integral domain different from its quotient field and x a nonzero element of D then $\mathbb{B}_x(D)$ cannot be Prüfer because it is contained in $D[(X-a)/x]$, and $D[(X-a)/x] \cong D[X]$ which is not Prüfer. Thus, the only case, in which it can occur that $\mathbb{B}_x(D)$ is Prüfer, is when $\mathbb{B}_x(D) = \mathrm{Int}(D)$.

E x a m p l e 2.24. Let $D = \mathbb{Z}[\{T/p_n\}_{n=1}^{\infty}]$, where $\{p_n\}_{n=1}^{\infty}$ is the set of all positive prime integers and T is an indeterminate over $\mathbb{Z}$. By [18], Example 166, D is a two-dimensional locally Noetherian integrally closed domain which is neither Noetherian nor Krull. Then D is an almost Krull domain and hence, by Corollary 2.2, $\mathbb{B}_x(D)$ is locally free as a D-module for any element x of D. Moreover, by Lemma 2.16, $\mathbb{B}_x(D)$ is a Jaffard domain of dimension 3 because locally Noetherian domains are Jaffard.

E x a m p l e 2.25. Let A be the domain of all algebraic integers and $\{p_n\}_{n=1}^{\infty}$ be the set of all positive prime integers. For each n choose a maximal ideal $\mathfrak{m}_n$ of A lying over $p_n\mathbb{Z}$ and set $D = A_S$, where $S = A \setminus \bigcup_{n=1}^{\infty} \mathfrak{m}_n$.

In [14], Example 1, page 338, Gilmer showed that D is a one-dimensional Prüfer domain which is not almost Dedekind. Then, by Corollary 2.3, $\mathbb{B}_x(D)$ is locally free as a D-module for any element x of D. Moreover, Pirtle in [24], page 439 pointed out that D is a generalized Krull domain and then it follows from Proposition 2.21 that $\mathbb{B}_x(D)$ is a PvMD of t-dimension one for any nonzero element x of D.

E x a m p l e 2.26. Let $\mathcal{E}$ be the ring of entire functions and set $D := \mathcal{E} + T\mathcal{E}_S[T]$, where T is an indeterminate over $\mathcal{E}$ and S is the set generated by the principal primes of $\mathcal{E}$.

According to [30], Example 2.6, D is a locally essential domain which is neither PvMD nor almost Krull. Thus, by Theorem 2.1, the D-module $\mathbb{B}_x(D)$ is faithfully flat for any element x of D.

We close this paper with the following two open questions.

(Q1) Is there an integral domain D such that $\mathbb{B}_x(D)$ is not (faithfully) flat as a D-module for some $x \in D$?

(Q2) Does the flatness of $\mathbb{B}_x(D)$ over $D[X]$ for some $x \in D$ force that $\mathbb{B}_x(D) = D[X]$? In particular, is there a nonessential domain D such that $\mathbb{B}_x(D)$ is flat over $D[X]$ but $\mathbb{B}_x(D) \neq D[X]$ for some $x \in D$?

A c k n o w l e d g e m e n t s. The authors would like to thank the referee for careful reading and for providing Remark 2.23.

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