ON UNITS OF SOME FIELDS OF THE FORM $\mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{-l})$

Mohamed Mahmoud Chems-Eddin, Fez

Abstract. Let $p \equiv 1 \pmod 8$ and $q \equiv 3 \pmod 8$ be two prime integers and let $l \notin \{-1, p, q\}$ be a positive odd square-free integer. Assuming that the fundamental unit of $\mathbb{Q}(\sqrt{2p})$ has a negative norm, we investigate the unit group of the fields $\mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{-l})$.

Keywords: multiquadratic number field; unit group; fundamental system of units

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1. Introduction

Let k be a number field of degree n (i.e., $[k : \mathbb{Q}] = n$). Denote by E_k the unit group of k that is the group of the invertible elements of the ring $\mathcal{O}_k$ of algebraic integers of the number field k. By the well known Dirichlet's unit theorem, if $n = r_1 + 2r_2$, where r_1 is the number of real embeddings and r_2 the number of conjugate pairs of complex embeddings of k, then there exist $r = r_1 + r_2 - 1$ units of $\mathcal{O}_k$ that generate E_k (modulo the roots of unity), and these r units are called the *fundamental system of units of k*. Therefore, it is well known that

$$E_k \simeq \mu(k) \times \mathbb{Z}^{r_1 + r_2 - 1},$$

where $\mu(k)$ is the group of roots of unity contained in k.

One major problem in algebraic number theory (more precisely in the theory of units of number fields which is related to almost all areas of algebraic number theory) is the computation of the fundamental system of units. For quadratic fields, the problem is easily solved. An early study of unit groups of multiquadratic fields was established by Varmon in [7]. For quartic bicyclic fields, Kubota (see [6]) gave a method for finding a fundamental system of units. Wada in [8] generalized Kubota's method, creating an algorithm for computing fundamental units in any given multiquadratic

field. However, in general, it is not easy to compute the unit group of a number field, especially for number fields of degree more than 4. Actually, in the literature there are only a few examples of computation of the unit group of a given number field k of degree 16 (see [3], [4], [5]). Let ε_l denote the *fundamental unit* of the quadratic field $\mathbb{Q}(\sqrt{l})$. The main goal of this paper is the determination of the seven generators of the torsion-free subgroup of E_k for an infinite family of number fields k of degree 16 of the form $\mathbb{L} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{-l})$, where $p \equiv 1 \pmod 8$ (with $N(\varepsilon_{2p}) = -1$) and $q \equiv 3 \pmod 8$ are two prime integers and $l \notin \{-1, p, q\}$ is a positive odd square-free integer. Before stating our main result, we need to fix some notations. Let $q(k)$ be the unit index of a multiquadratic number field k and $\eta_l = \zeta_6$ or -1 according to whether $l = 3$ or not. Let $\mathbb{L}^+$ denote the *maximal real subfield* of $\mathbb{L}$.

Finally, the letters x and y (or v and w) denote two integers such that $\varepsilon_{2pq} = x + y\sqrt{2pq}$ (or $\varepsilon_{pq} = v + w\sqrt{pq}$). By [2], Lemma 3.4, $(x-1)$, $p(x-1)$ or $2p(x+1)$ (respectively, $(v-1)$, $p(v-1)$ or $2p(v+1)$) is a square in $\mathbb{N}$. The main theorem of this paper is the following.

Theorem 1.1. *Let $p \equiv 1 \pmod 8$ and $q \equiv 3 \pmod 8$ be two primes. Assume furthermore that $N(\varepsilon_{2p}) = -1$. Put $\mathbb{L} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{-l})$. Then*
(1) *If $x - 1$ and $v - 1$ are squares in $\mathbb{N}$ and $q(\mathbb{L}^+) = 6$, we have:*

$$E_\mathbb{L} = \left\langle \eta_l, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}, \sqrt{\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}}, \sqrt{-\varepsilon_p^\alpha \sqrt{\varepsilon_q}^{1+\gamma}\sqrt{\varepsilon_{pq}}^\alpha} \right\rangle,$$

where $\alpha, \gamma \in \{0, 1\}$ such that $\alpha \neq \gamma$ and $\alpha = 1$ if and only if $l\varepsilon_p\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}}$ is a square in $\mathbb{L}^+$.
(2) *If $p(x - 1)$ and $2p(v + 1)$ are squares in $\mathbb{N}$ and $q(\mathbb{L}^+) = 6$, we have*

$$E_\mathbb{L} = \left\langle \eta_l, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}, \sqrt{\varepsilon_p\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}} \right\rangle.$$

(3) *If $2p(x + 1)$ and $p(v - 1)$ are squares in $\mathbb{N}$ and $q(\mathbb{L}^+) = 6$, we have*

$$E_\mathbb{L} = \left\langle \eta_l, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}, \sqrt{\varepsilon_p\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}} \right\rangle.$$

(4) *If $x - 1$ and $p(v - 1)$ are squares in $\mathbb{N}$, then*

$$E_\mathbb{L} = \left\langle \eta_l, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}, \sqrt{-\varepsilon_p^\alpha \sqrt{\varepsilon_{2q}}^{1+\gamma}\sqrt{\varepsilon_{2pq}}^\alpha} \right\rangle,$$

where $\alpha, \gamma \in \{0, 1\}$ such that $\alpha \neq \gamma$ and $\alpha = 1$ if and only if $l\varepsilon_p\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{2pq}}$ is a square in $\mathbb{L}^+$.
(5) *If $x - 1$ and $2p(v + 1)$ are squares in $\mathbb{N}$, then*

$$E_\mathbb{L} = \left\langle \eta_l, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}, \sqrt{-\varepsilon_p^\alpha \sqrt{\varepsilon_{2q}}^{1+\gamma}\sqrt{\varepsilon_{2pq}}^\alpha} \right\rangle,$$

where $\alpha, \gamma \in \{0, 1\}$ such that $\alpha \neq \gamma$ and $\alpha = 1$ if and only if $l\varepsilon_p\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{2pq}}$ is a square in $\mathbb{L}^+$.

(6) If $p(x-1)$ and $(v-1)$ are squares in $\mathbb{N}$, then

$$E_{\mathbb{L}} = \left\langle \eta_l, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}, \sqrt{-\varepsilon_p^\alpha \sqrt{\varepsilon_q}^{1+\gamma} \sqrt{\varepsilon_{pq}}^\alpha} \right\rangle,$$

where $\alpha, \gamma \in \{0,1\}$ such that $\alpha \neq \gamma$ and $\alpha = 1$ if and only if $l\varepsilon_p\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}}$ is a square in $\mathbb{L}^+$.

(7) If $2p(x+1)$ and $(v-1)$ are squares in $\mathbb{N}$, then

$$E_{\mathbb{L}} = \left\langle \eta_l, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}, \sqrt{-\varepsilon_p^\alpha \sqrt{\varepsilon_q}^{1+\gamma} \sqrt{\varepsilon_{pq}}^\alpha} \right\rangle,$$

where $\alpha, \gamma \in \{0,1\}$ such that $\alpha \neq \gamma$ and $\alpha = 1$ if and only if $l\varepsilon_p\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}}$ is a square in $\mathbb{L}^+$.

(8) If $2p(x+1)$ and $2p(v+1)$ are squares in $\mathbb{N}$, then

$$E_{\mathbb{L}} = \left\langle \eta_l, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}, \sqrt{-\sqrt{\varepsilon_{pq}}^{1+\gamma} \sqrt{\varepsilon_{2pq}}^\alpha} \right\rangle,$$

where $\alpha, \gamma \in \{0,1\}$ such that $\alpha \neq \gamma$ and $\alpha = 1$ if and only if $l\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}$ is a square in $\mathbb{L}^+$.

2. Some preparations for the proof of Theorem 1.1

We start by collecting and fixing some notations which will be very useful for the next section. The technique which we shall use is based on the following lemma.

Lemma 2.1. *Let $K_0/\mathbb{Q}$ be an abelian extension such that K_0 is real and β is a positive square-free algebraic integer of K_0. Assume that $K = K_0\big(\sqrt{-\beta}\big)$ is a quadratic extension of K_0, which is abelian over $\mathbb{Q}$. Assume furthermore that $\mathrm{i} = \sqrt{-1} \notin K$. Let $\{\varepsilon_1, \ldots, \varepsilon_r\}$ be a fundamental system of units of K_0. Without loss of generality, we may suppose that the units ε_i are positives. Let ε be a unit of K_0 such that $\beta\varepsilon$ is a square in K_0 (if it exists). Then a fundamental system of units of K is one of the following systems:*
(a) $\{\varepsilon_1, \ldots, \varepsilon_{r-1}, \sqrt{-\varepsilon}\}$ *if ε exists, in this case $\varepsilon = \varepsilon_1^{j_1} \ldots \varepsilon_{r-1}^{j_1}\varepsilon_r$, where $j_i \in \{0,1\}$.*
(b) $\{\varepsilon_1, \ldots, \varepsilon_r\}$ *else.*

P r o o f.   See [1], Proposition 3. $\qquad\qquad\square$

As it is stated in the above lemma, the unit group of $\mathbb{L}$ comes from that of $\mathbb{L}^+$ (the maximal real subfield of $\mathbb{L}$). However, this is not easy and demands some technical computations and eliminations, as we will see bellow. We need to define

the following morphisms: Let τ_1, τ_2 and τ_3 be the elements of $\mathrm{Gal}(\mathbb{L}^+/\mathbb{Q})$ defined by

$$\tau_1(\sqrt{2}) = -\sqrt{2}, \quad \tau_1(\sqrt{p}) = \sqrt{p}, \quad \tau_1(\sqrt{q}) = \sqrt{q},$$
$$\tau_2(\sqrt{2}) = \sqrt{2}, \quad \tau_2(\sqrt{p}) = -\sqrt{p}, \quad \tau_2(\sqrt{q}) = \sqrt{q},$$
$$\tau_3(\sqrt{2}) = \sqrt{2}, \quad \tau_3(\sqrt{p}) = \sqrt{p}, \quad \tau_3(\sqrt{q}) = -\sqrt{q}.$$

Note that $\mathrm{Gal}(\mathbb{L}^+/\mathbb{Q}) = \langle \tau_1, \tau_2, \tau_3 \rangle$ and the subfields k_1, k_2 and k_3 are fixed by $\langle \tau_3 \rangle$, $\langle \tau_2 \rangle$ and $\langle \tau_2\tau_3 \rangle$, respectively. With these notations we have the following tables which we shall use frequently (cf. [2], Tables 1 and 2):

ε	ε_2	ε_p	$\sqrt{\varepsilon_q}$	$\sqrt{\varepsilon_{2q}}$
$\varepsilon^{1+\tau_1}$	-1	ε_p^2	$-\varepsilon_q$	1
$\varepsilon^{1+\tau_2}$	ε_2^2	-1	ε_q	ε_{2q}
$\varepsilon^{1+\tau_3}$	ε_2^2	ε_p^2	-1	-1
$\varepsilon^{1+\tau_1\tau_2}$	-1	-1	$-\varepsilon_q$	1
$\varepsilon^{1+\tau_1\tau_3}$	-1	ε_p^2	1	$-\varepsilon_{2q}$
$\varepsilon^{1+\tau_2\tau_3}$	ε_2^2	-1	-1	-1

Table 1. Norm maps on units.

ε	Conditions	$\varepsilon^{1+\tau_2}$	$\varepsilon^{1+\tau_1\tau_2}$	$\varepsilon^{1+\tau_1\tau_3}$	$\varepsilon^{1+\tau_2\tau_3}$	$\varepsilon^{1+\tau_1}$
	$(x-1)$ is a square in $\mathbb{N}$	-1	$-\varepsilon_{2pq}$	$-\varepsilon_{2pq}$	ε_{2pq}	1
$\sqrt{\varepsilon_{2pq}}$	$p(x-1)$ is a square in $\mathbb{N}$	1	ε_{2pq}	$-\varepsilon_{2pq}$	$-\varepsilon_{2pq}$	1
	$2p(x+1)$ is a square in $\mathbb{N}$	-1	$-\varepsilon_{2pq}$	ε_{2pq}	$-\varepsilon_{2pq}$	1
	$(v-1)$ is a square in $\mathbb{N}$	-1	1	1	ε_{pq}	$-\varepsilon_{pq}$
$\sqrt{\varepsilon_{pq}}$	$p(v-1)$ is a square in $\mathbb{N}$	1	-1	1	$-\varepsilon_{pq}$	$-\varepsilon_{pq}$
	$2p(v+1)$ is a square in $\mathbb{N}$	-1	-1	1	$-\varepsilon_{pq}$	ε_{pq}

Table 2. Norm maps on units.

3. Proof of Theorem 1.1

The proof of our main result is very long, so in this section, we shall prove some items and clarify the techniques used therein. A patient reader can proceed analogously and prove the rest. Let us start with the first item:

(1) Assume that the conditions of the first point hold. According to [2], Theorems 3.3 and 3.5, a fundamental system of units of $\mathbb{L}^+$ is

$$\left\{ \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}, \sqrt{\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}} \right\}.$$

Therefore, according to Lemma 2.1, we should find an element χ, if it exists, which is in $\mathbb{L}^+$ such that

$$\chi^2 = l\varepsilon_2^a \varepsilon_p^b \sqrt{\varepsilon_q}^c \sqrt{\varepsilon_{2q}}^d \sqrt{\varepsilon_{pq}}^e \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^f \sqrt{\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}}^g,$$

where a, b, c, d, e, f and g are in $\{0,1\}$. Now we use the norm maps from $\mathbb{L}^+$ to its biquadratic subfields. These norms are given by Tables 1 and 2.

▷ Let us start by applying $N_{\mathbb{L}^+/K_1} = 1 + \tau_2$, where $K_1 = \mathbb{Q}(\sqrt{2}, \sqrt{q})$. By Tables 1 and 2, we have $N_{\mathbb{L}^+/K_1}\left(\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}\right) = \varepsilon_q\varepsilon_{2q}$. Thus,

$$N_{\mathbb{L}^+/K_1}\left(\sqrt{\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}}\right) = (-1)^s\sqrt{\varepsilon_q\varepsilon_{2q}},$$

and similarly $N_{\mathbb{L}^+/K_1}\left(\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\right) = (-1)^v\varepsilon_2$, for some s and v in $\{0,1\}$. Therefore, we have:

$$N_{\mathbb{L}^+/K_1}(\chi^2) = l^2\varepsilon_2^{2a}(-1)^b\varepsilon_q^c\varepsilon_{2q}^d(-1)^e(-1)^{fv}\varepsilon_2^f(-1)^{gs}\sqrt{\varepsilon_q\varepsilon_{2q}}^g$$
$$= l^2\varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d(-1)^{b+e+fv+gs}\varepsilon_2^f\sqrt{\varepsilon_q\varepsilon_{2q}}^g > 0,$$

Therefore $b + e + fv + gs = 0 \pmod 2$. By [2], Lemma 3.2, ε_q and ε_{2q} are squares in K_1, but ε_2, $\varepsilon_q\sqrt{\varepsilon_{2q}}$ and $\varepsilon_2\varepsilon_q\sqrt{\varepsilon_{2q}}$ are not squares in K_1. Thus, we have $f = g = 0$ and so $b = e$. It follows that

$$\chi^2 = l\varepsilon_2^a\varepsilon_p^e\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e.$$

▷ Let us apply $N_{\mathbb{L}^+/K_2} = 1 + \tau_1$ with $K_2 = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. So

$$N_{\mathbb{L}^+/K_2}(\chi^2) = l^2(-1)^a\varepsilon_p^{2e}(-1)^c\varepsilon_q^c1(-1)^e\varepsilon_{pq}^e = l^2\varepsilon_p^{2e}(-1)^{a+c+e}\varepsilon_q^c\varepsilon_{pq}^e > 0.$$

Thus $a+c+e = 0 \pmod 2$. Note that by [2], Lemmas 3.2 and 3.4, ε_q and ε_{pq} are not squares in K_2 but their product is. So $c = e$. Therefore $a = 0$ and

$$\chi^2 = l\varepsilon_p^e\sqrt{\varepsilon_q}^e\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e.$$

▷ Let us now apply $N_{\mathbb{L}^+/K_3} = 1 + \tau_2\tau_3$, with $K_3 = \mathbb{Q}(\sqrt{2}, \sqrt{pq})$. We have

$$N_{\mathbb{L}^+/K_3}(\chi^2) = l^2(-1)^e(-1)^e(-1)^d\varepsilon_{pq}^e = l^2\varepsilon_{pq}^e(-1)^d > 0.$$

Thus $d = 0$ and

$$\chi^2 = l\varepsilon_p^e\sqrt{\varepsilon_q}^e\sqrt{\varepsilon_{pq}}^e.$$

By applying the other norms, we deduce nothing new. So the first item by Lemma 2.1 holds.

(2) Assume that the conditions of the second point hold. According to [2], Theorem 3.10, a fundamental system of units of $\mathbb{L}^+$ is

$$\left\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}, \sqrt{\varepsilon_p\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}}\right\}.$$

So we consider

$$\chi^2 = l\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^f\sqrt{\varepsilon_p\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}}^g,$$

where a, b, c, d, e, f and g are in $\{0, 1\}$. We shall proceed as in the previous item.

▷ Let us start by applying $N_{\mathbb{L}+/K_1} = 1 + \tau_2$, where $K_1 = \mathbb{Q}\big(\sqrt{2}, \sqrt{q}\big)$. We have:

$$N_{\mathbb{L}+/K_1}(\chi^2) = l^2 \varepsilon_2^{2a} (-1)^b \varepsilon_q^c \varepsilon_{2q}^d (-1)^e (-1)^{fv} \varepsilon_2^f (-1)^{gs} \sqrt{\varepsilon_{2q}}^g$$
$$= l^2 \varepsilon_2^{2a} \varepsilon_q^c \varepsilon_{2q}^d (-1)^{b+e+fv+gs} \varepsilon_2^f \sqrt{\varepsilon_{2q}}^g > 0,$$

Thus, we have $f = g = 0$ and so $b = e$. It follows that

$$\chi^2 = l \varepsilon_2^a \varepsilon_p^e \sqrt{\varepsilon_q}^c \sqrt{\varepsilon_{2q}}^d \sqrt{\varepsilon_{pq}}^e.$$

▷ Let us apply $N_{\mathbb{L}+/K_2} = 1 + \tau_1$ with $K_2 = \mathbb{Q}\big(\sqrt{p}, \sqrt{q}\big)$. So

$$N_{\mathbb{L}+/K_2}(\chi^2) = l^2 (-1)^a \varepsilon_p^{2e} (-1)^c \varepsilon_q^c 1 \varepsilon_{pq}^e = l^2 \varepsilon_p^{2e} \varepsilon_{pq}^e (-1)^{a+c} \varepsilon_q^c > 0.$$

Therefore $a = c = 0$ and $\chi^2 = l \varepsilon_p^e \sqrt{\varepsilon_{2q}}^d \sqrt{\varepsilon_{pq}}^e$.

▷ Let us now apply $N_{\mathbb{L}+/K_3} = 1 + \tau_2 \tau_3$, with $K_3 = \mathbb{Q}\big(\sqrt{2}, \sqrt{pq}\big)$. We have

$$N_{\mathbb{L}+/K_3}(\chi^2) = l^2 (-1)^e (-1)^d (-1)^e \varepsilon_{pq}^e = l^2 (-1)^d \varepsilon_{pq}^e > 0.$$

Thus $d = e = 0$. So the second item by Lemma 2.1 holds.

Using the same techniques, a patient reader can construct a detailed proof for the rest.

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Author's address: *Mohamed Mahmoud Chems-Eddin*, Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz, Fez, Morocco e-mail: `2m.chemseddin@gmail.com`.