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ON GOLDIE ABSOLUTE DIRECT SUMMANDS IN MODULAR LATTICES

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Abstract. Absolute direct summand in lattices is defined and some of its properties in modular lattices are studied. It is shown that in a certain class of modular lattices, the direct sum of two elements has absolute direct summand if and only if the elements are relatively injective. As a generalization of absolute direct summand (ADS for short), the concept of Goldie absolute direct summand in lattices is introduced and studied. It is shown that Goldie ADS property is inherited by direct summands. A necessary and sufficient condition is given for an element of modular lattice to have Goldie ADS.

Keywords: injective element; ejective element; Goldie extending element; absolute direct summand; Goldie absolute direct summand

MSC 2020: 06B05, 06C05, 06B99

1. Introduction

The purpose of this paper is to introduce and study absolute direct summands in a certain class of lattices. The concept of absolute direct summands was first studied in modules by Fuchs in 1970 (see [4]). He called such modules ADS modules. Then ADS modules were investigated in [2]. Takil in her Ph.D. thesis collected results in this direction (see [10]). Some characterizations of ADS modules and rings have been studied by Alahmadi et al. in [1]. They provided equivalent conditions for a module to have ADS. In 2015, Mutlu in [11] studied properties of ADS modules with respect to summand intersection property.

In 2018, Quynh et al. (see [9]) introduced and studied Goldie absolute direct summand, which is a generalization of Goldie extending modules and ADS modules. They analyzed when a direct sum of Goldie absolute direct summands is Goldie absolute direct summand in a module.

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In the present paper, an analogue of ADS module is defined and studied as an ADS lattice. As a generalization of absolute direct summand property, Goldie absolute direct summand is defined and some characteristics of Goldie absolute direct summands are analyzed by using the concept of mutual ejectivity. For later concepts see [12].

The undefined concepts of lattice theory used in this paper are from Grätzer (see [5]). The following definitions are from Călugăreanu (see [3]). Let $a, b \in L$ and $a \leqslant b$. Then a is said to be *essential* in b (or b is an essential extension of a) if there is no nonzero $c \leqslant b$ such that $a \wedge c = 0$. It is denoted by $a \leqslant_e b$. If $a \leqslant_e b$ and there is no $c > b$ such that $a \leqslant_e c$, then b is called a *maximal essential extension* of a. An element $a \in L$ is *closed* (or essentially closed) in b if a has no proper essential extension in b. It is denoted by $a \leqslant_{cl} b$.

The following concepts are defined by Nimbhorkar and Shroff in [7]. If $a, b \in L$ and b is a maximal element in the set $\{x\colon\ x \in L, a \wedge x = 0\}$, then b is said to be a *max-semicomplement* of a. This concept is different from that of a pseudocomplement of an element in a lattice. For example, in the lattice L shown in Figure 1, b, c are max-semicomplements of a but a does not have a pseudocomplement in L.

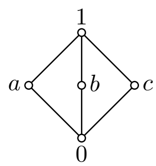

Figure 1.

Throughout this paper, L denotes a lattice with the least element 0 and wherever necessary, it is assumed that L satisfies one or more of the following conditions:

(C1) For any $a \leqslant b$ in L there exists a maximal essential extension of a in b.

(C2) For any $a \leqslant b$ in L, $c \leqslant b$ with $c \wedge a = 0$ there exists a max-semicomplement $d \geqslant c$ of a in b.

A familiar and important class of lattices with these conditions is upper continuous modular lattices, in particular, the lattice of all ideals of a modular lattice with 0. If $a, b, c \in L$ are such that $a \vee b = c$ and $a \wedge b = 0$, then a and b are called *direct summands* of c and it is denoted by $c = a \oplus b$. Here c is a direct sum of a and b. In a modular lattice L, the direct summands of $c \in L$ are closed in c and if $a, b, c \in L$ are such that $a \leqslant b \leqslant c$ and a is a direct summand of c, then a is also a direct summand of b. An element c of a lattice L is called *indecomposable* if $c = a \oplus b$ implies either $a = 0$ or $b = 0$. If for any two direct summands b, c of $a \in L$, $b \vee c$ is a direct summand of a, then a satisfies the *summand sum property*. Also, if for any two direct summands b, c of $a \in L$ with $b \wedge c \neq 0$, $b \wedge c$ is a direct summand of a, then a satisfies the *summand intersection property*. In a modular lattice, the summand sum (intersection) property is inherited by direct summand.

2. Absolute direct summands

In this section, absolute direct summand of an element of a lattice is defined.

Definition 2.1. Let L be a lattice with 0. An element $a \in L$ is said to have *absolute direct summands* if for every decomposition $a = a_1 \oplus a_2$ of a and every max-semicomplement b of a_1 in a, $a = a_1 \oplus b$.

In short, it is said that the element a has ADS property. A lattice L is said to have ADS property if every element a of L has ADS property.

Example 2.1. Consider the element f in the lattice shown in Figure 2. Here $f = a \oplus d$ and e is a max-semicomplement of a such that $f = a \oplus e$. Similarly, it can be checked for other decompositions of f. Hence, f has ADS property.

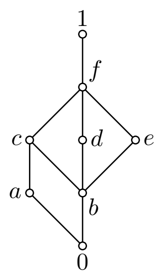

Figure 2.

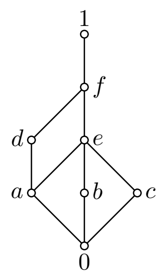

Figure 3.

Note that in a distributive lattice, every element has ADS but for modular lattice this fails. Consider the modular lattice L shown in Figure 3. In this lattice element $f = d \oplus c$ but b is a max-semicomplement of c such that $f \neq b \oplus c$. Hence, f does not have the ADS property.

In the following lemma it is proved that ADS property is inherited by direct summands.

Lemma 2.1. *Let L be a modular lattice satisfying conditions* (C1) *and* (C2) *and $a \in L$. If a has ADS, then every nonzero direct summand of a has ADS.*

Proof. Let $a = b \oplus c$ and b_1 be a direct summand of b. Let b_2 be a max-semicomplement of b_1 in b. Then by Theorem 2.2 of [6], $b_2 \oplus c$ is a max-semicomplement of b_1 in a. By ADS property of a, $a = b_1 \oplus (b_2 \oplus c)$. Now by using modularity of L,

$$b = a \wedge b = (b_1 \oplus b_2 \oplus c) \wedge b = (b_1 \oplus b_2) \oplus (c \wedge b) = b_1 \oplus b_2.$$

Hence, b has ADS property. □

Nimbhorkar and Shroff in [8] defined the injectivity in lattices as follows.

Definition 2.2. Let $a, b, c \in L$ be such that $a = b \oplus c$. Then c is said to be *b-injective* in a if for every $d \leqslant a$ with $d \wedge c = 0$ there exists $e \leqslant a$ such that $a = e \oplus c$ and $d \leqslant e$. If c is b-injective and b is c-injective in a, then b and c are said to be *relatively injective.*

In the following lemma, a necessary and sufficient condition is given for an element of a lattice to have ADS property.

Lemma 2.2. *Let L be a modular lattice satisfying conditions* (C1) *and* (C2). *An element $a \in L$ has ADS if and only if for every decomposition $a = a_1 \oplus a_2$ of a, a_1, a_2 are relatively injective.*

Proof. Let $a \in L$ have ADS and $a = a_1 \oplus a_2$. To show that a_1, a_2 are relatively injective let $d \leqslant a$ be such that $d \wedge a_2 = 0$. If d is a max-semicomplement of a_2, then by ADS property, $d \oplus a_2 = a$.

If d is not a max-semicomplement of a_2, then by condition (C2) there exists a max-semicomplement e of a_2 such that $d \leqslant e$. Again by ADS property, $e \oplus a_2 = a$. Hence, a_1 is a_2-injective.

Similarly, a_1-injectivity of a_2 can be proved.

Conversely, suppose that for each decomposition $a = a_1 \oplus a_2$ of a, a_1, a_2 are relatively injective. To show that a has ADS let $b \leqslant a$ be a max-semicomplement of a_1 in a. Since a_1 is a_2-injective, $b \wedge a_1 = 0$ implies that there exists $b_1 \leqslant a$ such that $a = a_1 \oplus b_1$, $b \leqslant b_1$. But b is maximal with the property that $b \wedge a_1 = 0$ yields that $b = b_1$. Hence a has ADS. □

The following remark is obvious from the above lemma.

Remark 2.1. Let L be a modular lattice satisfying conditions (C1) and (C2). An element $a \in L$ has ADS if and only if for any direct summand b of a such that $b \wedge c = 0$ for $c \leqslant a$, b is c-injective.

Proposition 2.1. *Let L be a modular lattice satisfying conditions* (C1) *and* (C2) *and $a, b, c \in L$ be such that a and b are closed in c. If $a \wedge b = 0$ and c has ADS, then $a \oplus b$ is closed in c.*

Proof. Let $a \oplus b$ be not closed in c, then by condition (C1), there exists a maximal essential extension d of $a \oplus b$ in c such that $a \oplus b \leqslant_e d$.

Now, $a \wedge b = 0$ implies that there exists a max-semicomplement k of a such that $b \leqslant k$. By ADS, $c = a \oplus k$ and by modularity of L for $b \leqslant k$,

$$a \oplus b \leqslant_e d \Rightarrow (a \oplus b) \wedge k \leqslant_e d \wedge k \Rightarrow b \leqslant_e d \wedge k.$$

But b is closed in c, so $b = d \wedge k$. Again, by modularity of L for $a \leqslant d$,

$$a \oplus b = a \oplus (k \wedge d) = (a \oplus k) \wedge d = c \wedge d = d.$$

Hence, $a \oplus b$ is closed in c. □

R e m a r k 2.2. Let L be a modular lattice satisfying conditions (C1) and (C2) and $a, b, c \in L$ be such that a and b are closed in c. If $a, a \wedge b$ are direct summands and c has ADS, then $a \vee b$ is closed in c.

P r o o f. Let k be a complement of $a \wedge b$ in c. Then by ADS, $c = (a \wedge b) \oplus k$. Now by modularity of L for $a \wedge b \leqslant b$,

$$b = c \wedge b = [(a \wedge b) \oplus k] \wedge b = (a \wedge b) \oplus (k \wedge b).$$

Then

$$a \vee b = a \vee [(a \wedge b) \vee (k \wedge b)] = [a \vee (a \wedge b)] \vee (k \wedge b) = a \vee (k \wedge b) = a \oplus (k \wedge b).$$

Hence, by Proposition 2.1, the proof is complete. □

Proposition 2.2. *Let L be a modular lattice satisfying conditions* (C1) *and* (C2) *and $a, b, c \in L$ be such that a, b are direct summands of c. If a has ADS and SIP, then $a \vee b$ is closed in c.*

P r o o f. Since c has SIP, $a \wedge b$ is a direct summand of a. Then by Remark 2.2, $a \vee b$ is closed in c. □

From [7], recall that for any finite number of nonzero elements $a_1, a_2, \ldots, a_n \in L$, $a_1 \vee \ldots \vee a_n$ is a direct sum if a_i's are join independent, i.e., $a_j \wedge \Big(\bigvee_{i=1, i \neq j}^{n} a_i\Big) = 0$ for each j. The following theorem follows from Lemma 2.2.

Theorem 2.1. *Let L be a modular lattice satisfying conditions* (C1) *and* (C2) *and $a, a_i \in L$, $i \in I$ be such that $a = \bigoplus_{i \in I} a_i$, where all a_i are indecomposable. If a has ADS, then a_i is a_j-injective for $i \neq j$, $i, j \in I$.*

Next result motivates the following question: 'Do the absolute direct summand property (ADS) and the summand intersection property (SIP) necessitate the other?'

To answer this, consider the lattice given in the Figure 2. It has already been discussed in Example 2.1 that f has ADS. The element f has direct summands d and e such that $d \wedge e = b$ is not a direct summand. Hence, f does not satisfy summand intersection property.

Theorem 2.2. *Let L be a modular lattice satisfying conditions* (C1) *and* (C2) *and $a, b \in L$ be such that a and b has ADS. If every $x \in L$ can be expressed as $x = x_1 \oplus x_2$ for some $x_1 \leqslant y_1$, $x_2 \leqslant y_2$ with $x \leqslant y$ and $y = y_1 \oplus y_2$, then $a \oplus b$ has ADS.*

P r o o f. Let p be a direct summand of $a \oplus b$. Then $a \oplus b = p \oplus q$ for a direct summand q of $a \oplus b$. By assumption, $p = a_1 \oplus b_1$ and $q = a_2 \oplus b_2$ for some $a_1, a_2 \leqslant a$, $b_1, b_2 \leqslant b$. It is clear that a_1, a_2 are direct summands of a and b_1, b_2 are direct summands of b. Let k be a max-semicomplement of p in $a \oplus b$. Again by assumption, $k = a_3 \oplus b_3$ for some $a_3 \leqslant a$, $b_3 \leqslant b$. Also, $p \oplus k \leqslant_e a \oplus b$,

$$\begin{aligned} p \oplus k \leqslant_e a \oplus b &\Rightarrow (a_1 \oplus b_1) \oplus (a_3 \oplus b_3) \leqslant_e a \oplus b \\ &\Rightarrow (a_1 \oplus a_3) \oplus (b_1 \oplus b_3) \leqslant_e a \oplus b \\ &\Leftrightarrow a_1 \oplus a_3 \leqslant_e a, b_1 \oplus b_3 \leqslant_e b. \end{aligned}$$

Since a_1, a_3 are direct summands of a, a_1 is a max-semicoplement of a_3 and b_1, b_3 are direct summands of b, so b_1 is a max-semicoplement of b_3. Since a and b have ADS, $a_1 \oplus a_3 = a$, $b_1 \oplus b_3 = b$. Hence, $p \oplus k = a \oplus b$ and $a \oplus b$ has ADS. □

An element a of a lattice L is called *extending* if every nonzero $b \leqslant a$ is essential in a direct summand of a. Note that in a modular lattice L satisfying conditions (C1) and (C2), $a \in L$ is extending if every max-semicomplement in a is a direct summand of a. Also, if $a \in L$ is extending, then every direct summand of a is extending.

Definition 2.3. Let L be a lattice with 0. An element $a \in L$ is called *SSP extending* if it is extending and satisfies summand sum property.

Theorem 2.3. *Let L be a modular lattice satisfying conditions* (C1) *and* (C2) *and $a \in L$ be a SSP extending. Then a has a unique maximal essential extension if and only if a has ADS and SIP.*

P r o o f. Let a have a unique maximal essential extension. Since a is extending, by Proposition 4.2 in [8], a is G-extending. Now, a has a unique maximal essential extension and is G-extending, so a satisfies SIP.

Let $a = a_1 \oplus a_2$. Since a is extending with SSP, by Lemma 3.6 in [8], a_1, a_2 are relatively injective. Hence, by Lemma 2.2, a has ADS.

Conversely let a have ADS and SIP. Let $b, c, d \leqslant a$ be such that $b \leqslant_e c \leqslant_{cl} a$ and $b \leqslant_e d \leqslant_{cl} a$. Since a is extending, c and d are direct summands of a. By assumption, $c \wedge d$ is a direct summand of a. Then by ADS, $a = k \oplus (c \wedge d)$ for a max-semicomplement k of $c \wedge d$ in a. By using modularity of L for $c \wedge d \leqslant d$,

$$c = c \wedge a = c \wedge [k \oplus (c \wedge d)] = (c \wedge k) \oplus (c \wedge d).$$

Also, $b \leqslant_e d \Rightarrow b \wedge c \leqslant_e d \wedge c$ and $b \wedge c \leqslant_e d \wedge c$ with $k \wedge (c \wedge d) = 0$ implies that $k \wedge (b \wedge c) = 0 \Rightarrow b \wedge (k \wedge c)$. Finally, $b \leqslant_e c$, $b \wedge (k \wedge c) = 0 \Rightarrow k \wedge c = 0$, therefore $c = c \wedge d \Rightarrow c \leqslant d$. Similarly, $d \leqslant c$ can be obtained. Hence $c = d$. □

In this section, Goldie absolute direct summand in lattices is defined, it is called Goldie ADS. It is a generalization of absolute direct summands.

Nimbhorkar and Shroff in [8] defined a β relation as follows:

▷ Let $a, b \in L$. Then $a \,\beta\, b$ if and only if $a \wedge b \leqslant_e a$ and $a \wedge b \leqslant_e b$. Note that $a \,\beta\, b$ is an equivalence relation on L.

By using relation β, Goldie absolute direct summand is defined as follows:

Definition 3.1. Let L be a lattice with 0 and $a, b, c \in L$ be such that $c = a \oplus b$. The element c is said to have *Goldie absolute direct summands* if for every decomposition $c = a \oplus b$ of c and every max-semicomplement k of a in c there exists $d \leqslant c$ such that $c = a \oplus d$, $k \,\beta\, d$. In short it is called Goldie ADS.

A bounded lattice L is said to have Goldie ADS if for every decomposition $a \oplus b = 1$, $a, b \in L$ and every max-semicomplement k of a in L there exists $d \in L$ such that $1 = a \oplus d$, $k \,\beta\, d$.

Remark 3.1. Consider the element $g \in L$ in the lattice shown in Figure 4. Here $g = a \oplus f$ and a has max-semicomplements d, e (other than f). For max-semicomplement d there exists $e \leqslant g$ such that $d \,\beta\, e$ and $g = a \oplus e$. Also, for max-semicomplement e there exists $d \leqslant g$ such that $e \,\beta\, d$ and $g = a \oplus d$. Similarly, it can be checked for all decompositions of g. Hence, g has Goldie ADS property.

Note that $x \,\beta\, x$ for every element of the lattice. In the lattice shown in Figure 5, $g = d \oplus c$, b is a max-semicomplement of c such that $g \neq b \oplus c$. Hence, g does not have Goldie ADS property.

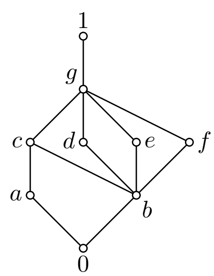

Figure 4.

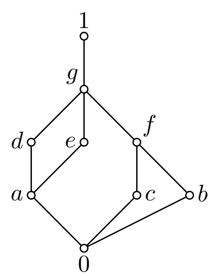

Figure 5.

Lemma 3.1. *Let L be a modular lattice satisfying conditions* (C1) *and* (C2) *and $a \in L$. If a has Goldie ADS, then every nonzero direct summand of a has Goldie ADS.*

P r o o f. Let $d \neq 0$ be a direct summand of a. Then there exist $c \leqslant a$ such that $d \oplus c = a$. Let $c = c_1 \oplus c_2$ and k be a max-semicomplement of c_1 in c. Then $a = (c_1 \oplus c_2) \oplus d = c_1 \oplus (c_2 \oplus d)$. Since c is a direct summand of a and k is a max-semicomplement of c_1 in c, by Theorem 2.2 in [6], k is a max-semicomplement of $c_1 \oplus d$ in a. But a has Goldie ADS, therefore there exist $t \leqslant a$ such that $k \,\beta\, t$ and $a = c_1 \oplus d \oplus t$. Now, by modularity of L,

$$c = a \wedge c = (c_1 \oplus d \oplus t) \wedge c = c_1 \oplus [(d \oplus t) \wedge c].$$

It remains to show that $k \beta [(d \oplus t) \wedge c]$, i.e., $[k \wedge (d \oplus t)] \leqslant_e k$, $[k \wedge (d \oplus t)] \leqslant_e [(d \oplus t) \wedge c]$.

Note that

$$t \wedge k \leqslant (d \oplus t) \wedge k \leqslant k, \quad t \wedge k \leqslant_e k \Rightarrow (d \oplus t) \wedge k \leqslant_e k.$$

Let $p \leqslant [(d \oplus t) \wedge c]$ such that $p \wedge [(d \oplus t) \wedge k] = 0$. Then $p \wedge (t \wedge k) \leqslant p \wedge [(d \oplus t) \wedge k] = 0$. Since k is a max-semicomplement of $c_1 \oplus d$ in a, by using modularity of L,

$$\begin{aligned} k \oplus (c_1 \oplus d) \leqslant_e a = c_1 \oplus d \oplus t &\Rightarrow [k \oplus (c_1 \oplus d)] \wedge c \leqslant_e [c_1 \oplus d \oplus t] \wedge c \\ &\Rightarrow (k \oplus c_1) \oplus (d \wedge c) \leqslant_e c_1 \oplus [(d \oplus t) \wedge c] \\ &\Rightarrow (k \oplus c_1) \leqslant_e c_1 \oplus [(d \oplus t) \wedge c] \\ &\Rightarrow k \leqslant_e [(d \oplus t) \wedge c]. \end{aligned}$$

Then $k \leqslant_e [(d \oplus t) \wedge c]$, $(d \oplus t) \wedge k \leqslant_e k \Rightarrow (d \oplus t) \wedge k \leqslant_e (d \oplus t) \wedge c$.

Lemma 3.2. *Let L be a lattice with 0 and $c \in L$. Then the following statements are equivalent.*

(1) *c has a Goldie ADS.*

(2) *For every decomposition $c = a \oplus b$ and every max-semicomplement k of a there exists $d \leqslant c$ and $x \leqslant c$ such that $x \leqslant_e k$ and $x \leqslant_e d$ and $c = a \oplus d$.*

P r o o f. (1) $\Rightarrow$ (2): Let c have Goldie ADS. Then for every decomposition $c = a \oplus b$ and every max-semicomplement k of a there exists $d \leqslant c$ such that $k \,\beta\, d$, i.e., $k \wedge d \leqslant_e k$ and $k \wedge d \leqslant_e d$ and $c = a \oplus d$. By putting $x = k \wedge d$, (2) follows.

(2) $\Rightarrow$ (1): Suppose (2) holds. Let $p \leqslant d$ be such that $(k \wedge d) \wedge p = 0$. Then

$$0 = p \wedge (k \wedge d) = (p \wedge k) \wedge d = p \wedge k$$
$$p \wedge x \leqslant p \wedge k = 0, \quad x \leqslant_e k \Rightarrow p = 0 \Rightarrow k \wedge d \leqslant_e d.$$

Using similar argument, $k \wedge d \leqslant_e k$ can be obtained. $\square$

Before stating the next result, recall the definition of ejective element in lattice from [8].

Definition 3.2. Let $a, b, c \in L$ be such that $a = b \oplus c$. Then b is said to be *c-ejective* in a if for every $d \leqslant a$ such that $d \wedge b = 0$ there exists $f \leqslant a$ such that $a = b \oplus f$ and $d \wedge f \leqslant_e d$. If b is c-ejective and c is b-ejective, then b and c are said to be *relatively ejective.*

In the following lemma, a necessary and sufficient condition is given for an element of a lattice to have Goldie ADS.

Theorem 3.1. *Let L be a modular lattice satisfying conditions* (C1) *and* (C2) *and $a \in L$. The element a has Goldie ADS if and only if for every decomposition $a = a_1 \oplus a_2$ of a, a_1 and a_2 are mutually ejective.*

P r o o f. Let $a = a_1 \oplus a_2$ have Goldie ADS. To show that a_1 is a_2-ejective let $k \leqslant a$ be such that $k \wedge a_1 = 0$. Then there exists max-semicomplement p of a_1 such that $k \leqslant p$. Since a has ADS, there exists $d \leqslant a$ such that $p \,\beta\, d$ and $a = a_1 \oplus d$. Now, $p \,\beta\, d \Rightarrow p \wedge d \leqslant_e d$, $p \wedge d \leqslant_e p$. It is now sufficient to show that $k \wedge d \leqslant_e k$. If $c \leqslant k$ be such that $(k \wedge d) \wedge c = 0$, then

$$(k \wedge d) \wedge c = 0 \Rightarrow d \wedge c = 0 \Rightarrow (p \wedge d) \wedge c = 0.$$

Now,

$$(p \wedge d) \wedge c = 0, \quad c \leqslant k \leqslant p, \quad p \wedge d \leqslant_e p \Rightarrow c = 0.$$

Hence, $k \wedge d \leqslant_e k$ and a_1 is a_2-ejective. Similarly, a_1-ejectivity of a_2 can be obtained.

Conversely, suppose that for every decomposition $a = a_1 \oplus a_2$ of a, a_1 and a_2 are mutually ejective. Let k be a max-semicomplement of a_1. Then by ejectivity, there exists $d \leqslant a$ such that $a = a_1 \oplus d$ and $k \wedge d \leqslant_e k$. It remains to show that $k \wedge d \leqslant_e d$. Since k is a max-semicomplement of a_1, $k \oplus a_1 \leqslant_e a$. Also,

$$k \wedge d \leqslant_e k, \quad k \wedge a_1 = 0 \Rightarrow (k \wedge d) \oplus a_1 \leqslant_e k \oplus a_1 = a.$$

By modularity of L for $k \wedge d \leqslant d$,

$$(k \wedge d) \oplus a_1 \leqslant_e a \Rightarrow [(k \wedge d) \oplus a_1] \wedge d \leqslant_e a \wedge d \Rightarrow k \wedge d \leqslant_e d.$$

Hence, a has Goldie ADS. □

Corollary 3.1. *Let L be a modular lattice satisfying conditions* (C1) *and* (C2) *and $a \in L$. If a has Goldie ADS and $d \leqslant a$ is closed in a, then a has ADS.*

Proof. Let a have Goldie ADS. Then for every decomposition $a = a_1 \oplus a_2$, a_1 and a_2 are mutually ejective. By Lemma 4.3 of [8], a_1 and a_2 are mutually injective. Then by Lemma 2.2, a has ADS.

Lemma 3.3. *Let L be a modular lattice satisfying conditions* (C1) *and* (C2) *and $a \in L$ have Goldie ADS. Then for every decomposition $a = a_1 \oplus a_2$, a_1 is b-ejective for every nonzero $b \leqslant a_2$.*

Proof. Let $a = a_1 \oplus a_2$ and $0 \neq b \leqslant a_2$ be such that $k = a_1 \oplus b$ and $l \leqslant k$ be such that $l \wedge a_1 = 0$. By Theorem 3.1, a_1 is a_2-ejective, therefore

$$l \wedge a_1 = 0 \Rightarrow a = a_1 \oplus c, \quad l \wedge c \leqslant_e l \quad \text{for some } c \leqslant a.$$

Then by modularity of L, for $a_1 \leqslant k$, $k = a \wedge k = (a_1 \oplus c) \wedge k = a_1 \oplus (c \wedge k)$. Also, $(c \wedge k) \wedge l = c \wedge l \leqslant_e l$. Hence, a_1 is b-ejective $b \leqslant a_2$. □

As a generalization of an extending element, Nimbhorkar and Shroff in [8] defined a Goldie extending element in a lattice by using the relation β as follows:

Let L be a lattice and $a \in L$. If for every $b \leqslant a$ there exists a direct summand c of a such that $b \beta c$, then a is said to be a Goldie extending (G-extending) element. Equivalently, in a modular lattice L, an element $a \in L$ is called a *Goldie extending* if for every closed element $b \leqslant a$ there exists a direct summand c of a such that $b \beta c$ holds.

Definition 3.3. Let L be a lattice with 0 and an element $a \in L$ is said to be *Goldie SSP* if a is G-extending and satisfies summand sum property.

A lattice L with 0 is said to be Goldie SSP if every $a \in L$ is Goldie extending and satisfies summand sum property.

Proposition 3.1. *Let L be a modular lattice satisfying conditions* (C1) *and* (C2). *If L is Goldie SSP, then L is Goldie ADS.*

Proof. Let $a, a_1, a_2 \in L$ be such that $a = a_1 \oplus a_2$. Let $b \in L$ be a max-semicomplement of a_1 in a. Then $a_1 \oplus b \leqslant_e a$. Since L is G-extending, there exists a direct summand d of a such that $b \beta d$, i.e., $b \wedge d \leqslant_e b$ and $b \wedge d \leqslant_e d$. Here

$$b \wedge d \leqslant_e b \Rightarrow a_1 \oplus (b \wedge d) \leqslant_e a_1 \oplus b \Rightarrow a_1 \oplus (b \wedge d) \leqslant_e a,$$
$$a_1 \oplus (b \wedge d) \leqslant_e a, \quad a_1 \oplus (b \wedge d) \leqslant a_1 \oplus d \leqslant a \Rightarrow a_1 \oplus d \leqslant_e a.$$

Now,

$$0 = b \wedge a_1 = (b \wedge d) \wedge a_1 = (b \wedge d) \wedge (d \wedge a_1),$$
$$b \wedge d \leqslant_e d, \quad (b \wedge d) \wedge (d \wedge a_1) = 0 \Rightarrow d \wedge a_1 = 0.$$

Since L satisfies summand sum property and $d \oplus a_1$ is a direct summand of a, $a_1 \oplus d \leqslant_e a \Rightarrow a_1 \oplus d = a$. □

References

[1] *A. Alahmadi, S. K. Jain, A. Leroy*: ADS modules. J. Algebra *352* (2012), 215–222. zbl MR doi

[2] *W. D. Burgess, R. Raphael*: On modules with the absolute direct summand property. Ring Theory. World Scientific, Singapore, 1993, pp. 137–148. zbl MR

[3] *G. Călugăreanu*: Lattice Concepts of Module Theory. Kluwer Texts in the Mathematical Sciences 22. Kluwer, Dordrecht, 2000. zbl MR doi

[4] *L. Fuchs*: Infnite Abelian Groups. I. Pure and Applied Mathematics 36. Academic Press, New York, 1970. zbl MR

[5] *G. Grätzer*: General Lattice Theory. Birkhäuser, Basel, 1998. zbl MR

[6] *S. K. Nimbhorkar, R. C. Shroff*: Generalized extending ideals in modular lattices. J. Indian Math. Soc., New Ser. *82* (2015), 127–146. zbl MR

[7] *S. K. Nimbhorkar, R. C. Shroff*: Ojective ideals in modular lattices. Czech. Math. J. *65* (2015), 161–178. zbl MR doi

[8] *S. K. Nimbhorkar, R. C. Shroff*: Goldie extending elements in modular lattices. Math. Bohem. *142* (2017), 163–180. zbl MR doi

[9] *T. C. Quynh, S. Şahinkaya*: Goldie absolute direct summand in rings and modules. Stud. Univ. Babeş-Bolyai, Math. *63* (2018), 437–445. zbl MR doi

[10] *F. Takil Mutlu*: On Modules With the Absolute Direct Summand Property: Ph.D. Thesis. University of Anatolia, Eskişehir, 2003.

[11] *F. Takil Mutlu*: On AdS-modules with the SIP. Bull. Iran. Math. Soc. *41* (2015), 1355–1363. zbl MR

[12] *A. Tercan, C. C. Yücel*: Module Theory, Extending Modules and Generalizations. Frontiers in Mathematics. Birkhäuser, Basel, 2016. zbl MR doi

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