NULL CONTROLLABILITY OF A COUPLED MODEL IN POPULATION DYNAMICS

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Cordially dedicated to my family

Abstract. We are concerned with the null controllability of a linear coupled population dynamics system or the so-called prey-predator model with Holling type I functional response of predator wherein both equations are structured in age and space. It is worth mentioning that in our case, the space variable is viewed as the "gene type" of population. The studied system is with two different dispersion coefficients which depend on the gene type variable and degenerate in the boundary. This system will be governed by one control force. To reach our goal, we develop first a Carleman type inequality for its adjoint system and consequently the pertinent observability inequality. Note that such a system is obtained via the original paradigm using the Lagrangian method. Afterwards, with the help of a cost function we will be able to deduce the existence of a control acting on a subset of the gene type domain and which steers both populations of a certain class of age to extinction in a finite time.

Keywords: degenerate population dynamics model; Lotka-Volterra system; Carleman estimate; observability inequality; null controllability

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1. Introduction

We consider the coupled population dynamics system

$$(1)\qquad \begin{aligned} &\frac{\partial y}{\partial t}+\frac{\partial y}{\partial a}-(k_1(x)y_x)_x+\mu_1(t,a,x)y+b(t,a,x)yp=\vartheta\chi_\omega && \text{in } Q,\\ &\frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}-(k_2(x)p_x)_x+\mu_2(t,a,x)p-\mu_3(t,a,x)yp=0 && \text{in } Q,\\ &y(t,a,1)=y(t,a,0)=p(t,a,1)=p(t,a,0)=0 && \text{on } (0,T)\times(0,A),\\ &y(0,a,x)=y_0(a,x);\quad p(0,a,x)=p_0(a,x) && \text{in } Q_A, \end{aligned}$$

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$$y(t,0,x)=\int_0^A \beta_1(t,a,x)y(t,a,x)\,\mathrm{d}a \qquad \text{in } Q_T,$$
$$p(t,0,x)=\int_0^A \beta_2(t,a,x)p(t,a,x)\,\mathrm{d}a \qquad \text{in } Q_T,$$

where $Q=(0,T)\times(0,A)\times(0,1)$, $Q_A=(0,A)\times(0,1)$, $Q_T=(0,T)\times(0,1)$, $\omega\subset\subset(0,1)$ and we denote $q=(0,T)\times(0,A)\times\omega$. The variables $y(t,a,x)$ and $p(t,a,x)$, which are in interaction, represent, respectively, the distributions of prey and predator at time t, of age a and of gene type x. Recall that system (1) above models the dispersion of gene in the two populations prey and predator. A further details about what can our paradigm model will be provided later. The parameters $\beta_1(t,a,x)$ (or $\beta_2(t,a,x)$), $\mu_1(t,a,x)$ (or $\mu_2(t,a,x)$) can be interpreted as the natural fertility rate of the prey population (or the predator population) and the natural mortality rate of the prey population (or of the predator population) while b stands for capturing rate of predator called also the ratio between the searching efficiency and attack rate of predator and $\mu_3=\lambda_2 b$ is a positive constant where, λ_2 is the measure of the predator's efficiency to convert prey biomass to fertility (or productivity). On the other hand, the parameters k_1 and k_2 are, respectively, the coefficients of dispersion of prey and predator populations which depend on the gene type variable x; ϑ and ω are, respectively, the control that we are looking for and the region of gene type where it acts. Such a control can be viewed in our situation as the capture strategy and corresponds in general to an external supply or to removal of individuals on the sub-domain $\omega\subset\subset(0,1)$. Herein we emphasize that since our aim is to steer the two populations on extinction and taking into account the ecological relationship between prey and predator, it will be judicious that the control acts on the prey population. Besides, y_0 and p_0 are, respectively, the initial distributions of the prey and predator populations and $\int_0^A \beta_1(t,a,x)y(t,a,x)\,\mathrm{d}a$ and $\int_0^A \beta_2(t,a,x)p(t,a,x)\,\mathrm{d}a$ are the distributions of the newborns of, respectively, prey and predators. Finally, the two positive fixed constants T and A are, respectively, the time of control and the maximal age of expectancy that we suppose here is the same for both populations. A suitable and powerful condition will be required later on T.

The population dynamics models in their different aspects attracted many authors and were investigated from many sides (see for example [4], [10], [28], [29], [31], [32], [33], [36], [43], [49], [50], [52]). Among those questions, we find the null controllability or in general the controllability problems for age and space structured population dynamics models which where studied intensively in the literature like [1], [2], [3], [7]. In [1], [2], the author tried to prove both the exact and approximate controllability for a population dynamics model where the coefficient is a positive constant. More precisely, to prove the first type of controllability, Ainseba used the mean of observability inequality which is a consequence of Carleman estimates based on the computations done in [37] for a non-degenerate heat equation. The second result of the same paper is reached by using an argument of density of the reachable set of states at time T on L^2-space for an age class $(0, a_1)$, where $a_1 < A$. Notice that the exact controllability is equivalent to the null controllability of a linear model. Based on this rule, using again the Carleman type inequalities and with the help of the characteristics method, the workers in [3] proved under the assumptions of that the L^∞-norm of the initial data is small and the fact that the coefficient of dispersion is positive function for all point of space domain, that their population model is exact controllable. Earlier in [7], a result similar to the one in [1], [2] was shown but without the so-called Carleman estimate. In fact, the method used here is a combination between a contradiction process and the so-called Mizohata uniqueness theorem (see the reference for further details).

Nevertheless, the previous works were established with either a space independent or a non-degenerate dispersion coefficient contrary to our paper and the works realized in [6], [26], whose calculus are based on the papers investigating the degenerate heat equation [18], [20], [21], [22], [23]. In fact, in [18] Cannarsa et al. showed weaker properties than null controllability result for a nonlinear degenerate heat equation, namely “regional null controllability” and “persistent regional null controllability” (see also [23]). The degeneracy occurs on the left of the boundary of the space domain and the nonlinearity involves a gradient term. Abstractly, the two major results were derived firstly for an adequate linearized equation. Afterwards, they enchained their proof with a tool invoking the notion of regional approximate controllability instead of Carleman inequality. Finally, the purpose is achieved by a Schauder fixed point technique. In [22], the authors assessed the two previous kinds of controllability of a semi-linear degenerate heat equation in the case where the nonlinearity is not related to the gradient term. Herein the employed approach was totally different and based on the cut-off functions technique and the non-degeneracy of the studied equation on a strip of the space domain which is somehow “far” from the point of degeneracy (see also [21]). The weighted inequalities of Carleman estimates were well-exploited by Boussaouira et al. in [9] and improved the results of the last three references. To be more accurate, through a Caccioppoli and Hardy-Poincaré inequalities, a persistent observability is gotten and then the global null controllability for a linear degenerate heat equation and consequently for the semilinear case with the help of fixed point technique. Let us stress that in [9], the authors treated only the divergence form. But what happened if the degenerate operator is in the non-divergence form? Fortunately, Cannarsa et al. in [20] confirmed that the result remains the same again with the aid of the Carleman estimates dilemma. Staying on the same trend, we advise the reader to take a glance on the items [19], [24].

As we said before, the degenerate population dynamics model, as for the non-degenerate case, follows the issues taken for the heat equation. In this context, [6], [26] were the first to be concerned with such a problem, each of them used a different technique and imposed also different conditions on time control T. Indeed, in [6] the authors allowed the dispersion coefficient to depend on the variable x and verifies $k(0) = 0$, i.e., the coefficient of dispersion k degenerates at 0 and they tried to obtain the null controllability in such a situation with $\beta \in L^\infty(Q)$ following [9] via a new Carleman estimate for a suitable full adjoint system and afterwards his observability inequality. However, the main controllability result of [6] was shown under the condition $T \geqslant A$ as in [12] and this constitutes a restrictiveness on the "optimality" of the control time T since it means, for example, that for a pest population whose maximal age A may equal to many days (may be many months or years) we need much time to bring the population to the zero equilibrium. In [39], precisely in Theorem 1.1, the authors showed, with the help of spectral theory, an interesting result concerning the (4) property for all the age classes $(0, A)$ for one equation. Nevertheless, the theorem was brought out under a restrictive condition on the control time T, namely it may be greater than the age expectancy A, besides that the control ϑ is essentially bounded in Q and these two assumptions will be improved in this paper (see Theorem 4.4 and Remark 4.5) using the so-called Carleman estimates (see Section 3). In the same vocation and to overcome the condition $T \geqslant A$, Maniar et al. in [26] suggested the fixed point technique implemented in [56] and which requires that the fertility rate must belong to $C^2(Q)$ and consists briefly in demonstrating in a first time the null controllability for an intermediate system with a fertility function $f \in L^2(Q_T)$ instead of $\int_0^A \beta(t, a, x) y(t, a, x)\, \mathrm{d}a$ and in achieving the task via the Leray-Schauder theorem.

On the other hand, a huge amount of works are focused on the control problems of (1), among them we find [5], [8], [60] and the references therein. In [5], a prey-predator model is taken under a reaction-diffusion system describing interaction between prey and predator populations. The goal is to look for a suitable control supported on a small spatial subdomain which guarantees the stabilization of the predator population to zero. The objective of [60] was different. Actually, an age-dependent prey-predator system was considered and the authors proved the existence and uniqueness of an optimal control (called also "optimal effor") which gives the maximal harvest via the study of the optimal harvesting problem associated to their coupled model. Similarly to the case of one equation in the papers [1], [2], [3], [5], [7], [60] assumed that their coefficients of diffusion are constants. This motivated Ait Ben Hassi et al. in [8] to generalize these works, specially [5], and investigated a semilinear parabolic cascade systems with two different diffusion coefficients allowed to depend on the space variable and degenerate at the left boundary of the space domain. Moreover, the purpose of this paper was to bring out the null controllability via a

Carleman type inequality of the adjoint problem of the associated linearized system using the results of [9] or [24] and with the help of the Schauder fixed point theorem. Another interesting works in the trend of cascade degenerate parabolic systems (without and with singularities) can be found also in the references [15], [38], [53].

But up till now and to our best knowledge, little is known about the global null controllability question of the age-structured population dynamics coupled systems both in degenerate and non-degenerate cases and the two only items which deal with such a paradigm are the one of Boutaayamou et al. in [14] and the recent paper [27]. In [14], the authors assessed a degenerate cascade population dynamics model in a non-divergence form and proved its null controllability like the one in (4) using the classical procedure based on the observability inequality deduced from the weighted estimates of Carleman kind (see more details in the introduction of [27]). Let us stress that the results in [14] were obtained under continuity regularity and biological assumptions on the natural fertility and mortality rates. Such hypotheses lead to omit interesting mathematical computations and this steered Maniar et al. in [27] to study (4) of a degenerate population cascade model governed with one control force in a divergence form and with L^∞-regularity on natural rates using some powerful tools like Hardy-Poincaré and Caccioppoli's inequalities, Gronwall lemma and semi-groups theory, to get the observability inequality of the associated full-adjoint system which leads to the desired goal. In this work, we address the control problem (4) related to (1) and it will be a generalization of the results obtained in [6], [26] and [27]. More precisely, following the global strategy of [8] we expect in this contribution to prove the global null controllability of the structured age and space Lotka-Volterra system (1) with one control force and when

$$T \in (0, \delta) \tag{2}$$

with

$$\delta = \min(\delta_1, \delta_2), \tag{3}$$

wherein δ_1 and δ_2 are fixed small enough and belong to $(0, A)$ and are defined by the intervals $(0, \delta_1)$ and $(0, \delta_1)$ which represent the age class of the prey newborns and predator newborns, respectively. That is, we show that for all $y_0, p_0 \in L^2(Q_A)$ there exists a control $\vartheta \in L^2(Q)$ such that the associated solution of (1) verifies

$$\begin{cases} y(T, a, x) = 0, & \text{a.e. in } (\delta, A) \times (0, 1), \\ p(T, a, x) = 0, & \text{a.e. in } (\delta, A) \times (0, 1). \end{cases} \tag{4}$$

It deserves to mention that the researched control ϑ depends on δ and the two initial distributions y_0 and p_0. Let us return back to condition (2) imposed on the fixed time of control T. This assumption is required not only for a technical cause but is also meaningful in the cost of controllability in the sense that we will be able to drive a very wide age classes of both populations to extinction fastly and quickly instead of waiting for months or years like in [6] (see also [13], [30] for similar explanation) and this will be an advantage of the optimality of the control ϑ. By the way, the null controllability property (4) does not allow to control the age class of non-fertile individuals of prey and predator populations and this can be justified in the mathematical standpoint (see the further proofs). Also, ecologically speaking, the fact that all classes of age were not answered can find its justification, in our opinion, as follows: the capture and the killing of the whole populations both prey and predator in a given time can have a boomerang effects on the ecosystem surrounding both populations. On the other hand, the paradigm (1) can model other prey-predator systems but not in the context of both populations densities eradication. For more precision, if one wants just to steer one of two species to extinction in finite time, model (1) can be used for instance as a description of the"cottony cushion-vedalia beetle" system, wherein the role of the control is to improve the number of vedalia beetle to delete in plausible threshold, the cottony cushion or also in some cases of host-paritoid systems, where the goal is to eliminate the parasitoid population. These two examples are interested to kill the prey population, nevertheless the model of cereal species (prey) and destructive fungal rust (predator) can be considered to control and reduce considerably the predator density. In our situation, where we have to catch and capture the populations densities of prey and predator, we give into paragraph 2.2, as mentioned previously, two examples from two different scientific specializations.

The result (4) is gotten under the conditions that all natural rates possess L^∞-regularity and this prevent us the use of the fixed point technique needed in [26], [56]. Another striking difference with the cited references is that our prey-predator model is a coupled dynamics system combining at the same time an age and space structure and likewise the degeneracy occurring for the two different dispersion coefficients k_1 and k_2 on the left-hand side of the gene type domain, that is, $k_i(0) = 0$, $i = 1, 2$, e.g. $k_i(x) = x^\alpha$, where α can be taken in $[0, 1)$ if we impose the Dirichlet boundary conditions or in $[1, 2)$ if we consider the Newmann boundary conditions (see assumptions (5) beneath). In this case, we say that (1) is a degenerate populations dynamics cascade system. Genetically speaking, such a property is natural since it means that if each population is not of a gene type, it cannot be transmitted to its offspring. Another remark to do is that our system is a prey-predator model with Holling type I functional response of predator. Finally, we highlight that this work can be generalized in the case of interior degeneracy, i.e., $k_1(x_0) = 0$ and $k_2(y_0) = 0$ (e.g. $k_1(x) = |x - x_0|^\alpha$ and $k_2(x) = |x - y_0|^\alpha$,

$\alpha \in (0,1)$), where $x_0, y_0 \in (0,1)$ using the results proved in [13], which are based essentially on the method applied for controllability problem of interior degenerate parabolic equations [34] and can be extended to the non-smooth case in the light of item [35].

The remainder of this paper is organized as follows: in Section 2, we give an overview about some types of Holling functional responses of predators as well as what can our system (1) model. Section 3 will be devoted to a discussion about the well-posedness of (1) and establish a new Carleman estimate of an intermediate adjoint system which helps us to provide an evidence of the main Carleman type inequality of the associated full adjoint system. As an outcome of the latter, in Section 4, an observability inequality is proved with the help of the semi-groups theory, which allows us to obtain a non classical implicit formulas of the adjoint system solution (see [13], [30] for a similar procedure). The obtained observability inequality will play a crucial role is showing the main controllability result stated in (4). We close this work with Section 5, which takes the form of an appendix, wherein the proofs of some basic tools are provided.

2. An overview of some Holling types functional responses and modelling of system (1)

As mentioned in the introduction, this section will be concerned with discussion of the different Holling types functional responses and the modelling of (1). More precisely, we give some references treating different questions of the prey-predator models involving the different functional responses even if some of them do not belong to the control problems just to amplify the importance of study of such coupled systems. Concerning the modeling, we suggest two examples from different scientific fields, namely ecology and bacteriology, and justify why we must look for a strength control to govern the danger of the given populations on human in an optimal way in the sense that the natural environment of prey and predator populations is not destabilized.

2.1. An overview of some Holling types functional responses. In general, the prey-predator systems called also Lotka-Volterra models describe different interactions between two or more populations (maybe the competition can be included). To have a more real relationship with ecological domains, Holling introduced earlier what is known as a functional responses of predator, namely types I, II and III. Another kinds can be also involved depending on the dynamics behavior of the consumer (predator) and prey, like type IV and Ivlev's functional responses, which will be described concisely later on.

Abstractly, a functional response can be defined as the relationship between an individual's rate of consumption (here we talk about a consumption of predator) and food's density (i.e., prey's density). This amounts to saying that a functional response reflects the capture ability of the predator to prey or in other words, the functional response is introduced to describe the change in the rate of consumption of prey by predator when the density of prey varies. In the plotting point of view, each type of functional response I, II or III has a special characteristic. In fact, type I, or the linear case of the predator response, is the situation when the plot of the number of prey consumed (per unit of time) as a function of prey density shows a linear relationship between the number of prey consumed and the prey density. The Holling type II, called also concave upward response, is the case when the gradient of the curve decreases monotonically with increasing prey density, probably saturating at a constant value of prey consumption. For information, the Lotka-Volterra model involving this functional response is known as the Rosenzweig-MacArthur model. The type III response is known between the specialists of population dynamics as the sigmoid response having a concave downward part at low food density. Actually, for the Holling III, a sigmoidal behavior occurs when the gradient of the curve first increases and then decreases with increasing prey density. This behavior is due to the "learning behavior" in the predator population.

Now, we address some "ecological" interpretations of the three first Holling types functional responses. The type I response is the result of simple assumption that the probability of a given predator (usually the passive one) encountering prey in a fixed time interval $[0, T_t]$ within a fixed spatial region depends linearly on the prey density. This can be expressed under the form $Y = aT_sX$, where Y is the amount of prey consumed by one predator, X is the prey density, T_s is the time available for searching and a is a constant of proportionality, termed as the discovery rate (which is in our case represented by the parameter b). In the absence of need to spend time handling the prey, all the time can be used for searching, i.e., $T_s = T_t$, and we have the type I response: assuming that the predators (having the density P) act independently, in time T_t the total amount of prey will be reduced by quantity aT_tXP. In addition, if each predator requires a handling time h for each individual prey that is consumed, the time available for searching T_s is reduced: $T_s = T_t - hY$. Taking into account the expression of Y in response type I, this leads to $Y = aT_tX - ahXY$ and this implies $Y = aT_tX/(1 + ahX)$ and this is exactly the type II response. Therefore, in the interval $[0, T_t]$ the total amount of prey is reduced by the quantity $aT_tXP/(1 + ahX)$. Let us point out that the term "ah" is dimensionless and can be interpreted as the ability of a generic predator to kill and consume a generic prey and it possesses the following characteristics times: "ah" is large if the handling time h is much longer than the typical discovery time $1/a$ and "ah" is small in the opposite limit; in this case the type II response is reduced to type I. The Holling type III functional response can be viewed as a generalization of type II and takes the form $aT_tX^k/(1+ahX^k)$ with $k>1$. In literature, this response is stimulated by supposing that learning behavior occurs in the predator population with a consequent increase in the discovery rate as more encounters with prey occurs (see [25] for more details). To see to wingspan of the Lotka-Volterra models from many sides of investigations, we provide a non-exhaustive list of some works dealing with crucial questions, which are discussed widely. We begin with the system whose functional response is Holling I. One of the important problems which takes a special attention, is the study of the steady states and more accurately, in [40] a prey-predator system with nonlinear diffusion effects is considered. Such nonlinear diffusion effects have an impact on a biological species as well as their resource-biomass (i.e., the capacity of their environment). Herein, the workers assume that the dispersive force and the diffusion depend on population pressure from other species. The question of equilibrium of Lotka-Volterra systems with Holling type I functional type response takes also a broad study theoretically and numerically in [54], specially the interior one, as well as their dynamical behavior such as the cyclic-fold, saddle-fold, homoclinic saddle connection. The Holling I introduced here is from the range of the so-called Beddington-DeAngelis functional response. Remaining in the type I, the authors in [44] tried to prove the existence of an asymptotically stable pest-eradication for a prey-predator system modeled by an ordinary differential equation, when the impulsive period is some critical value less by implementing Floquet theorem and a small amplitude perturbation method. Such a solution of eradication is somehow the mixing between a synthetic strategy (insecticides or pesticides for instance) and biological control, e.g. the natural enemies "killing" the dangerous pests (the prey here) without causing a serious damages to the two population densities (see also [59] for a similar study).

Even the similarity appearance between their curves, functional responses of type I and type II have two considerable differences: the first one was pointed out before and it concerns the predator time handling of prey. Contrary to the Holling type II, the time handling is missed for predators in type I, which means that the consumers have a little difficulty capturing and assimilating prey but they switch their time to other activities once their ingestion rate is great enough to satisfy their energetic needs. The second difference is in the dynamical behavior. In fact, while the Holling type II displays the local Hopf bifurcation, the Holling type I makes clear a global cyclic-fold bifurcation. These differences between Holling I and Holling II, in particular the first, lead a numerous works to take into account the predator time handling in their different models. We emphasize here that Holling II possesses a generalization, which is exactly Beddington-DeAngelis functional response cited previously. This functional takes the form $\Phi_{BD}(N,P)=cN/(e+N+h_1P)$, where N

and P should be the densities of prey and predator, while e stands for the half-saturation constant, i.e., the amount of prey at which the per capita predation rate is half of its maximum c and h_1 is a positive constant (see [54] for further details about this functional response). Among the works interested in Lotka-Volterra with Holling type II we cite for instance [48], [55]. In [55], a statistical study was presented to see if one can replace Holling type II by functional response from the type of Beddington-DeAngelis, Crowly-Martin or Hassel-Valey model for a divers cases related to the predator feeding rate. Peng et al. in [48] were concerned with the question of the steady-states of some reaction-diffusion models and they established the non-existence of a non-constant equilibrium solutions of two prey-predator systems with Holling II when the interaction between the two populations is strong as they claimed and where the constant measuring the ability of generic predator to kill and consume generic prey is equal to 1.

By the way, a wide classical ecological literature assumed that mathematical models with Holling type II (or in general the non-sigmoid) functional response involving a diffusion terms match thoroughly in description of the pattern formation of a phytoplankton-zooplankton system. The affirmative answers are basically related to experiments realized in laboratories on zooplankton feeding, which are carried out in small-sized containers or bottles. But if one wants to investigate zooplankton grazing control in real ecosystems (may be the oceans), it will be more relevant to introduce the Holling type III response as stated in the introduction of [46]. Actually, the main focus of [46] was to set a generic model which explains the observed alteration of type between the different functional responses of plankton systems and gives, as he presumed, an evidence that for such a system the Holling type III is more adequate than other kinds. In the vocation of well-posedness, the global existence, uniqueness and the boundedness of a strong solution of partial differential equation with a special case of Holling III was brought out in [11]. This strong solution was approximated numerically using a spectral method and a Runge-Kutta time solver. The modelling using the Holling type III does not stop here, it can play also a crucial role to model the entomophagous species (see [16] for more details).

But when a functional response describes the interaction between predator and prey when the prey exhibits group defense (like buffalo) or has ability to hide itself (like chameleon), then we talk about the Holling type IV functional response or the so-called Monod-Haldane function. This function takes the form $mX/(\gamma + b_1X + X^2)$, where X is the prey density, $m > 0$ is the complete saturation, whereas γ and b_1 denote, respectively, the half-saturation constant and b_1 the prey environmental carrying capacity. A space independent system of Lotka-Volterra kind using type IV was under consideration in [45] and the principal purpose of this item is to assess the impact of the harvesting on equilibria of both prey and predator populations. The quandaries used here are, as the authors cited, based on the dynamical theory combined with a technique of Hopf bifurcation. A numerical analysis is provided to compare the dependence of the dynamical behavior on the harvesting effort for the prey between Holling types III and IV (see the reference for more information). Of Course, there is a difference between the first three types of functional responses and the fourth one is that the latter is non-monotonic for $X \geqslant 0$, contrary to type I, II and III.

We close this overview by the so-called Ivlev functional response which can be implicated to ecological applicabilities such as host-parasitoid system and animal coat pattern. The Ivlev response is classified as a Holling type II according to [57] and the references therein and its expression is given by $g(X) = 1 - \mathrm{e}^{-\lambda_1 X}$, wherein λ_1 is the efficiency of predator capture of prey and X is the density of prey. For an investigation of a limit cycle question of a two-dimensional prey-predator system with the response g, one can take a glance in [42].

2.2. Modelling of system (1)**.** In this paragraph, we try to answer to the following question: what can our system (1) model? Before responding, we recall, as mentioned in the abstract and introduction, that (1) is an age and space structured Lotka-Volterra with the Holling type I functional response. This choice of response is not random because in our knowledge it has an effect to stabilize the densities of the prey and predator populations.

To deal with our question, we must take into account that the two populations must be deleted with a suitable control, either industrial or biological which consists in growing up a common natural enemy's density without passing a given threshold. A close adaptation to this situation are two examples: the first one belongs to the ecological field and concerns the mosquito-spiders, when the first population (prey) must be killed since it is a cause of many harmful diseases which can lead to the death of human, like the virus named West Nile Virus (in literature called briefly WNV). The consumer, which are the spiders here have to be extincted because some kinds of them are very venomous and can kill human, like the Funnel-Web spiders; we cite for instance the Atrax-robustus and A. formidabilis.

The second example comes from bacteriology field, namely "Escherichia coli-Bdellovibrio" system. It is well-known that Escherichia coli (E. coli) is a bacteria which normally lives in the intestines of people and animals and most of them are harmless and actually are an important part of healthy human intestinal tracts. However, some of them become very dangerous for human life when one consumes for example an undercooked meat; we mean here E. coli O157:H7. This type of E. coli bacteria produces a toxin that causes hemolytic uremic syndrome (HUS), a disease that can steer to a permanent kidney damage among very young and old people and even can kill them (see [41]). In general, the pathogenic E. coli can be split into six pathotypes which can cause diarrhea: the shiga toxin-producing E. coli, enterotoxigenic E. coli, enteropathogenic E. coli, enteroaggregative E. coli, enteroinvasive E. coli and diffusely adherent E. coli. On the other hand, we must be careful on the use of the powerful strategy to stop emergence of these toxic bacteria in the intestines like enhancing the amount of bdellovibrio bacteriovorus since this latter is a predator of the so-called "gram-negative bacteria" specially the harmless and healthy E. coli. Thus, it turns also out to control this bacterial predator.

3. Well-posedness and Carleman estimates

3.1. Well-posedness result. For this section and for the sequel, we assume that the dispersion coefficients k_i, $i = 1, 2$, satisfy the hypotheses

$$\begin{cases} k_i \in C([0,1]) \cap C^1((0,1]),\ k_i > 0 \text{ in } (0,1] \text{ and } k_i(0) = 0, \\ \exists\, \gamma \in [0,1)\colon\ x k_i'(x) \leqslant \gamma k_i(x),\ x \in (0,1]. \end{cases} \tag{5}$$

The last condition on k_i means in the case of $k_i(x) = x^{\alpha_i}$ that $0 \leqslant \alpha_i < 1$. Similarly, all results of this paper can be obtained also in the case of $1 \leqslant \alpha_i < 2$ taking, instead of Dirichlet conditions on $x = 0$, the Newmann condition $(k_i(x) u_x)_x(0) = 0$. At this level, we emphasize that our current analysis of the null controllability property (4) of paradigm (1) with the Dirichlet boundary condition on $x = 0$ (i.e., the weakly degenerate case) is similar to the study with the Newmann boundary condition on $x = 0$ (i.e., the strongly degenerate case). However, there is a slight difference between the two situations which is the only one and arises in the proof of Proposition 3.4 (see the sketch beneath).

On the other hand, we assume that the rates μ_1, μ_2, b, μ_3, β_1 and β_2 verify

$$\begin{cases} \mu_1, \mu_2, b, \mu_3, \beta_1, \beta_2 \in L^\infty(\overline{Q}), \\ \mu_1, \mu_2, b, \mu_3, \beta_1, \beta_2 \geqslant 0 \text{ a.e. in } Q. \end{cases} \tag{6}$$

Here, we open parentheses to say that contrary to some references like [1], [2], [6], we do not require the mortality rates to satisfy $\int_0^A \mu_i(t-s, A-s, x)\,\mathrm{d}s = \infty$, $i = 1, 2$, since these conditions do not play any role in the well-posedness of the result or in the null controllability computations.

In summary, to justify that our model (1) is well-posed we rewrite it under an abstract Cauchy problem, then we combine some references, namely [9], [17], [24], [30], [47], [51], [58], to get our result. This result needs the introduction of a pertinent framework represented by a weighted Sobolev spaces defined for $i=1,2$ by

$$
(7)\qquad \begin{cases} H^1_{k_i}(0,1)=\{u\in L^2(0,1)\colon\ u \text{ is abs. cont. in } [0,1]\colon \\ \qquad\qquad\qquad\qquad \sqrt{k_i}u_x\in L^2(0,1),\ u(1)=u(0)=0\}, \\ H^2_{k_i}(0,1)=\{u\in H^1_{k_i}(0,1)\colon\ k_iu_x\in H^1(0,1)\}, \end{cases}
$$

endowed, respectively, with the norms

$$
\begin{cases} \|u\|^2_{H^1_{k_i}(0,1)}=\|u\|^2_{L^2(0,1)}+\big\|\sqrt{k_i}u_x\big\|^2_{L^2(0,1)}, & u\in H^1_{k_i}(0,1), \\ \|u\|^2_{H^2_{k_i}(0,1)}=\|u\|^2_{H^1_{k_i}(0,1)}+\|(k_iu_x)_x\|^2_{L^2(0,1)}, & u\in H^2_{k_i}(0,1), \end{cases}
$$

with $i=1,2$ (see [9], [17], [24] or the references therein for the properties of such a spaces).

Now, put for $i=1,2$, $\mathrm{A}_i\theta=(k_i(x)\theta_x)_x$ with k_i verifying (5). The domains of the operators A_i, $i=1,2$, are exactly $H^2_{k_i}(0,1)$, $i=1,2$, given in (7) and it is well-known that such an operators are closed, self-adjoint and negative with dense domains in $L^2(0,1)$, which implies that they generate a $\mathcal{C}_0$-semigroups in $L^2(0,1)$ (see [9], [17], [24] for precise proofs).

On the other hand, consider the operators $\mathcal{A}_i$, $i=1,2$, defined by

$$
(8)\qquad \begin{cases} \mathcal{A}_i\theta=-\dfrac{\partial\theta}{\partial a}+\mathrm{A}_i\theta \quad \forall\,\theta\in D(\mathcal{A}_i), \\ D(\mathcal{A}_i)=\Big\{u\in L^2(0,A;D(\mathrm{A}_i));\ \dfrac{\partial u}{\partial a}\in L^2(0,A;H^1_{k_i}(0,1)); \\ \qquad\qquad\qquad\qquad u(0,x)=\displaystyle\int_0^A \beta_i(a,x)u(a,x)\,\mathrm{d}a\Big\}. \end{cases}
$$

From [58], Theorem 4, page 23 or [58], Theorem 5, page 26 and since $(\mathrm{A}_i,D(\mathrm{A}_i))$, $i=1,2$, are infinitesimal generators of $\mathcal{C}_0$-semigroups as mentioned before, one can conclude that $(\mathcal{A}_i,D(\mathcal{A}_i))$, $i=1,2$, generate $\mathcal{C}_0$-semigroups in $L^2(Q_A)$. In this context, we advise the reader to take a glance for a similar discussion of the well-posedness result into [30], Theorem 2.1.

Adapting these notations, the abstract Cauchy problem associated to (1) is formulated as

$$
(9)\qquad X'(t)=(\mathbb{A}+B(t))X(t)+F(X(t))+f(t),\quad X(0)=\begin{pmatrix} y_0 \\ p_0 \end{pmatrix},
$$

where

$$
X(t)=\begin{pmatrix} y(t) \\ p(t) \end{pmatrix},\quad \mathbb{A}=\begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix},\quad D(\mathbb{A})=D(\mathcal{A}_1)\times D(\mathcal{A}_2),
$$
$$
B(t)=\begin{pmatrix} M_{\mu_1} & 0 \\ 0 & M_{\mu_2} \end{pmatrix},\quad F(X(t))=\begin{pmatrix} M_b \\ M_{\mu_3} \end{pmatrix},\quad f(t)=\begin{pmatrix} \vartheta\chi_\omega \\ 0 \end{pmatrix}
$$

with the generators $\mathcal{A}_i$, $i = 1, 2$, defined by (8),

$$M_{\mu_i} w = -\mu_i w, \ i = 1, 2, \quad M_b(y, p) = -byp \quad \text{and} \quad M_{\mu_3}(y, p) = \mu_3 yp.$$

As we can see, the operator $(\mathbb{A}, D(\mathbb{A}))$ is a diagonal matrix of generators of $\mathcal{C}_0$-semigroups; as a consequence, $(\mathbb{A}, D(\mathbb{A}))$ is also a generator of a $\mathcal{C}_0$-semigroup in $L^2(Q)$. On the other hand, the operator $B(t)$ can be viewed as a bounded perturbation of $\mathbb{A}$ so that one has $D(\mathbb{A} + B(t)) = D(\mathbb{A})$.

Gathering all these arguments with the results in [47], Lemma 3.1 and in [51], Theorem 2.1 we somehow justify our theorem of well-posedness:

Theorem 3.1. *The following points hold:*

(1) *The operator* $(\mathbb{A} + B(t), D(\mathbb{A}))$ *generates a* $\mathcal{C}_0$*-semigroup in* $L^2(Q)$.
(2) *Under assumptions* (5) *and* (6) *and for all* $\vartheta \in L^2(Q)$ *and* $(y_0, p_0) \in D(\mathcal{A}_1) \times D(\mathcal{A}_2)$, *system* (9) *admits a unique mild solution* X *belonging to* $C([0, T]; D(\mathcal{A}_1) \times D(\mathcal{A}_2))$ *and verifies the integral equation*

$$\forall\, t \in [0, T], \quad X(t) = \mathrm{e}^{(\mathbb{A}+B)t} X_0 + \int_0^t \mathrm{e}^{(\mathbb{A}+B)(t-s)} (F(X(s)) + f(s))\, \mathrm{d}s. \tag{10}$$

Before to continuing, we shall make the following remark:

Remark 3.2. Since $D(\mathcal{A}_i)$, $i = 1, 2$, are dense in $L^2(Q_A)$, Theorem 3.1 can be extended to the space $L^2(Q_A)$ for the initial data (y_0, p_0) as well as our null controllability result (4).

3.2. Carleman inequality results. In this paragraph, we focus on the so-called Carleman estimates. Generally speaking, Carleman estimates are a priori estimates for the solutions of the adjoint systems and their derivatives. The first result of this section concerns the adjoint system of the Lotka-Volterra system (1). Classically, the adjoint system is derived by multiplying the governing equations of the direct problem by Lagrange multipliers which, means that the adjoint state is the Lagrange multiplier for the studied PDE. To obtain this model, we afterwards integrate over the domains of the existing variables (herein, the time, the gene type and the age variables). Note that it is not necessary to multiply the boundary and the initial conditions of the direct problem by Lagrange multipliers because they become identically null.

In our case, the associated adjoint model of (1) is stated in the following proposition.

Proposition 3.3. *The adjoint system of* (1) *is given by*

$$
(11)\quad
\begin{aligned}
&\frac{\partial u}{\partial t}+\frac{\partial u}{\partial a}+(k_1(x)u_x)_x-\mu_1(t,a,x)u-b(t,a,x)up\\
&\qquad=-\,\beta_1 u(t,0,x) && \text{in } Q,\\
&\frac{\partial v}{\partial t}+\frac{\partial v}{\partial a}+(k_2(x)v_x)_x-\mu_2(t,a,x)v+\mu_3(t,a,x)vy\\
&\qquad=-\,\beta_2 v(t,0,x) && \text{in } Q,\\
&u(t,a,1)=u(t,a,0)=v(t,a,1)=v(t,a,0)=0 && \text{on } (0,T)\times(0,A),\\
&u(T,a,x)=u_T(a,x);\ v(T,a,x)=v_T(a,x) && \text{in } Q_A,\\
&u(t,A,x)=v(t,A,x)=0 && \text{in } Q_T,
\end{aligned}
$$

where (y,p) *is the solution of* (1) *and* u *and* v *stand, respectively, for the adjoint variables of* y *and* p.

Proof. Firstly, we define the Lagrangian L related to (1) by

$$
(12)\quad
\begin{aligned}
&L(y,p,u,v,\vartheta,u_0,v_0)\\
&\quad=J(y,p,\vartheta)\\
&\qquad+\int_Q u\Big(\frac{\partial y}{\partial t}+\frac{\partial y}{\partial a}-(k_1(x)y_x)_x+\mu_1(t,a,x)y+b(t,a,x)yp-\vartheta\chi_\omega\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\
&\qquad+\int_Q v\Big(\frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}-(k_2(x)p_x)_x+\mu_2(t,a,x)p-\mu_3(t,a,x)yp\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\
&\qquad+\int_{Q_A}u_0(y(0)-y_0)\,\mathrm{d}a\,\mathrm{d}x+\int_{Q_A}v_0(p(0)-p_0)\,\mathrm{d}a\,\mathrm{d}x,
\end{aligned}
$$

where the functional J is given by

$$
J(y,p,\vartheta)=\frac{1}{2}\int_0^1\int_\delta^A(y^2(T,a,x)+p^2(T,a,x))\,\mathrm{d}a\,\mathrm{d}x+\frac{1}{2}\int_Q\vartheta^2\chi_\omega\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x.
$$

Now, put

$$
I_1=\int_Q u\Big(\frac{\partial y}{\partial t}+\frac{\partial y}{\partial a}-(k_1(x)y_x)_x+\mu_1(t,a,x)y+b(t,a,x)yp\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x
$$

and

$$
I_2=\int_Q v\Big(\frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}-(k_2(x)p_x)_x+\mu_2(t,a,x)p-\mu_3(t,a,x)yp\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x.
$$

With the aid of the integration by parts technique, taking into account the first newborns equation of (1) and assuming that

$$\text{(13)}\qquad \begin{cases} u(t,A,x) = v(t,A,x) = 0 & \text{in } (0,T)\times(0,1),\\ u(t,a,0) = u(t,a,1) = v(t,a,0) = v(t,a,1) = 0 & \text{on } (0,T)\times(0,A), \end{cases}$$

we obtain

$$\text{(14)}\qquad \begin{aligned} I_1 = &\int_{Q_A} u(T,a,x)y(T,a,x)\,\mathrm{d}a\,\mathrm{d}x\\ &- \int_{Q_A} u(0,a,x)y(0,a,x)\,\mathrm{d}a\,\mathrm{d}x\\ &- \int_Q y\Big(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (k_1 u_x)_x - \mu_1 u + \beta_1 u(t,0,x) - bup\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x, \end{aligned}$$

and

$$\text{(15)}\qquad \begin{aligned} I_2 = &\int_{Q_A} v(T,a,x)p(T,a,x)\,\mathrm{d}a\,\mathrm{d}x\\ &- \int_{Q_A} v(0,a,x)p(0,a,x)\,\mathrm{d}a\,\mathrm{d}x\\ &- \int_Q p\Big(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + (k_2 v_x)_x - \mu_2 v + \beta_2 v(t,0,x) + \mu_3 uy\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x. \end{aligned}$$

Combining (12) with (14) and (15) we get the following formula of L:

$$\begin{aligned} &L(y,p,u,v,\vartheta,u_0,v_0)\\ &\quad = \frac{1}{2}\int_{Q_A} (y^2(T,a,x) + p^2(T,a,x))\,\mathrm{d}a\,\mathrm{d}x\\ &\qquad + \frac{1}{2}\int_Q \vartheta^2\chi_\omega\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x - \int_Q \vartheta u\chi_\omega\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\ &\qquad + \int_{Q_A} (u_0(y(0) - y_0) + v_0(p(0) - p_0))\,\mathrm{d}a\,\mathrm{d}x\\ &\qquad + \int_{Q_A} u(T,a,x)y(T,a,x)\,\mathrm{d}a\,\mathrm{d}x - \int_{Q_A} u(0,a,x)y(0,a,x)\,\mathrm{d}a\,\mathrm{d}x\\ &\qquad - \int_Q y\Big(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (k_1 u_x)_x - \mu_1 u + \beta_1 u(t,0,x) - bup\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\ &\qquad + \int_{Q_A} v(T,a,x)p(T,a,x)\,\mathrm{d}a\,\mathrm{d}x - \int_{Q_A} v(0,a,x)p(0,a,x)\,\mathrm{d}a\,\mathrm{d}x\\ &\qquad - \int_Q p\Big(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + (k_2 v_x)_x - \mu_2 v + \beta_2 v(t,0,x) + \mu_3 uy\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x. \end{aligned}$$

The above expression of L can be rewritten as

$$
\begin{aligned}
(16)\quad & L(y,p,u,v,\vartheta,u_0,v_0)\\
&= \int_{Q_A}\Big(\frac{1}{2}y^2(T,a,x)\chi_{(\delta,A)}+u(T,a,x)y(T,a,x)\Big)\,\mathrm{d}a\,\mathrm{d}x\\
&\quad+\int_{Q_A}\Big(\frac{1}{2}p^2(T,a,x)\chi_{(\delta,A)}+v(T,a,x)p(T,a,x)\Big)\,\mathrm{d}a\,\mathrm{d}x\\
&\quad+\int_{Q}\Big(\frac{1}{2}\vartheta^2\chi_\omega-\vartheta u\chi_\omega\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x+\int_{Q_A}y(0)(u_0-u(0,a,x))\,\mathrm{d}a\,\mathrm{d}x\\
&\quad-\int_{Q_A}u_0y_0\,\mathrm{d}a\,\mathrm{d}x+\int_{Q_A}p(0)(v_0-v(0,a,x))\,\mathrm{d}a\,\mathrm{d}x-\int_{Q_A}v_0p_0\,\mathrm{d}a\,\mathrm{d}x\\
&\quad-\int_{Q}y\Big(\frac{\partial u}{\partial t}+\frac{\partial u}{\partial a}+(k_1u_x)_x-\mu_1u+\beta_1u(t,0,x)-bup\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\
&\quad-\int_{Q}p\Big(\frac{\partial v}{\partial t}+\frac{\partial v}{\partial a}+(k_2v_x)_x-\mu_2v+\beta_2v(t,0,x)+\mu_3uy\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x.
\end{aligned}
$$

Thus, for any $h\in L^2(Q)$, one has $dLh=\frac{\partial L}{\partial y}h+\frac{\partial L}{\partial p}h+\frac{\partial L}{\partial u}h+\frac{\partial L}{\partial v}h+\frac{\partial L}{\partial u_0}h+\frac{\partial L}{\partial v_0}h+\frac{\partial L}{\partial \vartheta}h$.

Keeping in mind that $\frac{\partial L}{\partial u}$ and $\frac{\partial L}{\partial u_0}$ are, respectively, the main equation and the initial condition satisfied by y, as well as $\frac{\partial L}{\partial v}$ and $\frac{\partial L}{\partial v_0}$ are, respectively, the main equation and the initial condition satisfied by p, $dLh=\frac{\partial L}{\partial y}h+\frac{\partial L}{\partial p}h+\frac{\partial L}{\partial \vartheta}h$.

Now, to reach an optimum of L, one must resolve the equation $dLh=0$ for all $h\in L^2(Q)$. Generally, in our situation we will impose sufficient conditions like $\frac{\partial L}{\partial y}h=\frac{\partial L}{\partial p}h=\frac{\partial L}{\partial \vartheta}h=0$.

Actually, to attempt the formula of the adjoint system (11), we just need to have $\frac{\partial L}{\partial y}h=\frac{\partial L}{\partial p}h=0$. The third equation will be used later to express the control ϑ.

On the other hand, recall that for all $h\in L^2(Q)$ we have

$$
\begin{aligned}
\frac{\partial L}{\partial y}h&=\int_{Q_A}(y(T,a,x)\chi_{(\delta,A)}+u(T,a,x))h(T,a,x)\,\mathrm{d}a\,\mathrm{d}x\\
&\quad+\int_{Q_A}(u_0-u(0,a,x))h(0,a,x)\,\mathrm{d}a\,\mathrm{d}x\\
&\quad-\int_{Q}h\Big(\frac{\partial u}{\partial t}+\frac{\partial u}{\partial a}+(k_1u_x)_x-\mu_1u+\beta_1u(t,0,x)-bup\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial L}{\partial p}h&=\int_{Q_A}(p(T,a,x)\chi_{(\delta,A)}+v(T,a,x))h(T,a,x)\,\mathrm{d}a\,\mathrm{d}x\\
&\quad+\int_{Q_A}(v_0-v(0,a,x))h(0,a,x)\,\mathrm{d}a\,\mathrm{d}x\\
&\quad-\int_{Q}h\Big(\frac{\partial v}{\partial t}+\frac{\partial v}{\partial a}+(k_2v_x)_x-\mu_2v+\beta_2v(t,0,x)+\mu_3vy\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x.
\end{aligned}
$$

Sufficient conditions which can be applied to get both $\frac{\partial L}{\partial y}h = \frac{\partial L}{\partial p}h = 0$ are, respectively,

$$
\begin{cases}
u(T,a,x) = -y(T,a,x)\chi_{(\delta,A)} & \text{in } (0,A)\times(0,1),\\
u(0,a,x) = u_0(a,x) & \text{in } (0,A)\times(0,1),\\
\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} + (k_1 u_x)_x - \mu_1 u - bup = -\beta_1 u(t,0,x) & \text{in } Q,
\end{cases} \tag{17}
$$

and

$$
\begin{cases}
v(T,a,x) = -p(T,a,x)\chi_{(\delta,A)} & \text{in } (0,A)\times(0,1),\\
v(0,a,x) = v_0(a,x) & \text{in } (0,A)\times(0,1),\\
\dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial a} + (k_2 v_x)_x - \mu_2 v + \mu_3 vy = -\beta_2 v(t,0,x) & \text{in } Q.
\end{cases} \tag{18}
$$

Finally, the thesis follows gathering (13), (17) and (18) with

$$u_T(a,x) = -y(T,a,x)\chi_{(\delta,A)}(a) \quad \text{in } (0,A)\times(0,1)$$

and

$$v_T(a,x) = -p(T,a,x)\chi_{(\delta,A)}(a) \quad \text{in } (0,A)\times(0,1).$$

□

For the strongly degenerate case, we follow the same computations to obtain the adjoint system with the Newmann boundary condition at $x = 0$ (i.e., $(k_i(x)u_x)_x(0) = 0$, $i = 1, 2$) instead of Dirichlet boundary conditions at $x = 0$ in system (11).

Traditionally, the proof of the Carleman estimates of the full adjoint system (11) is based tightly on the choice of the so-called weight functions. In our case, these functions are set for all $(t, a, x) \in Q$ as

$$
\begin{cases}
\varphi_i := \Theta(t,a)\psi_i(x), & i = 1,2,\\
\Theta(t,a) := \dfrac{1}{(t(T-t))^4 a^4},\\
\psi_i(x) := \lambda_i\Big(\displaystyle\int_0^x \frac{r}{k_i(r)}\,\mathrm{d}r - d_i\Big), & i = 1,2,\\
\varphi(t,a,x) := \Theta(t,a)\mathrm{e}^{\kappa\sigma(x)},\\
\Phi(t,a,x) := \Theta(t,a)\Psi(x),\\
\Psi(x) := \mathrm{e}^{\kappa\sigma(x)} - \mathrm{e}^{2\kappa\|\sigma\|_\infty},
\end{cases} \tag{19}
$$

where σ is the function given by

$$
\begin{cases}
\sigma \in C^2([0,1]),\\
\sigma(x) > 0 & \text{in } (0,1),\ \sigma(0) = \sigma(1) = 0,\\
\sigma_x(x) \neq 0 & \text{in } [0,1]\setminus\omega_0,
\end{cases} \tag{20}
$$

where $\omega_0 \subset\subset \omega$ is an open subset. The existence of σ is proved in [37], Lemma 1.1 using a device of differential geometry. λ_i, d_i and κ are supposed to verify the assumptions

$$
\begin{cases}
d_1 > \dfrac{1}{k_1(1)(2-\gamma)}, \\
\dfrac{\lambda_1}{\lambda_2} \geqslant \dfrac{d_2}{d_1 - \int_0^1 r k_1^{-1}(r)\,\mathrm{d}r}, \\
\kappa \geqslant \dfrac{4\ln(2)}{\|\sigma\|_\infty}, \\
d_2 \geqslant \dfrac{5}{k_2(1)(2-\gamma)},
\end{cases} \tag{21}
$$

with

$$
\lambda_2 \in I = \Big[\frac{k_2(1)(2-\gamma)(\mathrm{e}^{2\kappa\|\sigma\|_\infty}-1)}{d_2 k_2(1)(2-\gamma)-1}, \frac{4(\mathrm{e}^{2\kappa\|\sigma\|_\infty}-\mathrm{e}^{\kappa\|\sigma\|_\infty})}{3d_2}\Big),
$$

which can be shown nonempty (see the proof of Lemma 5.3 in Appendix). On the other hand, in the light of the first and the fourth conditions in (21) on d_1 and d_2, one can observe that $\psi_i(x) < 0$, $i = 1,2$, for all $x \in [0,1]$ and $\Theta(t,a) \to \infty$ as $t \to 0^+, T^-$ and $a \to 0^+$.

The first step being our full ω-Carleman estimate is to show an intermediate Carleman type inequality stated in Theorem 3.9 beneath. To this end, one needs two basic propositions concerned with Carleman type inequalities in both the degenerate and non-degenerate case for one equation model. The first one is:

Proposition 3.4. *Consider the following system with $h \in L^2(Q)$, $\mu \in L^\infty(Q)$ and k verifying* (5)*:*

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (k(x)u_x)_x - \mu(t,a,x)u = h && \text{in } Q,\\
&u(t,a,1) = u(t,a,0) = 0 && \text{on } (0,T)\times(0,A),\\
&u(T,a,x) = u_T(a,x) && \text{in } Q_A,\\
&u(t,A,x) = 0 && \text{in } Q_T.
\end{aligned} \tag{22}
$$

Then there exist two positive constants C and s_0 such that every solution u of (22) *satisfies for all $s \geqslant s_0$ the inequality*

$$
\begin{aligned}
&\int_Q s^3\Theta^3 \frac{x^2}{k(x)} u^2 \mathrm{e}^{2s\varphi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \int_Q s\Theta k(x) u_x^2 \mathrm{e}^{2s\varphi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\
&\quad \leqslant C\Big(\int_Q h^2 \mathrm{e}^{2s\varphi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + sk(1)\int_0^A\int_0^T \Theta u_x(t,a,1)^2 \mathrm{e}^{2s\varphi(t,a,1)}\,\mathrm{d}t\,\mathrm{d}a\Big),
\end{aligned} \tag{23}
$$

where φ *and* Θ *are the weight functions defined by*

$$\begin{cases} \varphi := \Theta(t,a)\psi(x), \\ \Theta(t,a) := \dfrac{1}{(t(T-t))^4 a^4}, \\ \psi(x) := c_1\Big(\displaystyle\int_0^x \frac{r}{k(r)}\,\mathrm{d}r - c_2\Big), \end{cases} \tag{24}$$

with $c_2 > 1/(k(1)(2-\gamma))$, $c_1 > 0$ *and* γ *being the parameter defined by* (5).

P r o o f. For the proof of Proposition 3.4, we refer the reader to [26], Proposition 3.1. By the way, if we want to study the strong degenerate case, i.e., the dispersion coefficient k verifies instead of (5), the assumptions

$$\begin{cases} k \in W^{1,\infty}([0,1]),\ k > 0 \text{ in } (0,1] \text{ and } k(0) = 0, \\ \exists\, \gamma_5 \in [1,2)\colon\ xk'(x) \leqslant \gamma_5 k(x),\ x \in (0,1]. \end{cases} \tag{25}$$

In this case and as is mentioned before, we must replace the Dirichlet boundary condition on $x = 0$ in system (22) by the Newmann boundary condition on $x = 0$ given by $(k(x)u_x)_x(0) = 0$. On the other hand, if one takes a glance on the proof of [26], Proposition 3.1, we see that the procedure is similar for the weakly and strongly degenerate cases except the integral denoted I_{33} given by

$$I_{33} = -2s^3 \int_Q (k(x)\varphi_x \nu_x)(\varphi_x^2 k(x)\nu)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x, \tag{26}$$

where φ is given in (24) and $\nu = \mathrm{e}^{s\varphi}u$. Since $(k(x))^2(\varphi_x)^3 = (c_1)^3(\Theta)^3 x^3/k(x)$, we deduce from (26) that

$$\begin{aligned} I_{33} = &- s^3(c_1)^3 \int_Q (\Theta)^3 \frac{x^3}{k(x)} \frac{d\nu^2}{dx}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ = &- s^3(c_1)^3 \int_0^A \int_0^T (\Theta)^3 \Big[\frac{x^3}{k(x)}\nu^2\Big]_0^1 \mathrm{d}t\,\mathrm{d}a \\ &+ s^3(c_1)^3 \int_Q (\Theta)^3 \frac{x^2(3k(x) - xk'(x))}{k^2(x)}\nu^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x. \end{aligned}$$

Taking into account that $\nu(t,a,1) = 0$, a.e. in $(0,T)\times(0,A)$, we infer that

$$\begin{aligned} I_{33} = &- s^3(c_1)^3 \int_0^A \int_0^T (\Theta)^3 \Big[\frac{x^3}{k(x)}\nu^2\Big]_{x=0} \mathrm{d}t\,\mathrm{d}a \\ &+ s^3(c_1)^3 \int_Q (\Theta)^3 \frac{x^2(3k(x) - xk'(x))}{k^2(x)}\nu^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x. \end{aligned} \tag{27}$$

In the light of the hypotheses (25), the function $x \mapsto x^{\gamma_5}/k(x)$ is nondecreasing in $(0,1)$. Therefore,

$$\forall\, x \in (0,1), \quad 0 < \frac{x^{\gamma_5}}{k(x)} \leqslant \frac{1}{k(1)}.$$

Consequently,

$$\forall\, x \in (0,1), \quad 0 < \frac{x^3}{k(x)} \leqslant \frac{x^{3-\gamma_5}}{k(1)}.$$

Subsequently,

$$\forall\, x \in (0,1), \quad 0 < \frac{x^3}{k(x)}\nu^2 \leqslant \frac{x^{3-\gamma_5}}{k(1)}\nu^2.$$

Using (25) again, we conclude that $\lim\limits_{x\to 0^+} x^{3-\gamma_5} = 0$. Hence,

$$\lim_{x\to 0^+} \frac{x^{3-\gamma_5}}{k(1)}\nu^2(\cdot,\cdot,x) = 0,$$

which involves that

$$\lim_{x\to 0^+} \frac{x^3}{k(x)}\nu^2(\cdot,\cdot,x) = 0.$$

Finally, with the help of (27) we deduce that

$$I_{33} = s^3(c_1)^3 \int_Q (\Theta)^3 \frac{x^2(3k(x) - xk'(x))}{k^2(x)}\nu^2 \,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x.$$

The remainder of the proof of [26], Proposition 3.1 can be treated similarly as for the weakly degenerate case. □

Proposition 3.5. *Let us consider the system*

$$(28)\qquad \begin{aligned} &\frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} + (k(x)z_x)_x - \mu(t,a,x)z = h && \text{in } Q_1,\\ &z(t,a,b_1) = z(t,a,b_2) = 0 && \text{on } (0,T)\times(0,A), \end{aligned}$$

where $Q_1 = (0,T)\times(0,A)\times(b_1,b_2)$, $(b_1,b_2) \subset (0,1)$, $h \in L^2(Q)$, $k \in C^1([0,1])$ is a positive function and $\mu \in L^\infty(Q_1)$. Then there exist two positive constants C and s_0 such that for any $s \geqslant s_0$, the solution z of (28) verifies the estimate

$$(29)\qquad \int_{Q_1} (s^3\varphi^3 z^2 + s\varphi z_x^2)\mathrm{e}^{2s\Phi} \,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \leqslant C\Big(\int_{Q_1} h^2 \mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \int_\omega\int_0^A\int_0^T s^3\varphi^3 z^2 \mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big),$$

where the weight functions φ, Θ and Φ are defined by (19) and σ by (20).

For the proof of Proposition 3.5, careful computations allow us to adapt the same procedure of [1], Lemma 2.1 to show (29) in the case where k is a positive general non-degenerate coefficient, with our weight function Θ given by (19) and the source term h. Besides the two last Propositions 3.4 and 3.5, we must bring out another important result.

Lemma 3.6. *Under assumptions* (21), *the functions* φ_1, φ_2 *and* Φ *stated in* (19) *satisfy the following inequalities:*

$$\begin{cases} \varphi_1 \leqslant \varphi_2, \\ \frac{4}{3}\Phi < \varphi_2 \leqslant \Phi. \end{cases} \tag{30}$$

P r o o f. By the definitions of φ_i, $i = 1, 2$, and Φ and taking into account that Θ is positive, showing the results of (30) is equivalent to showing

$$\begin{cases} \psi_1 \leqslant \psi_2, \\ \frac{4}{3}\Psi < \psi_2 \leqslant \Psi. \end{cases} \tag{31}$$

The first inequality in (31) is assured by the second assumption in (21), while the second one is deduced from $\lambda_2 \in I$ and this completes the proof. □

Due to the nontrivial form of the adjoint system (11) and to ensure a "good" calculus of its Carleman estimates and then the associated observability inequality, the forthcoming notion, which we must introduce, is mollification. Abstractly, a mollification of a given function f is an operator of convolution of f and a C^∞-function called "a mollifier", whose support is exactly the closed ball of center 0 and radius $\tilde{\varepsilon} > 0$ small enough. We recall abstractly and without evidence some characteristics of a mollification.

Firstly, we call "a mollifier" in $\mathbb{R}^n$, $n \geqslant 1$, each function ψ_3 satisfying the following properties:

$$\begin{cases} \operatorname{Supp}(\psi_3) = \overline{B(0,1)}, & \\ \psi_3(X) \geqslant 0 & \forall\, X \in \mathbb{R}^n, \\ \displaystyle\int_{\mathbb{R}^n} \psi_3(X)\,\mathrm{d}X = 1, & \end{cases} \tag{32}$$

where $\overline{B(0,1)}$ stands for the closed ball of center 0 and radius 1. Many examples of functions verifying (32) can be built but one of the most popular mollifiers is given by

$$\psi_3(X) = \begin{cases} \dfrac{1}{\widetilde{M}} \mathrm{e}^{-1/(1-|X|^2)} & \text{if } |X| < 1, \\ 0 & \text{if } |X| \geqslant 1. \end{cases} \tag{33}$$

Here, $\widetilde{M} = \int_{|X|<1} \mathrm{e}^{-1/(1-|X|^2)}\, \mathrm{d}X$ and $|\cdot|$ denotes one of the usual norms of $\mathbb{R}^n$ (for example, take the euclidian norm).

Now, put for all $X \in \mathbb{R}^n$, the function

$$\psi_{3,\tilde{\varepsilon}}(X) = \tilde{\varepsilon}^{-n} \psi_3\Big(\frac{X}{\tilde{\varepsilon}}\Big) \tag{34}$$

with $\tilde{\varepsilon} > 0$ small enough and ψ_3 being the mollifier of (33). Usually and according to the properties of mollifiers, it is well-known that $\lim\limits_{\varepsilon\to 0} \psi_{3,\tilde{\varepsilon}}(X) = \delta(X)$, where δ is exactly the Dirac's function.

Using expression (34), we have the following definition.

Definition 3.7. Let $G \subset \mathbb{R}^n$ be an open set and $f \in L^1_{\mathrm{loc}}(G)$. We call the mollification of f denoted by $f_{\tilde{\varepsilon}}$ the convolution operator $\psi_{3,\tilde{\varepsilon}} * f$ given by

$$\forall\, X \in G, \quad f_{\tilde{\varepsilon}}(X) = \psi_{3,\tilde{\varepsilon}} * f(X) = \int_G \psi_{3,\tilde{\varepsilon}}(X - r) f(r)\, \mathrm{d}r. \tag{35}$$

It is worth pointing out that if $G \subset \mathbb{R}^n$ is an open set, one can extend by the value 0 any function $f \in L^q(G)$, $q \in [1, \infty)$ (or $f \in C_c^\infty(G)$) to $L^q(\mathbb{R}^n)$, $q \in [1, \infty)$ (or $f \in C_c^\infty(\mathbb{R}^n)$).

The definition of mollification (35) generates some interesting and useful properties for the sequel, which are listed in the following proposition.

Proposition 3.8. *Let $G \subset \mathbb{R}^n$ be an open set.*

(1) *If $f \in L^q(G)$, $q \in [1, \infty]$, then $f_{\tilde{\varepsilon}} \in C^\infty(G)$ and $\|f_{\tilde{\varepsilon}}\|_{L^q(G)} \leqslant \|f\|_{L^q(G)}$;*

(2) *If $f \in C_0(G)$, then $f_{\tilde{\varepsilon}} \to f$ as $\tilde{\varepsilon} \to 0$ uniformly on G; i.e.,*

$$\lim_{\tilde{\varepsilon}\to 0} \sup_{X \in G} |f_{\tilde{\varepsilon}}(X) - f(X)| = 0;$$

(3) *If $f \in L^q(G)$, $q \in [1, \infty[$, then $\lim\limits_{\tilde{\varepsilon}\to 0} \|f_{\tilde{\varepsilon}} - f\|_{L^q(G)} = 0$.*

At the end, we recall that the space $C^\infty(G)$ with $G \subset \mathbb{R}^n$ is an open subset endowed with the topology of the semi-norm

$$\forall\, f \in C^\infty(G), \quad p_{K,j}(f) = \sup_{X \in K, |m| \leqslant j} |\partial^m f(X)|, \tag{36}$$

herein, $m = (m_1, m_2, \dots, m_n)$ is a multi-index, $|m| := \sum_{i=1}^{n} m_i$, $K \subset G$ is a compact subset and $j \in \mathbb{N}$ arbitrarily large. For the sake of simplicity we will denote this semi-norm by $\|\cdot\|_{j,K}$. This principal of mollification will be applied in our case via two situations:

The first one is when $n = 3$, $G = Q = (0,T) \times (0,A) \times (0,1)$ and $q = 2$. Since the solution $(y,p) \in L^2(Q) \times L^2(Q)$. Thus, if X and r are vectors from Q, then the mollifications associated, respectively, to y and p will be defined for all $X \in Q$ by

$$\begin{cases} y_{\bar{\varepsilon}}(X) = \int_Q \psi_{3,\bar{\varepsilon}}(X-r)y(r)\,\mathrm{d}r, \\ p_{\bar{\varepsilon}}(X) = \int_Q \psi_{3,\bar{\varepsilon}}(X-r)p(r)\,\mathrm{d}r, \end{cases} \tag{37}$$

where $\psi_{3,\bar{\varepsilon}}$ is defined by (34). Besides, since $y_{\bar{\varepsilon}}$ and $p_{\bar{\varepsilon}}$ are $C^\infty(Q)$-functions, through (36) we have

$$\begin{cases} \|y_{\bar{\varepsilon}}\|_{j,K} := \sup\limits_{(t,a,x)\in K, |m|\leqslant j} |\partial^m y_{\bar{\varepsilon}}(t,a,x)|, \\ \|p_{\bar{\varepsilon}}\|_{j,K} := \sup\limits_{(t,a,x)\in K, |m|\leqslant j} |\partial^m p_{\bar{\varepsilon}}(t,a,x)|, \end{cases} \tag{38}$$

where $K \subset Q$ is a compact subset, j is still an integer arbitrarily large and m is the multi-index such that $|m| = \sum_{i=1}^{3} m_i$.

The second one is when $n = 2$, $G = Q_T = (0,T) \times (0,1)$ and $q = 2$. In fact, the quantities $y(t,0,x)$ and $p(t,0,x)$ for all $(t,x) \in Q_T$ given in the renewal equations of (1) belong to $L^2(Q_T)$ because $(y,p) \in L^2(Q) \times L^2(Q)$ and the fertility rates β_i, $i = 1,2$, are in $L^\infty(Q)$. The associated mollifications of $y(t,0,x)$ and $p(t,0,x)$ are exactly the following convolution operators on Q_T:

$$\begin{cases} g_{y,\bar{\varepsilon}}(t,x) := y_{\bar{\varepsilon}}(t,0,x) = \psi_{3,\bar{\varepsilon}} * y(t,0,x), & (t,x) \in Q_T, \\ g_{p,\bar{\varepsilon}}(t,x) := p_{\bar{\varepsilon}}(t,0,x) = \psi_{3,\bar{\varepsilon}} * p(t,0,x), & (t,x) \in Q_T. \end{cases} \tag{39}$$

Like the first situation, since $g_{y,\bar{\varepsilon}}$ and $g_{p,\bar{\varepsilon}}$ are $C^\infty(Q_T)$-functions, we have

$$\begin{cases} \|g_{y,\bar{\varepsilon}}\|_{j,K} := \sup\limits_{(t,x)\in K, |m|\leqslant j} |\partial^m y_{\bar{\varepsilon}}(t,0,x)|, \\ \|g_{p,\bar{\varepsilon}}\|_{j,K} := \sup\limits_{(t,x)\in K, |m|\leqslant j} |\partial^m p_{\bar{\varepsilon}}(t,0,x)|, \end{cases} \tag{40}$$

where $K \subset Q_T$ is a compact subset, j is still an integer arbitrarily large and m is the multi-index such that $|m| = \sum_{i=1}^{2} m_i$.

Henceforth, it will be suitable and pertinent, in the light of the first point of Proposition 3.8 (especially its first part) that the study of Carleman estimates and then observability inequality will be focused in a first time on the following intermediate adjoint system instead of (11):

$$
(41)\quad \begin{cases}
\dfrac{\partial u}{\partial t}+\dfrac{\partial u}{\partial a}+(k_1(x)u_x)_x-\mu_1(t,a,x)u-b(t,a,x)up_{\tilde\varepsilon}=-\beta_1u(t,0,x) & \text{in } Q,\\
\dfrac{\partial v}{\partial t}+\dfrac{\partial v}{\partial a}+(k_2(x)v_x)_x-\mu_2(t,a,x)v+\mu_3(t,a,x)vy_{\tilde\varepsilon}=-\beta_2v(t,0,x) & \text{in } Q,\\
u(t,a,1)=u(t,a,0)=v(t,a,1)=v(t,a,0)=0 & \text{on } (0,T)\times(0,A),\\
u(T,a,x)=u_T(a,x);\ v(T,a,x)=v_T(a,x) & \text{in } Q_A,\\
u(t,A,x)=v(t,A,x)=0 & \text{in } Q_T.
\end{cases}
$$

Obtaining the observability inequality of (41), the observability inequality of (11) is an immediate consequence of the third point of Proposition 3.8.

Putting this arsenal of devices, we begin firstly by performing an intermediate Carleman type estimate replacing the functions "$-\beta_1u(t,0,x)$" and "$-\beta_2u(t,0,x)$" in (41) with two $L^2(Q)$-functions h_1 and h_2, respectively. This result is stated in the following theorem.

Theorem 3.9. *Consider the system*

$$
(42)\quad \begin{cases}
\dfrac{\partial u}{\partial t}+\dfrac{\partial u}{\partial a}+(k_1(x)u_x)_x-\mu_1(t,a,x)u-b(t,a,x)up_{\tilde\varepsilon}=h_1 & \textit{in } Q,\\
\dfrac{\partial v}{\partial t}+\dfrac{\partial v}{\partial a}+(k_2(x)v_x)_x-\mu_2(t,a,x)v+\mu_3(t,a,x)vy_{\tilde\varepsilon}=h_2 & \textit{in } Q,\\
u(t,a,1)=u(t,a,0)=v(t,a,1)=v(t,a,0)=0 & \textit{on } (0,T)\times(0,A),\\
u(T,a,x)=u_T(a,x);\ v(T,a,x)=v_T(a,x) & \textit{in } Q_A,\\
u(t,A,x)=v(t,A,x)=0 & \textit{in } Q_T,
\end{cases}
$$

where h_1 and h_2 are $L^2(Q)$-functions and $y_{\tilde\varepsilon}$ and $p_{\tilde\varepsilon}$ are given by (37). *Assume that the dispersion coefficients k_i, $i=1,2$, satisfy* (5) *(or* (25)*) and let $A,T>0$ be fixed. Then, there exist two positive constants $C_{\tilde\varepsilon}$ and s_0 such that every solution (u,v) of* (42) *(or with Newmann conditions on $x=0$) verifies, for all $s\geqslant s_0$, the inequality*

$$
(43)\quad \int_Q\Big(s^3\Theta^3\frac{x^2}{k_1(x)}u^2+s\Theta k_1(x)u_x^2\Big)e^{2s\varphi_1}\,dt\,da\,dx
+\int_Q\Big(s^3\Theta^3\frac{x^2}{k_2(x)}v^2+s\Theta k_2(x)v_x^2\Big)e^{2s\varphi_2}\,dt\,da\,dx
\leqslant C_{\tilde\varepsilon}\Big(\int_Q(h_1^2+h_2^2)e^{2s\Phi}\,dt\,da\,dx+\int_q s^3\Theta^3(u^2+v^2)e^{2s\Phi}\,dt\,da\,dx\Big),
$$

where all the weight functions are defined by (19) *and $\tilde\varepsilon$ is defined by* (34).

P r o o f. Let us introduce the smooth cut-off function $\xi\colon \mathbb{R}\to\mathbb{R}$ defined by

$$\tag{44} \begin{cases} 0\leqslant \xi(x)\leqslant 1, & x\in\mathbb{R},\\ \xi(x)=1, & x\in[0,\frac13(2x_1+x_2)],\\ \xi(x)=0, & x\in[\frac13(2x_2+x_1),1], \end{cases}$$

where $(x_1,x_2)\subset\omega$. Let (u,v) be the solution of (42) and set $w:=\xi u$ and $z:=\xi v$ and put $\omega' := (\frac13(2x_1+x_2),\frac13(2x_2+x_1))$. Then (w,z) satisfies the following system:

$$\tag{45} \begin{aligned} &\frac{\partial w}{\partial t}+\frac{\partial w}{\partial a}+(k_1(x)w_x)_x-\mu_1(t,a,x)w\\ &\qquad =\xi h_1+b(t,a,x)wp_{\tilde\varepsilon}+\xi_x k_1u_x+(k_1\xi_x u)_x && \text{in } Q,\\ &\frac{\partial z}{\partial t}+\frac{\partial z}{\partial a}+(k_2(x)z_x)_x-\mu_2(t,a,x)z\\ &\qquad =\xi h_2-\mu_3(t,a,x)zy_{\tilde\varepsilon}+\xi_x k_2v_x+(k_2\xi_x v)_x && \text{in } Q,\\ &w(t,a,1)=w(t,a,0)=z(t,a,1)=z(t,a,0)=0 && \text{on } (0,T)\times(0,A),\\ &w(T,a,x)=w_T(a,x);\ z(T,a,x)=z_T(a,x) && \text{in } Q_A,\\ &w(t,A,x)=z(t,A,x)=0 && \text{in } Q_T. \end{aligned}$$

Using Proposition 3.4 for the inhomogeneous term "$\xi h_1+b(t,a,x)wp_{\tilde\varepsilon}+\xi_x k_1u_x+(k_1\xi_x u)_x$", the definition of ξ and Young's inequality, we get the following inequality:

$$\tag{46} \begin{aligned} \int_Q\Big(s\Theta k_1w_x^2+s^3\Theta^3\frac{x^2}{k_1}w^2\Big)\mathrm{e}^{2s\varphi_1}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x &\leqslant C\bigg(\int_Q(\xi^2(h_1+bup_{\tilde\varepsilon})^2+(\xi_xk_1u_x+(k_1\xi_xu)_x)^2)\mathrm{e}^{2s\varphi_1}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\ &\quad+sk_1(1)\int_0^A\int_0^T\Theta w_x^2(t,a,1)\mathrm{e}^{2s\varphi_1(t,a,1)}\,\mathrm{d}t\,\mathrm{d}a\bigg)\\ &\leqslant \overline{C}\int_Q(\xi^2h_1^2+b^2u^2p_{\tilde\varepsilon}^2+(\xi_xk_1u_x+(k_1\xi_xu)_x)^2)\mathrm{e}^{2s\varphi_1}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x. \end{aligned}$$

Thanks again to the definition of ξ, we have

$$\tag{47} \begin{aligned} \int_0^1(\xi_xk_1u_x+(k_1\xi_xu)_x)^2\mathrm{e}^{2s\varphi_1}\,\mathrm{d}x &\leqslant \int_{\omega'}(8(k_1\xi_x)^2u_x^2+2(k_1\xi_x)_x^2u^2)\mathrm{e}^{2s\varphi_1}\,\mathrm{d}x\\ &\leqslant C\int_{\omega'}(u^2+u_x^2)\mathrm{e}^{2s\varphi_1}\,\mathrm{d}x. \end{aligned}$$

The third assumption in (5) (or the second assumption in (25)) implies that the function $x\mapsto x^2/k_1(x)$ is nondecreasing.

On the other hand, keeping in mind that $p_{\tilde\varepsilon} \in C^\infty(Q)$ (see the first point of Proposition 3.8) and the hypothesis on b in (6), the Hardy-Poincaré inequality in [9] for the function $u\mathrm{e}^{s\varphi_1}$ implies

$$\begin{aligned}\int_0^1 b^2u^2p_{\tilde\varepsilon}^2\mathrm{e}^{2s\varphi_1}\,\mathrm{d}x &\leqslant \frac{\|b\|_\infty^2}{k_1(1)}\|p_{\tilde\varepsilon}\|_{j,K}^2\int_0^1\frac{k_1(x)}{x^2}(u\mathrm{e}^{s\varphi_1})^2\,\mathrm{d}x\\ &\leqslant C_{\mathrm{HP}}\frac{\|b\|_\infty^2}{k_1(1)}\|p_{\tilde\varepsilon}\|_{j,K}^2\int_0^1 k_1(x)(u\mathrm{e}^{s\varphi_1})_x^2\,\mathrm{d}x,\end{aligned}$$

where $C_{\mathrm{HP}} > 0$ is the constant of Hardy-Poincaré and $\|p_{\tilde\varepsilon}\|_{j,K}$ is defined by relation (38). Thus, from the definition of ψ_1 in (19), we obtain

$$\int_0^1 b^2u^2p_{\tilde\varepsilon}^2\mathrm{e}^{2s\varphi_1}\,\mathrm{d}x \leqslant C\int_0^1 k_1(x)u_x^2\mathrm{e}^{2s\varphi_1}\,\mathrm{d}x + C\int_0^1 s^2\Theta^2\frac{x^2}{k_1(x)}u^2\mathrm{e}^{2s\varphi_1}\,\mathrm{d}x.$$

Hence, for s quite large we have

$$\int_0^1 b^2u^2p_{\tilde\varepsilon}^2\mathrm{e}^{2s\varphi_1}\,\mathrm{d}x \leqslant \frac{1}{2}\int_0^1 s\Theta k_1(x)u_x^2\mathrm{e}^{2s\varphi_1}\,\mathrm{d}x + \frac{1}{2}\int_0^1 s^3\Theta^3\frac{x^2}{k_1(x)}u^2\mathrm{e}^{2s\varphi_1}\,\mathrm{d}x. \tag{48}$$

Gathering inequalities (46)–(48) for s quite large, the following inequality holds:

$$\begin{aligned}&\int_Q\Big(s\Theta k_1w_x^2 + s^3\Theta^3\frac{x^2}{k_1}w^2\Big)\mathrm{e}^{2s\varphi_1}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\ &\quad\leqslant \overline{C}\int_Q h_1^2\mathrm{e}^{2s\varphi_1}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + C_1\int_{\omega'}\int_0^A\int_0^T(u^2+u_x^2)\mathrm{e}^{2s\varphi_1}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\ &\qquad+\frac{1}{2}\Big(\int_Q s\Theta k_1(x)u_x^2\mathrm{e}^{2s\varphi_1}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \int_Q s^3\Theta^3\frac{x^2}{k_1(x)}u^2\mathrm{e}^{2s\varphi_1}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big).\end{aligned} \tag{49}$$

Applying the same on "$\xi h_2 - \mu_3(t,a,x)zy_{\tilde\varepsilon} + \xi_xk_2v_x + (k_2\xi_xv)_x$" and taking into account that $y_{\tilde\varepsilon} \in C^\infty(Q)$, we conclude

$$\begin{aligned}&\int_Q\Big(s\Theta k_2z_x^2 + s^3\Theta^3\frac{x^2}{k_2}z^2\Big)\mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\ &\quad\leqslant \overline{C_1}\int_Q h_2^2\mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + C_2\int_{\omega'}\int_0^A\int_0^T(v^2+v_x^2)\mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\ &\qquad+\frac{1}{2}\Big(\int_Q s\Theta k_2(x)v_x^2\mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \int_Q s^3\Theta^3\frac{x^2}{k_2(x)}v^2\mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big).\end{aligned} \tag{50}$$

Summing side by side (49) and (50), using the fact that $\varphi_1 \leqslant \varphi_2$ (see Lemma 3.6) we can see that for s quite large,

$$\begin{aligned}
\int_Q \Big(s\Theta k_1 w_x^2 + s^3\Theta^3 \frac{x^2}{k_1} w^2\Big) \mathrm{e}^{2s\varphi_1}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \int_Q \Big(s\Theta k_2 z_x^2 + s^3\Theta^3 \frac{x^2}{k_2} z^2\Big) \mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x& \\
\leqslant C_4 \int_Q (h_1^2 + h_2^2)\mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x& \\
+ C_5 \int_{\omega'} \int_0^A \int_0^T (u^2 + v^2 + u_x^2 + v_x^2)\mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x& \\
+ \frac{1}{2}\Big(\int_Q s\Theta k_1(x) u_x^2 \mathrm{e}^{2s\varphi_1}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \int_Q s^3\Theta^3 \frac{x^2}{k_1(x)} u^2 \mathrm{e}^{2s\varphi_1}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big)& \\
+ \frac{1}{2}\Big(\int_Q s\Theta k_2(x) v_x^2 \mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \int_Q s^3\Theta^3 \frac{x^2}{k_2(x)} v^2 \mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big).&
\end{aligned}$$

In the light of Caccioppoli's inequality (127), the last inequality becomes

$$\begin{aligned}
(51)\quad \int_Q \Big(s\Theta k_1 w_x^2 + s^3\Theta^3 \frac{x^2}{k_1} w^2\Big) \mathrm{e}^{2s\varphi_1}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x& \\
+ \int_Q \Big(s\Theta k_2 z_x^2 + s^3\Theta^3 \frac{x^2}{k_2} z^2\Big) \mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x& \\
\leqslant C_6\Big(\int_Q (h_1^2 + h_2^2)\mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \int_q s^2\Theta^2 (u^2 + v^2)\mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big).&
\end{aligned}$$

Now, let $W := \eta u$ and $Z := \eta v$ with $\eta = 1 - \xi$. Then W and Z are supported in $(x_1, 1)$ and verify the following system:

$$(52)\quad \begin{cases}
\dfrac{\partial W}{\partial t} + \dfrac{\partial W}{\partial a} + (k_1(x) W_x)_x - \mu_1(t,a,x) W & \\
\qquad = \eta h_1 + b(t,a,x) W p_{\tilde{\varepsilon}} + \eta_x k_1 u_x + (k_1 \eta_x u)_x & \text{in } Q_{x_1}, \\
\dfrac{\partial Z}{\partial t} + \dfrac{\partial Z}{\partial a} + (k_2(x) Z_x)_x - \mu_2(t,a,x) Z & \\
\qquad = \eta h_2 - \mu_3(t,a,x) Z y_{\tilde{\varepsilon}} + \eta_x k_2 v_x + (k_2 \eta_x v)_x & \text{in } Q_{x_1}, \\
W(t,a,1) = W(t,a,x_1) = Z(t,a,1) = Z(t,a,x_1) = 0 & \text{on } (0,T)\times(0,A), \\
W(T,a,x) = W_T(a,x);\ Z(T,a,x) = Z_T(a,x) & \text{in } Q_A, \\
W(t,A,x) = Z(t,A,x) = 0 & \text{in } Q_T,
\end{cases}$$

where $Q_{x_1} = (0,T)\times(0,A)\times(x_1,1)$. Then the system satisfied by W and Z is non-degenerate.

Hence, applying Proposition 3.5 to the first equation of (52) for $b_1 = x_1$ and $b_2 = 1$ and $h = \eta h_1 + b(t,a,x)Wp + \eta_x k_1 u_x + (k_1\eta_x u)_x$, the following estimate occurs:

$$\begin{aligned}\int_Q (s^3\varphi^3 W^2 + s\varphi W_x^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ \leqslant C\Big(\int_Q (\eta h_1 + b(t,a,x)Wp_{\tilde{\varepsilon}} + \eta_x k_1 u_x + (k_1\eta_x u)_x)^2 \mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ + \int_\omega\int_0^A\int_0^T s^3\Theta^3 u^2 \mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big).\end{aligned}$$

Accordingly, Caccioppoli's inequality stated in [26], Lemma 5.1, the definition of the cut-off function η, Young's inequality and s quite large lead to

$$\begin{aligned}\int_Q (s^3\varphi^3 W^2 + s\varphi W_x^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ \leqslant \widetilde{C}\Big(\int_Q (\eta^2 h_1^2 \mathrm{e}^{2s\Phi} + b^2 W^2 p_{\tilde{\varepsilon}}^2 \mathrm{e}^{2s\Phi} + (\eta_x k_1 u_x + (k_1\eta_x u)_x)^2 \mathrm{e}^{2s\Phi})\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ + \int_\omega\int_0^A\int_0^T s^3\Theta^3 u^2\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big) \\ \leqslant \widetilde{C}_1\Big(\int_Q (\eta^2 h_1^2\mathrm{e}^{2s\Phi} + b^2W^2p_{\tilde{\varepsilon}}^2\mathrm{e}^{2s\Phi})\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ + \int_{\omega'}\int_0^A\int_0^T (8(k_1\eta_x)^2u_x^2 + 2((k_1\eta_x)_x)^2u^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big) \\ + \widetilde{C}\int_\omega\int_0^A\int_0^T s^3\Theta^3u^2\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ \leqslant \widetilde{C}_2\Big(\int_Q (\eta^2 h_1^2\mathrm{e}^{2s\Phi} + b^2W^2p_{\tilde{\varepsilon}}^2\mathrm{e}^{2s\Phi})\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ + \int_{\omega'}\int_0^A\int_0^T (u^2+u_x^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big) \\ + \widetilde{C}\int_\omega\int_0^A\int_0^T s^3\Theta^3u^2\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ \leqslant \widetilde{C}_3\Big(\int_Q (\eta^2 h_1^2\mathrm{e}^{2s\Phi} + b^2W^2p_{\tilde{\varepsilon}}^2\mathrm{e}^{2s\Phi})\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ + \int_\omega\int_0^A\int_0^T s^3\Theta^3u^2\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big),\end{aligned} \tag{53}$$

where Φ and φ are defined in (19) and ω' is given at the beginning of this proof.

On the other hand, using the fact that $x \mapsto x^2/k_2(x)$ is nondecreasing, $p_{\tilde{\varepsilon}} \in C^\infty(Q)$, applying the Hardy-Poincaré inequality to the function $W\mathrm{e}^{s\Phi}$ and taking s quite large, the same procedure employed to obtain (48) steers to

$$\begin{aligned}(54)\qquad &\int_Q b^2 W^2 p_{\tilde{\varepsilon}}^2 \mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\leqslant c\Big(\int_Q k_2 W_x^2 \mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \int_Q s^2\Theta^2 \frac{x^2}{k_2(x)} W^2 \mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big) \\ &\leqslant \frac{1}{2}\int_Q (s^3\varphi^3 W^2 + s\varphi W_x^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x.\end{aligned}$$

Therefore, injecting (54) in (53) we arrive to

$$\begin{aligned}(55)\qquad &\int_Q (s^3\varphi^3 W^2 + s\varphi W_x^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\leqslant \widetilde{C}_4\Big(\int_Q h_1^2 \mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \int_\omega\int_0^A\int_0^T s^3\Theta^3 u^2 \mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big).\end{aligned}$$

Replying the same device for the source term $h := \eta h_2 - \mu_3(t,a,x)Zy_{\tilde{\varepsilon}} + \eta_x k_2 v_x + (k_2\eta_x v)_x$ and thanks again to the argument $y_{\tilde{\varepsilon}} \in C^\infty(Q)$ we infer that

$$\begin{aligned}(56)\qquad &\int_Q (s^3\varphi^3 Z^2 + s\varphi Z_x^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\leqslant \widetilde{C}_5\Big(\int_Q h_2^2 \mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \int_\omega\int_0^A\int_0^T s^3\Theta^3 v^2 \mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big).\end{aligned}$$

Subsequently, adding (55) to (56) side by side, we merely observe that

$$\begin{aligned}(57)\qquad &\int_Q (s^3\varphi^3(W^2+Z^2)s\varphi(W_x^2+Z_x^2))\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\leqslant \widetilde{C}_6\Big(\int_Q (h_1^2+h_2^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\quad + \int_\omega\int_0^A\int_0^T s^3\Theta^3(u^2+v^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big).\end{aligned}$$

Using the fact that $u = w + W$ and $v = z + Z$, $\varphi_1 \leqslant \varphi_2 \leqslant \Phi$, estimates (51) and (57) lead to estimate (43). □

For special functions h_1 and h_2, Theorem 3.9 play a crucial role in demonstrating the following intermediate Carleman estimate.

Theorem 3.10. *Assume that assumptions* (5) *(or* (25)*) and* (6) *hold. Let* $T, A > 0$ *be fixed such that* $T \in (0, \delta)$ *with* δ *satisfying* (3). *Then there exist two positive constants* $C_{\tilde\varepsilon}$ *(independent of* δ*) and* s_0 *such that for all* $s \geqslant s_0$, *every solution of* (41) *(or with Newmann conditions on* $x = 0$*)* (u, v) *satisfies*

$$\begin{aligned}(58)\qquad &\int_Q \Big(s^3\Theta^3 \frac{x^2}{k_1(x)} u^2 + s\Theta k_1(x) u_x^2\Big) \mathrm{e}^{2s\varphi_1}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\quad + \int_Q \Big(s^3\Theta^3 \frac{x^2}{k_2(x)} v^2 + s\Theta k_2(x) v_x^2\Big) \mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\leqslant C_{\tilde\varepsilon}\Big(\int_q s^3\Theta^3 (u^2+v^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\quad + \int_0^1\int_0^\delta (u_T^2(a,x) + v_T^2(a,x))\,\mathrm{d}a\,\mathrm{d}x\Big),\end{aligned}$$

where $\tilde\varepsilon$ *is given by* (34).

P r o o f. Let $h_1 := -\beta_1 u(t,0,x)$ and $h_2 := -\beta_2 v(t,0,x)$. Therefore, thanks to hypotheses (6) on β_1 and β_2 and estimate (43) we have the existence of two positive constants C and s_0 such that for all $s \geqslant s_0$ the following inequality holds:

$$\begin{aligned}(59)\quad s^3 &\int_Q \Theta^3\Big(\frac{x^2}{k_1(x)} u^2 \mathrm{e}^{2s\varphi_1} + \frac{x^2}{k_2(x)} v^2 \mathrm{e}^{2s\varphi_2}\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\quad + s\int_Q \Theta(k_1(x)u_x^2 \mathrm{e}^{2s\varphi_1} + k_2(x) v_x^2 \mathrm{e}^{2s\varphi_2})\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\leqslant C_{\tilde\varepsilon}\Big(\int_Q ((\beta_1 u(t,0,x))^2 + (\beta_2 v(t,0,x))^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\quad + \int_q s^3\Theta^3(u^2+v^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big) \\ &\leqslant C_{7,\tilde\varepsilon}\Big(\int_0^1\int_0^T (u^2(t,0,x) + v^2(t,0,x))\,\mathrm{d}t\,\mathrm{d}x \\ &\quad + \int_q s^3\Theta^3(u^2+v^2)\mathrm{e}^{2s\Phi}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big).\end{aligned}$$

Set for all $(t,a,x) \in Q$, $U(t,a,x) = u(T-t, A-a, x)$, $V(t,a,x) = v(T-t, A-a, x)$, $Y_{\tilde\varepsilon}(t,a,x) = y_{\tilde\varepsilon}(T-t, A-a, x)$, $P_{\tilde\varepsilon}(t,a,x) = p_{\tilde\varepsilon}(T-t, A-a, x)$. Then one has

$$\begin{aligned}(60)\quad &\frac{\partial U}{\partial t} + \frac{\partial U}{\partial a} - (k_1(x)U_x)_x + \mu_1(T-t, A-a, x)U + b(T-t, A-a, x)UP_{\tilde\varepsilon} \\ &\qquad = \beta_1(T-t, A-a, x)U(t, A, x) \qquad \text{in } Q, \\ &\frac{\partial V}{\partial t} + \frac{\partial V}{\partial a} + (k_2(x)V_x)_x + \mu_2(T-t, A-a, x)V - \mu_3(T-t, A-a, x)VY_{\tilde\varepsilon} \\ &\qquad = \beta_2(T-t, A-a, x)V(t, A, x) \qquad \text{in } Q,\end{aligned}$$

$$
\begin{aligned}
U(t,a,1) &= U(t,a,0) = V(t,a,1) = V(t,a,0) = 0 && \text{on } (0,T)\times(0,A),\\
U(t,0,x) &= V(t,0,x) = 0 && \text{in } Q_T,\\
U(0,a,x) &= U_0(a,x) = u_T(A-a,x);\\
V(0,a,x) &= V_0(a,x) = v_T(A-a,x) && \text{in } Q_A.
\end{aligned}
$$

We emphasize here that similar implicit formulas of u and v given beneath are already used in the proof of observability inequality in [30] and before in the one of the main Carleman estimates and the observability inequality in [13].

In fact, integrating along the characteristic lines we get

$$
\begin{cases}
U(t,a,\cdot) = \displaystyle\int_0^a S(a-l)(\beta_1(T-t,A-l,\cdot)U(t,A,\cdot)\\
\qquad\qquad -b(T-t,A-l,\cdot)U(t,l,\cdot)P_{\tilde{\varepsilon}}(t,l\cdot))\,\mathrm{d}l & \text{if } t>a,\\
U(t,a,\cdot) = S(t)U_0(a-t,\cdot) + \displaystyle\int_0^t S(t-l)(\beta_1(T-l,A-a,\cdot)U(l,A,\cdot)\\
\qquad\qquad -b(T-l,A-a,\cdot)U(l,a,\cdot)P_{\tilde{\varepsilon}}(l,a,\cdot))\,\mathrm{d}l & \text{if } t\leqslant a,
\end{cases}
\tag{61}
$$

and

$$
\begin{cases}
V(t,a,\cdot) = \displaystyle\int_0^a L(a-l)(\beta_2(T-t,A-l,\cdot)V(t,A,\cdot)\\
\qquad\qquad +\mu_3(T-t,A-l,\cdot)V(t,l,\cdot)Y_{\tilde{\varepsilon}}(t,l\cdot))\,\mathrm{d}l & \text{if } t>a,\\
V(t,a,\cdot) = L(t)V_0(a-t,\cdot) + \displaystyle\int_0^t L(t-l)(\beta_2(T-l,A-a,\cdot)V(l,A,\cdot)\\
\qquad\qquad +\mu_3(T-l,A-a,\cdot)V(l,a,\cdot)Y_{\tilde{\varepsilon}}(l,a\cdot))\,\mathrm{d}l & \text{if } t\leqslant a,
\end{cases}
\tag{62}
$$

where $S(t)_{t\geqslant 0}$ and $L(t)_{t\geqslant 0}$ are the bounded semigroups generated, respectively, by the operators $A_3U := -(k_1(x)U_x)_x + \mu_1(T-t,A-a,x)U$ and $A_4V := -(k_2(x)V_x)_x + \mu_2(T-t,A-a,x)V$.

On the other hand, in the light of the transformations between U and u and also between V and v and if one replaces t by $T-t$ and a by $A-a$ in the implicit formulas (61) and (62) the functions u and v can be expressed as

$$
\begin{cases}
u(t,a,\cdot) = \displaystyle\int_0^{A-a} S(A-a-l)(\beta_1(t,A-l,\cdot)u(t,0,\cdot)\\
\qquad\qquad -b(t,A-l,\cdot)u(t,A-l,\cdot)p_{\tilde{\varepsilon}}(t,A-l\cdot))\,\mathrm{d}l & \text{if } a>t+(A-T),\\
u(t,a,\cdot) = S(T-t)u_T(a+T-t,\cdot) + \displaystyle\int_t^T S(l-t)(\beta_1(l,a,\cdot)u(l,0,\cdot)\\
\qquad\qquad -b(l,a,\cdot)u(l,a,\cdot)p_{\tilde{\varepsilon}}(l,a\cdot))\,\mathrm{d}l & \text{if } a\leqslant t+(A-T),
\end{cases}
\tag{63}
$$

and

$$
(64)\quad \begin{cases} v(t,a,\cdot) = \displaystyle\int_0^{A-a} L(A-a-l)(\beta_2(t,A-l,\cdot)v(t,0,\cdot) \\ \qquad +\mu_3(t,A-l,\cdot)v(t,A-l,\cdot)y_{\tilde\varepsilon}(t,A-l\cdot))\,\mathrm{d}l \qquad \text{if } a > t+(A-T), \\ v(t,a,\cdot) = L(T-t)v_T(a+T-t,\cdot) + \displaystyle\int_t^T L(l-t)(\beta_2(l,a,\cdot)v(l,0,\cdot) \\ \qquad +\mu_3(l,a,\cdot)v(l,a,\cdot)y_{\tilde\varepsilon}(l,a\cdot))\,\mathrm{d}l \qquad \text{if } a \leqslant t+(A-T). \end{cases}
$$

Thus, by relations (39) the following equalities hold:

$$
(65)\quad \begin{cases} u(t,0,\cdot) = S(T-t)u_T(T-t,\cdot) + \displaystyle\int_t^T S(l-t)(\beta_1(l,0,\cdot) - b(l,0,\cdot)g_{p,\tilde\varepsilon}(l,\cdot)) \\ \qquad \times u(l,0,\cdot)\,\mathrm{d}l, \\ v(t,0,\cdot) = L(T-t)v_T(T-t,\cdot) + \displaystyle\int_t^T L(l-t)(\beta_2(l,0,\cdot) + \mu_3(l,0,\cdot)g_{y,\tilde\varepsilon}(l,\cdot)) \\ \qquad \times v(l,0,\cdot)\,\mathrm{d}l. \end{cases}
$$

Passing to the absolute value of the first equality in (65), we get the following relation:

$$
\begin{aligned}
&|u(t,0,x)| \\
&\quad = \left| S(T-t)u_T(T-t,x) + \int_t^T S(l-t)(\beta_1(l,0,x) - b(l,0,x)g_{p,\tilde\varepsilon}(l,x))u(l,0,x)\,\mathrm{d}l \right| \\
&\quad \leqslant |S(T-t)u_T(T-t,x)| + \int_t^T |S(l-t)(\beta_1(l,0,x) - b(l,0,x)g_{p,\tilde\varepsilon}(l,x))u(l,0,x)|\,\mathrm{d}l.
\end{aligned}
$$

Combining the last inequality with the fact that $(S(t))_{t\geqslant 0}$ is a $\mathcal{C}_0$-semigroup, we deduce readily that

$$
\begin{aligned}
&|u(t,0,x)| \\
&\quad \leqslant |S(T-t)u_T(T-t,x)| + \int_t^T |M\mathrm{e}^{\lambda_1(l-t)}(\beta_1(l,0,x) - b(l,0,x)g_{p,\tilde\varepsilon}(l,x))u(l,0,x)|\,\mathrm{d}l \\
&\quad \leqslant |S(T-t)u_T(T-t,x)| + \int_t^T |M\mathrm{e}^{\lambda_1 T}(\beta_1(l,0,x) - b(l,0,x)g_{p,\tilde\varepsilon}(l,x))u(l,0,x)|\,\mathrm{d}l,
\end{aligned}
$$

where

$$
(66)\qquad M \geqslant 1 \quad \text{and} \quad \lambda_1 \in \mathbb{R}.
$$

Applying Hölder's inequality to the last estimate, we obtain

$$
\begin{aligned}
|u(t,0,x)|^2 &\leqslant 2|S(T-t)u_T(T-t,x)|^2 \\
&\quad + 2\Big(\int_t^T |M\mathrm{e}^{\lambda_1 T}(\beta_1(l,0,x) - b(l,0,x)g_{p,\tilde\varepsilon}(l,x))u(l,0,x)|\,\mathrm{d}l\Big)^2 \\
&\leqslant 2|S(T-t)u_T(T-t,x)|^2 \\
&\quad + \int_t^T 2TM^2\mathrm{e}^{2\lambda_1 T}|\beta_1(l,0,x) - b(l,0,x)g_{p,\tilde\varepsilon}(l,x)|^2|u(l,0,x)|^2\,\mathrm{d}l.
\end{aligned}
$$

Accordingly,

$$|u(t,0,x)|^2 \leqslant 2|S(T-t)u_T(T-t,x)|^2 + \int_t^T 4TM^2\mathrm{e}^{2\lambda_1 T}(\beta_1^2(l,0,x)+b^2(l,0,x)g_{p,\tilde\varepsilon}(l,x)^2)|u(l,0,x)|^2\,\mathrm{d}l.$$

Gronwall-Bellman's lemma applied with respect to the time variable time t in (67) implies

$$\begin{aligned}|u(t,0,x)|^2 \leqslant{}& 2|S(T-t)u_T(T-t,x)|^2 + \int_t^T 8TM^2\mathrm{e}^{2\lambda_1 T}(|S(T-s)u_T(T-s,x)|^2)\\ &\times(\beta_1^2(s,0,x)+b^2(s,0,x)g_{p,\tilde\varepsilon}^2(s,x))\\ &\times\exp\Big(\int_t^s 4TM^2\mathrm{e}^{2\lambda_1 T}(\beta_1^2(s,0,x)+b^2(s,0,x)g_{p,\tilde\varepsilon}^2(s,x))\,\mathrm{d}\tau\Big)\,\mathrm{d}s.\end{aligned}$$

Thanks to hypotheses (6) on the natural rates β_i, $i=1,2$, and b and using the second relation of (40) we conclude

$$|u(t,0,x)|^2 \leqslant 2|S(T-t)u_T(T-t,x)|^2 + \widetilde{M}_5\int_t^T |S(T-s)u_T(T-s,x)|^2\,\mathrm{d}s \tag{67}$$

with

$$\begin{aligned}\widetilde{M}_5 ={}& 8TM^2\mathrm{e}^{2\lambda_1 T}(\|\beta_1\|_\infty^2 + A\|b\|_\infty^2\|\beta_2\|_\infty^2\|g_{p,\tilde\varepsilon}\|_{j,K}^2)\\ &\times\exp(4T^2M^2\mathrm{e}^{2\lambda_1 T}(\|\beta_1\|_\infty^2 + A\|b\|_\infty^2\|\beta_2\|_\infty^2\|g_{p,\tilde\varepsilon}\|_{j,K}^2)).\end{aligned}$$

Integrating inequality (67) over $(0,T)\times(0,1)$, we can see that

$$\begin{aligned}\int_0^1\int_0^T |u(t,0,x)|^2\,\mathrm{d}t\,\mathrm{d}x \leqslant{}& 2\int_0^1\int_0^T |S(T-t)u_T(T-t,x)|^2\,\mathrm{d}t\,\mathrm{d}x\\ &+\widetilde{M}_5\int_0^1\int_0^T\int_t^T |S(T-s)u_T(T-s,x)|^2\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x\\ \leqslant{}& 2\int_0^1\int_0^T |S(r)u_T(r,x)|^2\,\mathrm{d}t\,\mathrm{d}x\\ &+\widetilde{M}_5\int_0^1\int_0^T\int_0^{T-t} |S(m)u_T(m,x)|^2\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x,\end{aligned} \tag{68}$$

herein, $m=T-s$ and $r=T-t$. Taking into account that $T\in(0,\delta)$, we have the existence of a positive constant $\widetilde{C}_8$ such that

$$\begin{aligned}\int_0^1\int_0^T |u(t,0,x)|^2\,\mathrm{d}t\,\mathrm{d}x \leqslant{}& \widetilde{C}_8\int_0^1\int_0^T |u_T(m,x)|^2\,\mathrm{d}m\,\mathrm{d}x\\ \leqslant{}& \widetilde{C}_8\int_0^1\int_0^\delta |u_T(m,x)|^2\,\mathrm{d}m\,\mathrm{d}x,\end{aligned} \tag{69}$$

where $\widetilde{C}_8 = M^2\mathrm{e}^{2\lambda_1 T}(2+T\widetilde{M}_5)$ and M and λ_1 are the same as in (66).

Similarly and with the help of the first point of (40), we can prove that

$$\int_0^1\int_0^T |v(t,0,x)|^2\,\mathrm{d}t\,\mathrm{d}x \leqslant \widetilde{C}_9\int_0^1\int_0^T |v_T(l,x)|^2\,\mathrm{d}l\,\mathrm{d}x \leqslant \widetilde{C}_9\int_0^1\int_0^\delta |v_T(l,x)|^2\,\mathrm{d}l\,\mathrm{d}x. \tag{70}$$

Implementing (69) and (70) in (59) we reach the Carleman estimate (58). □

Before continuing, we shall evoke the following remark.

Remark 3.11. In general, if we want to express the implicit formula of a population dynamics model's solution, the characteristic method is the pertinent candidate. The principle of this method is to write the solution of the studied model covering all the whole $(0,T)\times(0,A)$ by deleting one of the two variables, time or age in a two main sub-parts of $(0,T)\times(0,A)$. Classically, these sub-parts are separated via a given line whose equation is $a=t+c$, where $c>0$, which in our case is equal to $A-T$.

Following this, we obtain the formula of our solution both in the two parts $a>t+c$ and $a\leqslant t+c$ and this is exactly what happened in the implicit formulas of u and v defined, respectively, by relations (63) and (64).

If $A=T$, i.e., $(0,T)\times(0,A)$ is a square, we get the classical implicit formulas in the two parts $a>t$ and $a\leqslant t$ by dividing the pavement $(0,T)\times(0,A)$ with respect to the first bisector given by the equation $a=t$.

Since our aim is to prove the null controllability property (4) for one control force problem, one must somehow "delete" the adjoint variable to the non-controlled solution, which is in our case v from the right-hand side of Carleman estimate (58). Hence, we presume the following result:

Theorem 3.12. *Let the assumptions on k_i, $i=1,2$, (5) (or (25)) and on the natural rates (6) be verified. Let $A,T>0$ be given and fixed such that $T\in(0,\delta)$, where δ verifies (3). Then every solution (u,v) of (41) (or with Newmann conditions on $x=0$) satisfies*

$$\begin{aligned}\int_Q\Big(s^3\Theta^3\frac{x^2}{k_1(x)}u^2+s\Theta k_1(x)u_x^2\Big)\mathrm{e}^{2s\varphi_1}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x &+\int_Q\Big(s^3\Theta^3\frac{x^2}{k_1(x)}v^2+s\Theta k_1(x)v_x^2\Big)\mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\ &\leqslant C_{\widetilde{\varepsilon}}\Big(\int_q u^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x+\int_0^1\int_0^\delta (u_T^2(a,x)+v_T^2(a,x))\,\mathrm{d}a\,\mathrm{d}x\Big).\end{aligned}\tag{71}$$

This theorem is an immediate outcome of Theorem 3.10 and the following lemma.

Lemma 3.13. *Assume that* (5) *(or* (25)*) and* (6) *hold and let* $A, T > 0$ *be given and fixed such that* $T \in (0, \delta)$*, where* δ *verifies* (3)*. Then there exists a positive constant* M *(which is independent from* δ*) such that for every solution* (u, v) *of* (41) *(or with Newmann conditions on* $x = 0$*) the following inequality occurs*

$$\int_\omega \int_0^A \int_0^T s^3 \Theta^3 v^2 \mathrm{e}^{2s\Phi} \,\mathrm{d}t \,\mathrm{d}a \,\mathrm{d}x \leqslant M \int_0^1 \int_0^\delta v_T^2(a, x) \,\mathrm{d}a \,\mathrm{d}x; \tag{72}$$

here M *is different from the one in* (66)*.*

Proof. The proof of Lemma 3.13 is similar to the one of observability inequality (73) in the step to prove inequality (85) beneath specially relations (86) (in the case $a > t + (A - T)$) and (113) and (114) (in the case $a \leqslant t + (A - T)$). We must also take into account inequality (116) in Remark 4.2 and set $R_1(t, a, x) := s^{3/2} \Theta^{3/2} \mathrm{e}^{s\Phi}$, where Θ and Φ are the weight functions given in (19).

Finally, it deserves to point out that the boundedness of R_1 (actually $(R_1)^2$) is a consequence of the fact that for all $\tilde{r} \in \mathbb{R}$, $\sup_Q s^{\tilde{r}} \Theta^{\tilde{r}} \mathrm{e}^{2s\Phi} < \infty$. □

The full ω-Carleman estimate (71) can be used in a standard way to obtain a relevant observability inequality of system (41) (or with Newmann conditions on $x = 0$). This together with the third point of Proposition 3.8 lead to the observability inequality of (11) and afterwards the null controllability result for the coupled population dynamics system (1) with one control force.

4. Observability inequality and null controllability result

4.1. Observability inequality result. This paragraph is devoted to the observability inequality of system (41) (or with Newmann conditions on $x = 0$). The proof is based essentially on Carleman estimate (71) and Hardy-Poincaré inequality with the help of Gronwall-Bellman's lemma.

Proposition 4.1. *Assume that* (5) *(or* (25)*) and* (6) *hold. Let* $A, T > 0$ *be given and fixed such that* $T < \delta$ *with* δ *verifying* (3)*. Then there exists a positive constant* $C_{\delta,\tilde{\varepsilon}}$ *such that for every solution* (u, v) *of* (41) *(or with Newmann conditions on* $x = 0$*) the following observability inequality is satisfied:*

$$\begin{aligned} \int_0^1 \int_0^A &(u^2(0, a, x) + v^2(0, a, x)) \,\mathrm{d}a \,\mathrm{d}x \\ &\leqslant C_{\delta,\tilde{\varepsilon}} \Big(\int_q u^2 \,\mathrm{d}t \,\mathrm{d}a \,\mathrm{d}x + \int_0^1 \int_0^\delta (u_T^2(a, x) + v_T^2(a, x)) \,\mathrm{d}a \,\mathrm{d}x \Big). \end{aligned} \tag{73}$$

P r o o f. For $\kappa_1 > 0$ to be defined later, let $\tilde{u} = \mathrm{e}^{\kappa_1 t} u$ and $\tilde{v} = \mathrm{e}^{\kappa_1 t} v$, where (u, v) is the solution of (11). Then $(\tilde{u}, \tilde{v})$ verifies the system

$$\begin{aligned}
&\frac{\partial \tilde{u}}{\partial t} + \frac{\partial \tilde{u}}{\partial a} + (k_1(x)\tilde{u}_x)_x - (\mu_1 + \kappa_1 + bp_{\bar{\varepsilon}})\tilde{u} = -\beta_1 \tilde{u}(t,0,x) && \text{in } Q,\\
&\frac{\partial \tilde{v}}{\partial t} + \frac{\partial \tilde{v}}{\partial a} + (k_2(x)\tilde{v}_x)_x - (\mu_2 + \kappa_1 - \mu_3 y_{\bar{\varepsilon}})\tilde{v} = -\beta_2 \tilde{v}(t,0,x) && \text{in } Q,\\
&\tilde{u}(t,a,1) = \tilde{u}(t,a,0) = \tilde{v}(t,a,1) = \tilde{v}(t,a,0) = 0 && \text{on } (0,T)\times(0,A),\\
&\tilde{u}(T,a,x) = \mathrm{e}^{\kappa_1 T} u_T(a,x);\ \tilde{v}(T,a,x) = \mathrm{e}^{\kappa_1 T} v_T(a,x) && \text{in } Q_A,\\
&\tilde{u}(t,A,x) = \tilde{v}(t,A,x) = 0 && \text{in } Q_T.
\end{aligned} \tag{74}$$

Multiplying the first and the second equation of (74) by $\tilde{u}$ and $\tilde{v}$, respectively, and integrating by parts the new equations over $Q_t = (0,t)\times(0,A)\times(0,1)$, taking into account the rest of equations in (74), we get

(75)

$$\begin{aligned}
&-\frac{1}{2}\int_0^1\int_0^A \tilde{u}^2(t,a,x)\,\mathrm{d}a\,\mathrm{d}x + \frac{1}{2}\int_0^1\int_0^A \tilde{u}^2(0,a,x)\,\mathrm{d}a\,\mathrm{d}x + \frac{1}{2}\int_0^1\int_0^t \tilde{u}^2(\tau,0,x)\,\mathrm{d}\tau\,\mathrm{d}x\\
&+\int_{Q_t} k_1\tilde{u}_x^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x + \int_{Q_t}(\kappa_1 + \mu_1 + bp_{\bar{\varepsilon}})\tilde{u}^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x = \int_{Q_t}\beta_1\tilde{u}\tilde{u}(\tau,0,x)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x
\end{aligned}$$

and

(76)

$$\begin{aligned}
&-\frac{1}{2}\int_0^1\int_0^A \tilde{v}^2(t,a,x)\,\mathrm{d}a\,\mathrm{d}x + \frac{1}{2}\int_0^1\int_0^A \tilde{v}^2(0,a,x)\,\mathrm{d}a\,\mathrm{d}x + \frac{1}{2}\int_0^1\int_0^t \tilde{v}^2(\tau,0,x)\,\mathrm{d}\tau\,\mathrm{d}x\\
&+\int_{Q_t} k_2\tilde{v}_x^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x + \int_{Q_t}(\kappa_1 + \mu_1 - \mu_3 y_{\bar{\varepsilon}})\tilde{v}^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x = \int_{Q_t}\beta_2\tilde{v}\tilde{v}(\tau,0,x)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x.
\end{aligned}$$

Summing (75) and (76), we have

$$\begin{aligned}
\int_0^1\int_0^A (\tilde{u}^2(0,a,x) + \tilde{v}^2(0,a,x))\,\mathrm{d}a\,\mathrm{d}x &+ \int_0^1\int_0^t (\tilde{u}^2(\tau,0,x) + \tilde{v}^2(\tau,0,x))\,\mathrm{d}\tau\,\mathrm{d}x\\
&+ 2\int_{Q_t}(k_1\tilde{u}_x^2 + k_2\tilde{v}_x^2)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x + 2\int_{Q_t}(\mu_1 + bp_{\bar{\varepsilon}})\tilde{u}^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x\\
&+ 2\int_{Q_t}(\mu_2 - \mu_3 y_{\bar{\varepsilon}})\tilde{v}^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x + 2\int_{Q_t}\kappa_1(\tilde{u}^2 + \tilde{v}^2)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x\\
&= 2\Big(\int_{Q_t}\beta_1\tilde{u}\tilde{u}(\tau,0,x)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x + \int_{Q_t}\beta_2\tilde{v}\tilde{v}(\tau,0,x)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x\Big)\\
&+ \int_0^1\int_0^A (\tilde{u}^2(t,a,x) + \tilde{v}^2(t,a,x))\,\mathrm{d}a\,\mathrm{d}x.
\end{aligned}$$

Thus,

$$\begin{aligned}
(77)\quad & \int_0^1 \int_0^A (\tilde{u}^2(0,a,x) + \tilde{v}^2(0,a,x))\,\mathrm{d}a\,\mathrm{d}x \\
& + \int_0^1 \int_0^t (\tilde{u}^2(\tau,0,x) + \tilde{v}^2(\tau,0,x))\,\mathrm{d}\tau\,\mathrm{d}x + 2\int_{Q_t} bp_{\tilde{\varepsilon}}\tilde{u}^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x \\
& - 2\int_{Q_t} \mu_3 y_{\tilde{\varepsilon}}\tilde{v}^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x + 2\int_{Q_t} \kappa_1(\tilde{u}^2 + \tilde{v}^2)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x \\
& \leqslant 2\Big(\int_{Q_t} \beta_1\tilde{u}\tilde{u}(\tau,0,x)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x + \int_{Q_t} \beta_2\tilde{v}\tilde{v}(\tau,0,x)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x\Big) \\
& + \int_0^1 \int_0^A (\tilde{u}^2(t,a,x) + \tilde{v}^2(t,a,x))\,\mathrm{d}a\,\mathrm{d}x.
\end{aligned}$$

With the help of Young's inequality, one can check out the following relations:

$$\begin{aligned}
(78)\quad & 2\int_{Q_t} \beta_1\tilde{u}\tilde{u}(\tau,0,x)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x \\
& = 2\int_{Q_t} \frac{\beta_1}{4\sqrt{\varepsilon}}\tilde{u}4\sqrt{\varepsilon}\tilde{u}(\tau,0,x)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x \\
& \leqslant \frac{\|\beta_1\|_\infty^2}{16\varepsilon}\int_{Q_t} \tilde{u}^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x + 16\varepsilon\int_{Q_t} \tilde{u}^2(\tau,0,x)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
(79)\quad & 2\int_{Q_t} \beta_1\tilde{v}\tilde{v}(\tau,0,x)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x \\
& \leqslant \frac{\|\beta_2\|_\infty^2}{16\varepsilon}\int_{Q_t} \tilde{v}^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x + 16\varepsilon\int_{Q_t} \tilde{v}^2(\tau,0,x)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x.
\end{aligned}$$

As a consequence of (77), (78) and (79) one has

$$\begin{aligned}
(80)\quad & \int_0^1 \int_0^A (\tilde{u}^2(0,a,x) + \tilde{v}^2(0,a,x))\,\mathrm{d}a\,\mathrm{d}x + \int_0^1 \int_0^t (\tilde{u}^2(\tau,0,x) + \tilde{v}^2(\tau,0,x))\,\mathrm{d}\tau\,\mathrm{d}x \\
& + 2\int_{Q_t} bp_{\tilde{\varepsilon}}\tilde{u}^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x - 2\int_{Q_t} \mu_3 y_{\tilde{\varepsilon}}\tilde{v}^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x \\
& + 2\int_{Q_t} \kappa_1(\tilde{u}^2 + \tilde{v}^2)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x \\
& \leqslant \frac{\|\beta_1\|_\infty^2}{16\varepsilon}\int_{Q_t} \tilde{u}^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x + \frac{\|\beta_2\|_\infty^2}{16\varepsilon}\int_{Q_t} \tilde{v}^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x \\
& + 16A\varepsilon\int_0^1 \int_0^t (\tilde{u}^2(\tau,0,x) + \tilde{v}^2(\tau,0,x))\,\mathrm{d}\tau\,\mathrm{d}x \\
& + \int_0^1 \int_0^A (\tilde{u}^2(t,a,x) + \tilde{v}^2(t,a,x))\,\mathrm{d}a\,\mathrm{d}x.
\end{aligned}$$

For $\varepsilon < 1/(16A)$, we deduce from (80) that

$$\begin{aligned}\int_0^1\int_0^A(\tilde u^2(0,a,x)+\tilde v^2(0,a,x))\,\mathrm{d}a\,\mathrm{d}x+2\int_{Q_t}bp_{\tilde\varepsilon}\tilde u^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x&\\ -2\int_{Q_t}\mu_3y_{\tilde\varepsilon}\tilde v^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x+2\int_{Q_t}\kappa_1(\tilde u^2+\tilde v^2)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x&\\ \leqslant\frac{\max(\|\beta_1\|_\infty^2,\|\beta_2\|_\infty^2)}{16\varepsilon}\int_{Q_t}(\tilde u^2+\tilde v^2)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x&\\ +\int_0^1\int_0^A(\tilde u^2(t,a,x)+\tilde v^2(t,a,x))\,\mathrm{d}a\,\mathrm{d}x.&\end{aligned}$$

Subsequently,

$$\begin{aligned}\int_0^1\int_0^A(\tilde u^2(0,a,x)+\tilde v^2(0,a,x))\,\mathrm{d}a\,\mathrm{d}x+2\int_{Q_t}\kappa_1(\tilde u^2+\tilde v^2)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x&\\ \leqslant-2\int_{Q_t}bp_{\tilde\varepsilon}\tilde u^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x+2\int_{Q_t}\mu_3y_{\tilde\varepsilon}\tilde v^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x&\\ +\frac{\max(\|\beta_1\|_\infty^2,\|\beta_2\|_\infty^2)}{16\varepsilon}\int_{Q_t}(\tilde u^2+\tilde v^2)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x&\\ +\int_0^1\int_0^A(\tilde u^2(t,a,x)+\tilde v^2(t,a,x))\,\mathrm{d}a\,\mathrm{d}x.&\end{aligned}\tag{81}$$

In the light of assumptions (6) on b and μ_3 and using the fact that $y_{\tilde\varepsilon}, p_{\tilde\varepsilon}\in C^\infty(Q)$, relation (81) steers to

$$\begin{aligned}\int_0^1\int_0^A(\tilde u^2(0,a,x)+\tilde v^2(0,a,x))\,\mathrm{d}a\,\mathrm{d}x+2\int_{Q_t}\kappa_1(\tilde u^2+\tilde v^2)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x&\\ \leqslant2\|b\|_\infty\|p_{\tilde\varepsilon}\|_{j,K}^2\int_{Q_t}\tilde u^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x&\\ +2\|\ \mu_3\|_\infty\|y_{\tilde\varepsilon}\|_{j,K}^2\int_{Q_t}\tilde v^2\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x&\\ +\frac{\max(\|\beta_1\|_\infty^2,\|\beta_2\|_\infty^2)}{16\varepsilon}\int_{Q_t}(\tilde u^2+\tilde v^2)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x&\\ +\int_0^1\int_0^A(\tilde u^2(t,a,x)+\tilde v^2(t,a,x))\,\mathrm{d}a\,\mathrm{d}x&\\ \leqslant\Big(2\max(\|b\|_\infty\|p_{\tilde\varepsilon}\|_{j,K}^2,\|\mu_3\|_\infty\|y_{\tilde\varepsilon}\|_{j,K}^2)&\\ +\frac{\max(\|\beta_1\|_\infty^2,\|\beta_2\|_\infty^2)}{16\varepsilon}\Big)\int_{Q_t}(\tilde u^2+\tilde v^2)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}x&\\ +\int_0^1\int_0^A(\tilde u^2(t,a,x)+\tilde v^2(t,a,x))\,\mathrm{d}a\,\mathrm{d}x.&\end{aligned}\tag{82}$$

Taking now $\kappa_1 \geqslant \max(\|b\|_\infty \|p_{\tilde{\varepsilon}}\|^2_{j,K}, \|\mu_3\|_\infty \|y_{\tilde{\varepsilon}}\|^2_{j,K}) + \max(\|\beta_1\|^2_\infty, \|\beta_2\|^2_\infty)/(32\varepsilon)$ and thanks to the definitions of $\tilde{u}$ and $\tilde{v}$, inequality (82) is reduced to

(83)
$$\int_0^1 \int_0^A (u^2(0,a,x) + v^2(0,a,x))\,\mathrm{d}a\,\mathrm{d}x \leqslant \mathrm{e}^{\kappa_1 T} \int_0^1 \int_0^A (u^2(t,a,x) + v^2(t,a,x))\,\mathrm{d}a\,\mathrm{d}x.$$

Integrating (83) over $(\frac{1}{4}T, \frac{3}{4}T)$, we get

$$\begin{aligned}(84)\qquad &\int_0^1 \int_0^A (u^2(0,a,x) + v^2(0,a,x))\,\mathrm{d}a\,\mathrm{d}x \\ &\qquad \leqslant \frac{2\mathrm{e}^{\kappa_1 T}}{T} \int_0^1 \int_0^A \int_{T/4}^{3T/4} (u^2(t,a,x) + v^2(t,a,x))\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x.\end{aligned}$$

Henceforth, the crucial step to establish the observability inequality (73) is to show the existence of a positive constant $\widehat{C}$ such that

$$\begin{aligned}(85)\qquad &\int_0^1 \int_0^{\delta - 3T/4} \int_{T/4}^{3T/4} (u^2(t,a,x) + v^2(t,a,x))\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\qquad \leqslant \widehat{C} \int_0^1 \int_0^\delta (u_T^2(a,x) + v_T^2(a,x))\,\mathrm{d}a\,\mathrm{d}x.\end{aligned}$$

To this end, we use the implicit formulas of u and v given, respectively, by (63) and (64), the formulas of the initial datum with respect to the age (65) as well as the expressions of (39). Such a proof will be split on the cases when $a > t + (A - T)$ and $a \leqslant t + (A-T)$ (see again the two references [13] and [30] for a similar argumentation and also Remark 3.11). In fact, if $a > t + (A - T)$, after careful calculus one has

$$(86)\quad \begin{cases} u(t,a,\cdot) = \displaystyle\int_0^{A-a} S(A-a-l)\beta_1(t,A-l,\cdot)S(T-t)u_T(T-t,\cdot)\,\mathrm{d}l \\ \qquad + \displaystyle\int_0^{A-a} S(A-a-l)\Big(\beta_1(t,A-l,\cdot)\int_t^T S(m-t)(\beta_1(m,0,\cdot) \\ \qquad -b(m,0,\cdot)g_{p,\tilde{\varepsilon}}(m,\cdot))u(m,0,\cdot)\,\mathrm{d}m\Big)\,\mathrm{d}l \\ \qquad - \displaystyle\int_0^{A-a} S(A-a-l)b(t,A-l,\cdot)u(t,A-l,\cdot)p_{\tilde{\varepsilon}}(t,A-l,\cdot)\,\mathrm{d}l, \\ v(t,a,\cdot) = \displaystyle\int_0^{A-a} L(A-a-l)\beta_2(t,A-l,\cdot)L(T-t)v_T(T-t,\cdot)\,\mathrm{d}l \\ \qquad + \displaystyle\int_0^{A-a} L(A-a-l)\Big(\beta_2(t,A-l,\cdot)\int_t^T L(m-t)(\beta_2(m,0,\cdot) \\ \qquad +\mu_3(m,0,\cdot)g_{y,\tilde{\varepsilon}}(m,\cdot))v(m,0,\cdot)\,\mathrm{d}m\Big)\,\mathrm{d}l \\ \qquad + \displaystyle\int_0^{A-a} L(A-a-l)\mu_3(t,A-l,\cdot)v(t,A-l,x)y_{\tilde{\varepsilon}}(t,A-l,\cdot)\,\mathrm{d}l, \end{cases}$$

where $(S(t))_{t\geqslant 0}$ and $(L(t))_{t\geqslant 0}$ are the semi-groups defined after relations (61) and (62). Thus, we claim from (86) that exists $\widetilde{M_3}, \widetilde{M_4} > 0$ such that

$$\tag{87} \begin{cases} \displaystyle\int_0^1 \int_0^{\delta - 3T/4} \int_{T/4}^{3T/4} u^2(t,a,x)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \leqslant \widetilde{M_3} \int_0^1 \int_0^{\delta} u_T^2(a,x)\,\mathrm{d}a\,\mathrm{d}x, \\ \displaystyle\int_0^1 \int_0^{\delta - 3T/4} \int_{T/4}^{3T/4} v^2(t,a,x)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \leqslant \widetilde{M_4} \int_0^1 \int_0^{\delta} v_T^2(a,x)\,\mathrm{d}a\,\mathrm{d}x. \end{cases}$$

The proofs of the two last inequalities are similar, so we restrict ourselves to show the first one. From the first equality of (86) one has

$$\tag{88} \begin{aligned} |u(t,a,x)| = \bigg| &\int_0^{A-a} S(A-a-l)\beta_1(t,A-l,x)S(T-t)u_T(T-t,x)\,\mathrm{d}l \\ &+ \int_0^{A-a} S(A-a-l)\bigg(\beta_1(t,A-l,x)\int_t^T S(m-t)(\beta_1(m,0,x) \\ &- b(m,0,x)g_{p,\tilde{\varepsilon}}(m,x))u(m,0,x)\,\mathrm{d}m\bigg)\,\mathrm{d}l \\ &- \int_0^{A-a} S(A-a-l)b(t,A-l,x)u(t,A-l,x)p(t,A-l,x)\,\mathrm{d}l\bigg|. \end{aligned}$$

Subsequently,

$$\tag{89} \begin{aligned} |u(t,a,x)| &\leqslant \bigg|\int_0^{A-a} S(A-a-l)\beta_1(t,A-l,x)S(T-t)u_T(T-t,x)\,\mathrm{d}l\bigg| \\ &+ \bigg|\int_0^{A-a} S(A-a-l)\bigg(\beta_1(t,A-l,x)\int_t^T S(m-t)(\beta_1(m,0,x) \\ &- b(m,0,x)g_{p,\tilde{\varepsilon}}(m,x))u(m,0,x)\,\mathrm{d}m\bigg)\,\mathrm{d}l\bigg| \\ &+ \bigg|\int_0^{A-a} S(A-a-l)b(t,A-l,x)u(t,A-l,x)p_{\tilde{\varepsilon}}(t,A-l,x)\,\mathrm{d}l\bigg| \\ &\leqslant \int_0^{A-a} |S(A-a-l)\beta_1(t,A-l,x)S(T-t)u_T(T-t,x)|\,\mathrm{d}l \\ &+ \int_0^{A-a} \bigg|S(A-a-l)\bigg(\beta_1(t,A-l,x)\int_t^T S(m-t)(\beta_1(m,0,x) \\ &- b(m,0,x)g_{p,\tilde{\varepsilon}}(m,x))u(m,0,x)\,\mathrm{d}m\bigg)\bigg|\,\mathrm{d}l \\ &+ \int_0^{A-a} |S(A-a-l)b(t,A-l,x)u(t,A-l,x)p_{\tilde{\varepsilon}}(t,A-l,x)|\,\mathrm{d}l. \end{aligned}$$

With the variable change $r = A - l$, (89) becomes

$$|u(t,a,x)| \leqslant \int_a^A |S(r-a)\beta_1(t,r,x)S(T-t)u_T(T-t,x)|\,\mathrm{d}r + \int_a^A \biggl|S(r-a)\biggl(\beta_1(t,r,x)\int_t^T S(m-t)(\beta_1(m,0,x) - b(m,0,x)g_{p,\tilde{\varepsilon}}(m,x))u(m,0,x)\,\mathrm{d}m\biggr)\biggr|\,\mathrm{d}r + \int_a^A |S(r-a)b(t,r,x)u(t,r,x)p_{\tilde{\varepsilon}}(t,r,x)|\,\mathrm{d}r.$$

Since $(S(t))_{t\geqslant 0}$ is a $\mathcal{C}_0$-semigroup, then

$$\|S(r-a)\| \leqslant M\mathrm{e}^{\lambda_1(r-a)} \leqslant M\mathrm{e}^{A\lambda_1}, \tag{90}$$

where M and λ_1 are the same as in (66). Hence,

$$|u(t,a,x)| \leqslant \int_a^A |S(r-a)\beta_1(t,r,x)S(T-t)u_T(T-t,x)|\,\mathrm{d}r + \int_a^A M\mathrm{e}^{A\lambda_1}\biggl|\biggl(\beta_1(t,r,x)\int_t^T S(m-t)(\beta_1(m,0,x) - b(m,0,x)g_{p,\tilde{\varepsilon}}(m,x))u(m,0,x)\,\mathrm{d}m\biggr)\biggr|\,\mathrm{d}r + \int_a^A M\mathrm{e}^{A\lambda_1}|b(t,r,x)u(t,r,x)p_{\tilde{\varepsilon}}(t,r,x)|\,\mathrm{d}r.$$

Afterwards,

$$\begin{aligned}|u(t,a,x)|^2 &\leqslant \biggl(\int_a^A |S(r-a)\beta_1(t,r,x)S(T-t)u_T(T-t,x)|\,\mathrm{d}r + \int_a^A M\mathrm{e}^{A\lambda_1}\biggl|\biggl(\beta_1(t,r,x)\int_t^T S(m-t)(\beta_1(m,0,x) - b(m,0,x)g_{p,\tilde{\varepsilon}}(m,x))u(m,0,x)\,\mathrm{d}m\biggr)\biggr|\,\mathrm{d}r \\ &\quad + \int_a^A M\mathrm{e}^{A\lambda_1}|b(t,r,x)u(t,r,x)p_{\tilde{\varepsilon}}(t,r,x)|\,\mathrm{d}r\biggr)^2 \\ &\leqslant 3\biggl(\int_a^A |S(r-a)\beta_1(t,r,x)S(T-t)u_T(T-t,x)|\,\mathrm{d}r\biggr)^2 \\ &\quad + 3\biggl(\int_a^A M\mathrm{e}^{A\lambda_1}\biggl|\biggl(\beta_1(t,r,x)\int_t^T S(m-t)(\beta_1(m,0,x) - b(m,0,x)g_{p,\tilde{\varepsilon}}(m,x))u(m,0,x)\,\mathrm{d}m\biggr)\biggr|\,\mathrm{d}r\biggr)^2 \\ &\quad + 3\biggl(\int_a^A M\mathrm{e}^{A\lambda_1}|b(t,r,x)u(t,r,x)p_{\tilde{\varepsilon}}(t,r,x)|\,\mathrm{d}r\biggr)^2.\end{aligned} \tag{91}$$

Applying now Hölder's inequality to (91) we obtain

$$|u(t,a,x)|^2 \leqslant 3\int_a^A A|S(r-a)\beta_1(t,r,x)S(T-t)u_T(T-t,x)|^2\,\mathrm{d}m + 3\int_a^A AM^2\mathrm{e}^{2A\lambda_1}|b(t,r,x)u(t,r,x)p_{\tilde\varepsilon}(t,r,x)|^2\,\mathrm{d}r + 3\int_a^A AM^2\mathrm{e}^{2A\lambda_1}\biggl(\beta_1(t,r,x)\int_t^T S(m-t)(\beta_1(m,0,x) - b(m,0,x)g_{p,\tilde\varepsilon}(m,x))u(m,0,x)\,\mathrm{d}m\biggr)^2\,\mathrm{d}r. \tag{92}$$

On the other hand, the relation

$$\|S(m-t)\| \leqslant M\mathrm{e}^{\lambda_1(m-t)} \leqslant M\mathrm{e}^{T\lambda_1},$$

together with the fact that $g_{p,\tilde\varepsilon} \in C^\infty(Q_T)$, hypotheses (6) on β_1 and b and again Hölder inequality lead, respectively, to the following successive estimates

$$\begin{aligned}&3\int_a^A AM^2\mathrm{e}^{2A\lambda_1}\\ &\quad\times\biggl(\beta_1(t,r,x)\int_t^T |S(m-t)(\beta_1(m,0,x) - b(m,0,x)g_{p,\tilde\varepsilon}(m,x))u(m,0,x)|\,\mathrm{d}m\biggr)^2\,\mathrm{d}r\\ &\leqslant 3AM^4\mathrm{e}^{2(A+T)\lambda_1}\\ &\quad\times\int_a^A s\biggl(\int_t^T \beta_1(t,r,x)(\beta_1(m,0,x) - b(m,0,x)g_{p,\tilde\varepsilon}(m,x))u(m,0,x)\,\mathrm{d}m\biggr)^2\,\mathrm{d}r\\ &\leqslant 6AM^4\mathrm{e}^{2(A+T)\lambda_1}(\|\beta_1\|_\infty^2 + \|b\|_\infty^2\|g_{p,\tilde\varepsilon}\|_\infty^2)\int_a^A\biggl(\int_t^T \beta_1(t,r,x)u(m,0,x)\,\mathrm{d}m\biggr)^2\,\mathrm{d}r\\ &\leqslant 6ATM^4\mathrm{e}^{2(A+T)\lambda_1}(\|\beta_1\|_\infty^2 + \|b\|_\infty^2\|g_{p,\tilde\varepsilon}\|_\infty^2)\int_a^A\int_t^T \beta_1^2(t,r,x)u^2(m,0,x)\,\mathrm{d}m\,\mathrm{d}r,\end{aligned} \tag{93}$$

wherein M and λ_1 are the same as in (66).

The combination of (92) and (93) steers to the following inequality:

$$\begin{aligned}|u(t,a,x)|^2 \leqslant{}& 3\int_a^A A|S(r-a)\beta_1(t,r,x)S(T-t)u_T(T-t,x)|^2\,\mathrm{d}r\\ &+ 6ATM^4\mathrm{e}^{2(A+T)\lambda_1}(\|\beta_1\|_\infty^2 + \|b\|_\infty^2\|g_{p,\tilde\varepsilon}\|_\infty^2)\\ &\times\int_a^A\int_t^T \beta_1^2(t,r,x)u^2(m,0,x)\,\mathrm{d}m\,\mathrm{d}r\\ &+\int_a^A 3AM^2\mathrm{e}^{2A\lambda_1}|b(t,r,x)p_{\tilde\varepsilon}(t,r,x)|^2|u(t,r,x)|^2\,\mathrm{d}r.\end{aligned} \tag{94}$$

Put

$$(95)\quad \begin{cases} \widetilde{M}_{11} := 6ATM^4\mathrm{e}^{2(A+T)\lambda_1}(\|\beta_1\|_\infty^2 + \|b\|_\infty^2\|g_{p,\tilde{\varepsilon}}\|_\infty^2), \\ B(t,a,x) := 3\displaystyle\int_a^A A|S(r-a)\beta_1(t,r,x)S(T-t)u_T(T-t,x)|^2\,\mathrm{d}r \\ \qquad\qquad +\widetilde{M}_{11}\displaystyle\int_a^A\int_t^T \beta_1^2(t,r,x)u^2(m,0,x)\,\mathrm{d}m\,\mathrm{d}r \qquad\qquad \text{in } Q, \\ D(t,r,x) := 3AM^2\mathrm{e}^{2A\lambda_1}|b(t,r,x)p_{\tilde{\varepsilon}}(t,r,x)|^2. \end{cases}$$

Under these notations, (94) becomes

$$(96)\qquad |u(t,a,x)|^2 \leqslant B(t,a,x) + \int_a^A D(t,r,x)|u(t,r,x)|^2\,\mathrm{d}r.$$

Applying Gronwall-Bellman's lemma to (96) with respect to the age variable a we get

$$(97)\qquad |u(t,a,x)|^2 \leqslant B(t,a,x) + \int_a^A B(t,s,x)D(t,s,x)\mathrm{e}^{\int_a^s D(t,r,x)\,\mathrm{d}r}\,\mathrm{d}s,$$

where s denotes a variable of integration and not the parameter of Carleman estimates.

To approve our claim (87), it remains to find upper bounds of "$B(t,a,x)$" and "$\int_a^A B(t,s,x)D(t,s,x)\mathrm{e}^{\int_a^s D(t,r,x)\,\mathrm{d}r}\,\mathrm{d}s$".

Indeed, thanks again to (90), the formula of B given in (95) allows us to say that

$$(98)\qquad \begin{aligned} B(t,a,x) &\leqslant 3AM^2\mathrm{e}^{2\lambda_1 A}\|\beta_1\|_\infty^2\int_a^A |S(T-t)u_T(T-t,x)|^2\,\mathrm{d}r \\ &\quad + \widetilde{M}_{11}\|\beta_1\|_\infty^2\int_a^A\int_t^T u^2(m,0,x)\,\mathrm{d}m\,\mathrm{d}r \\ &\leqslant 3A^2M^2\mathrm{e}^{2\lambda_1 A}\|\beta_1\|_\infty^2|S(T-t)u_T(T-t,x)|^2 \\ &\quad + \widetilde{M}_{11}A\|\beta_1\|_\infty^2\int_t^T u^2(m,0,x)\,\mathrm{d}m, \end{aligned}$$

where the constant $\widetilde{M}_{11}$ is defined by (95). This involves

$$(99)\qquad \begin{aligned} \int_a^A B(t,s,x)\,\mathrm{d}s &\leqslant 3A^3M^2\mathrm{e}^{2\lambda_1 A}\|\beta_1\|_\infty^2|S(T-t)u_T(T-t,x)|^2 \\ &\quad + \widetilde{M}_{11}A^2\|\beta_1\|_\infty^2\int_t^T u^2(m,0,x)\,\mathrm{d}m. \end{aligned}$$

Also, one can check out in a straightforward way that

$$
\begin{aligned}
(100)\qquad \mathrm{e}^{\int_a^s D(t,r,x)\,\mathrm{d}r} &= \exp\left(3AM^2\mathrm{e}^{2A\lambda_1}\int_a^s |b(t,r,x)p_{\tilde{\varepsilon}}(t,r,x)|^2\,\mathrm{d}r\right)\\
&\leqslant \exp\left(3AM^2\mathrm{e}^{2A\lambda_1}\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2(s-a)\right)\\
&\leqslant \exp\left(3A^2M^2\mathrm{e}^{2A\lambda_1}\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2\right).
\end{aligned}
$$

Here in the first inequality, we employed the fact that $p_{\tilde{\varepsilon}} \in C^\infty(Q)$. The index j and the compact K are defined in (40). Recall again that s here is the symbol of an integration variable and not the parameter of Carleman estimates. Therefore, in the light of the expression of D defined in (95) and estimates (99) and (100) we can deduce that

$$
\begin{aligned}
(101)\quad & \int_a^A B(t,s,x)D(t,s,x)\mathrm{e}^{\int_a^s D(t,r,x)\,\mathrm{d}r}\,\mathrm{d}s\\
&\qquad \leqslant 3AM^2\mathrm{e}^{2A\lambda_1}\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2\\
&\qquad\quad \times \exp\left(3A^2M^2\mathrm{e}^{2A\lambda_1}\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2\right)\int_a^A B(t,s,x)\,\mathrm{d}s\\
&\qquad \leqslant 9A^4M^4\mathrm{e}^{4A\lambda_1}\|\beta_1\|_\infty^2\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2\\
&\qquad\quad \times \exp\left(3A^2M^2\mathrm{e}^{2A\lambda_1}\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2\right)|S(T-t)u_T(T-t,x)|^2\\
&\qquad\quad + 3\widetilde{M}_{11}A^3\|\beta_1\|_\infty^2\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2\mathrm{e}^{2A\lambda_1}\\
&\qquad\quad \times \exp\left(3A^2M^2\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2\mathrm{e}^{2A\lambda_1}\right)\int_t^T u^2(m,0,x)\,\mathrm{d}m.
\end{aligned}
$$

Accordingly, via inequalities (97), (98) and (101) the following result holds:

$$
\begin{aligned}
(102)\quad & |u(t,a,x)|^2\\
&\quad \leqslant \left(9A^4M^4\mathrm{e}^{4A\lambda_1}\|\beta_1\|_\infty^2\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2\exp\left(3A^2M^2\mathrm{e}^{2A\lambda_1}\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2\right)\right.\\
&\qquad \left.+\, 3A^2M^2\mathrm{e}^{2\lambda_1 A}\|\beta_1\|_\infty^2\right)|S(T-t)u_T(T-t,x)|^2\\
&\qquad + \left(3\widetilde{M}_{11}A^3\|\beta_1\|_\infty^2\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2\mathrm{e}^{2A\lambda_1}\exp\left(3A^2M^2\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2\mathrm{e}^{2A\lambda_1}\right)\right.\\
&\qquad \left.+\, \widetilde{M}_{11}A\|\beta_1\|_\infty^2\right)\int_t^T u^2(m,0,x)\,\mathrm{d}m.
\end{aligned}
$$

Set

$$
(103)\quad \begin{cases}
\widetilde{M}_{12} := 9A^4M^4\mathrm{e}^{4A\lambda_1}\|\beta_1\|_\infty^2\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2\exp\left(3A^2M^2\mathrm{e}^{2A\lambda_1}\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2\right)\\
\qquad\qquad +3A^2M^2\mathrm{e}^{2\lambda_1 A}\|\beta_1\|_\infty^2,\\
\widetilde{M}_{13} := 3\widetilde{M}_{11}A^3\|\beta_1\|_\infty^2\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2\mathrm{e}^{2A\lambda_1}\exp\left(3A^2M^2\|b\|_\infty^2\|p_{\tilde{\varepsilon}}\|_{j,K}^2\mathrm{e}^{2A\lambda_1}\right)\\
\qquad\qquad +\widetilde{M}_{11}A\|\beta_1\|_\infty^2.
\end{cases}
$$

Thus, inequality (102) is simplified in the following way:

$$|u(t,a,x)|^2 \leqslant \widetilde{M}_{12}|S(T-t)u_T(T-t,x)|^2 + \widetilde{M}_{13}\int_t^T u^2(m,0,x)\,\mathrm{d}m. \tag{104}$$

Integration of (104) over $(\frac{1}{4}T, \frac{3}{4}T) \times (0, \delta - \frac{3}{4}T) \times (0,1)$ steers to

$$\begin{aligned}\int_0^1\int_0^{\delta-3T/4}\int_{T/4}^{3T/4} &|u(t,a,x)|^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\leqslant \widetilde{M}_{12}\int_0^1\int_0^{\delta-3T/4}\int_{T/4}^{3T/4}|S(T-t)u_T(T-t,x)|^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\quad + \widetilde{M}_{13}\int_0^1\int_0^{\delta-3T/4}\int_{T/4}^{3T/4}\int_t^T u^2(m,0,x)\,\mathrm{d}m\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\leqslant \widetilde{M}_{12}\int_0^1\int_0^{\delta-3T/4}\int_{T/4}^{3T/4}|S(T-t)u_T(T-t,x)|^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\quad + \frac{T}{2}\Big(\delta - \frac{3T}{4}\Big)\widetilde{M}_{13}\int_0^1\int_0^T u^2(m,0,x)\,\mathrm{d}m\,\mathrm{d}x.\end{aligned} \tag{105}$$

The variable change $\tilde{s} = T - t$ in the first integral of the right-hand side of (105) allows us to say that

$$\begin{aligned}\int_0^1\int_0^{\delta-3T/4}\int_{T/4}^{3T/4} &|u(t,a,x)|^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\leqslant \Big(\delta - \frac{3T}{4}\Big)\widetilde{M}_{12}\int_0^1\int_{T/4}^{3T/4}|S(\tilde{s})u_T(\tilde{s},x)|^2\,\mathrm{d}\tilde{s}\,\mathrm{d}x \\ &\quad + \frac{T}{2}\Big(\delta - \frac{3T}{4}\Big)\widetilde{M}_{13}\int_0^1\int_0^T u^2(m,0,x)\,\mathrm{d}m\,\mathrm{d}x.\end{aligned} \tag{106}$$

Since $T < \delta$ and exploiting relation (69), one can transform (106) to

$$\begin{aligned}\int_0^1\int_0^{\delta-3T/4}\int_{T/4}^{3T/4} &|u(t,a,x)|^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\leqslant \Big(\delta - \frac{3T}{4}\Big)\widetilde{M}_{12}\int_0^1\int_{T/4}^{3T/4} M^2\mathrm{e}^{2\lambda_1 T}|u_T(\tilde{s},x)|^2\,\mathrm{d}\tilde{s}\,\mathrm{d}x \\ &\quad + \frac{T}{2}\Big(\delta - \frac{3T}{4}\Big)\widetilde{M}_{13}\int_0^1\int_0^\delta u^2(m,0,x)\,\mathrm{d}m\,\mathrm{d}x.\end{aligned} \tag{107}$$

Hence,

$$\int_0^1\int_0^{\delta-3T/4}\int_{T/4}^{3T/4}|u(t,a,x)|^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \leqslant \widetilde{M}_3\int_0^1\int_0^\delta |u_T(a,x)|^2\,\mathrm{d}a\,\mathrm{d}x, \tag{108}$$

where $\widetilde{M_3} = (\delta - \frac{3}{4}T)(M^2 e^{2\lambda_1 T}\widetilde{M_{12}} + \frac{1}{2}T\widetilde{C_8}\widetilde{M_{13}})$ with M and λ_1 being the same as in (66), $\widetilde{C_8}$ is given in (69), the positive constants $\widetilde{M_{12}}$ and $\widetilde{M_{13}}$ are given in (103) and $\widetilde{M_{11}}$ is defined by (95).

Consequently, the first relation of (87) is achieved. Likewise, we can prove the second inequality of (87) using the same procedure as we mentioned previously.

Finally, inequality (85) is true in the case where $a > t + (A - T)$.

Let us now address the case when $a \leqslant t + (A - T)$. Like the first case, mixing the implicit formula (63) and (64) (in the case when $a \leqslant t + (A - T)$) and (65) we obtain

(109)
$$\begin{cases} u(t,a,\cdot) = S(T-t)u_T(a+T-t,\cdot) \\ \qquad + \int_t^T S(l-t)\beta_1(l,a,\cdot)S(T-l)u_T(T-l,\cdot)\,\mathrm{d}l \\ \qquad + \int_t^T S(l-t)\beta_1(l,a,\cdot) \\ \qquad \times\Big(\int_l^T S(m-l)(\beta_1(m,0,\cdot) - b(m,0,\cdot)g_{p,\tilde{\varepsilon}}(m,\cdot))u(m,0,\cdot)\,\mathrm{d}m\Big)\,\mathrm{d}l \\ \qquad - \int_t^T S(l-t)b(l,a,\cdot)p_{\tilde{\varepsilon}}(l,a\cdot)u(l,a,\cdot)\,\mathrm{d}l, \\ v(t,a,\cdot) = L(T-t)v_T(a+T-t,\cdot) \\ \qquad + \int_t^T L(l-t)\beta_2(l,a,\cdot)L(T-l)v_T(T-l,\cdot)\,\mathrm{d}l \\ \qquad + \int_t^T L(l-t)\beta_2(l,a,\cdot) \\ \qquad \times\Big(\int_l^T L(m-l)(\beta_2(m,0,\cdot) + \mu_3(m,0,\cdot)g_{y,\tilde{\varepsilon}}(m,\cdot))v(m,0,\cdot)\,\mathrm{d}m\Big)\,\mathrm{d}l \\ \qquad + \int_t^T L(l-t)\mu_3(l,a,\cdot)y_{\tilde{\varepsilon}}(l,a\cdot)v(l,a,\cdot)\,\mathrm{d}l. \end{cases}$$

Set for all $(t,a,x) \in Q$, $R(t,a,x) := \int_t^T S(l-t)\beta_1(l,a,\cdot)S(T-l)u_T(T-l,\cdot)\,\mathrm{d}l + \int_t^T S(l-t)\beta_1(l,a,\cdot)\big(\int_l^T S(m-l)(\beta_1(m,0,\cdot) - b(m,0,\cdot)g_{p,\tilde{\varepsilon}}(m,\cdot))u(m,0,\cdot)\,\mathrm{d}m\big)\,\mathrm{d}l - \int_t^T S(l-t)b(l,a,\cdot)p_{\tilde{\varepsilon}}(l,a\cdot)u(l,a,\cdot)\,\mathrm{d}l$. Following this, the first solution u becomes

(110)
$$\forall\,(t,a,x) \in Q, \quad u(t,a,x) = S(T-t)u_T(a+T-t,\cdot) + R(t,a,x).$$

If one mimics the same blend of semi-group theory, Gronwall-Bellman's lemma and Hölder inequality used in the computations in the case "$a > t + (A - T)$", we can establish the existence of the positive constant $\widetilde{M_3}$ of (108) such that

(111)
$$\int_0^1 \int_0^{\delta - 3T/4} \int_{T/4}^{3T/4} |R(t,a,x)|^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \leqslant \widetilde{M_3} \int_0^1 \int_0^{\delta} |u_T(a,x)|^2\,\mathrm{d}a\,\mathrm{d}x.$$

On the other hand, we can observe that

$$\begin{aligned}\int_0^1\int_0^{\delta-3T/4}\int_{T/4}^{3T/4} &|S(T-t)u_T(a+T-t,x)|^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\ &=\int_0^1\int_0^{\delta-3T/4}\int_{T/4}^{3T/4}|S(\tilde t)u_T(a+\tilde t,x)|^2\,\mathrm{d}\tilde t\,\mathrm{d}a\,\mathrm{d}x\\ &\leqslant M^2\mathrm{e}^{3\lambda_1T/2}\int_0^1\int_0^{\delta-3T/4}\int_{T/4}^{3T/4}|u_T(a+\tilde t,x)|^2\,\mathrm{d}\tilde t\,\mathrm{d}a\,\mathrm{d}x,\end{aligned}$$

where M and λ_1 are the same as in (66) and $\tilde t := T-t$. With the variable change $\tilde a = a+\tilde t$, it follows that

$$\begin{aligned}(112)\quad \int_0^1\int_0^{\delta-3T/4}\int_{T/4}^{3T/4} &|S(T-t)u_T(a+T-t,x)|^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\ &\leqslant M^2\mathrm{e}^{3\lambda_1T/2}\int_0^1\int_0^{\delta-3T/4}\int_{a+T/4}^{a+3T/4}|u_T(\tilde a,x)|^2\,\mathrm{d}\tilde a\,\mathrm{d}a\,\mathrm{d}x\\ &\leqslant M^2\mathrm{e}^{3\lambda_1T/2}\int_0^1\int_0^{\delta-3T/4}\int_{T/4}^{A}|u_T(\tilde a,x)|^2\,\mathrm{d}\tilde a\,\mathrm{d}a\,\mathrm{d}x\\ &\leqslant \Big(\delta-\frac{3T}{4}\Big)M^2\mathrm{e}^{3\lambda_1T/2}\int_0^1\int_{T/4}^{A}|u_T(\tilde a,x)|^2\,\mathrm{d}\tilde a\,\mathrm{d}x\\ &\leqslant \Big(\delta-\frac{3T}{4}\Big)M^2\mathrm{e}^{3\lambda_1T/2}\int_0^1\int_0^{A}|u_T(\tilde a,x)|^2\,\mathrm{d}\tilde a\,\mathrm{d}x,\end{aligned}$$

the second inequality obtained in (112) is an outcome of the inclusion $(a+\frac14T, a+\frac34T)\subset(\frac14T,\delta)$, for all $a\in(0,\delta-\frac34T)$.

It is now clear from (110), (111) and (112) that there exists a positive constant $\widetilde{M_5}$ such that

$$(113)\qquad \int_0^1\int_0^{\delta-3T/4}\int_{T/4}^{3T/4}|u(t,a,x)|^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\leqslant \widetilde{M_5}\int_0^1\int_0^{\delta}|u_T(a,x)|^2\,\mathrm{d}a\,\mathrm{d}x,$$

accurately, $\widetilde{M_5}:=2(\widetilde{M_3}+(\delta-\frac34T)M^2\mathrm{e}^{3\lambda_1T/2})$.

In the same manner, one can bring out a similar inequality for the solution v through the second implicit formula in (109), i.e., the existence of $\widetilde{M_6}>0$ such that

$$(114)\qquad \int_0^1\int_0^{\delta-3T/4}\int_{T/4}^{3T/4}|v(t,a,x)|^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\leqslant \widetilde{M_6}\int_0^1\int_0^{\delta}|v_T(a,x)|^2\,\mathrm{d}a\,\mathrm{d}x.$$

Hence, summing up (113) and (114), inequality (85) holds in the current case. Abstractly,

$$\int_0^1\int_0^{\delta-3T/4}\int_{T/4}^{3T/4}(u^2(t,a,x)+v^2(t,a,x))\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \leqslant \widehat{C}\int_0^1\int_0^{\delta}(u_T^2(a,x)+v_T^2(a,x))\,\mathrm{d}a\,\mathrm{d}x$$

is satisfied in both cases $a > t+(A-T)$ and $a \leqslant t+(A-T)$ and when $\widehat{C}$ is a positive constant. The last inequality together with (84) imply that

$$\begin{aligned}\int_0^1\int_0^A &(u^2(0,a,x)+v^2(0,a,x))\,\mathrm{d}a\,\mathrm{d}x \\ &\leqslant \frac{2\mathrm{e}^{\kappa_1 T}}{T}\int_0^1\int_{\delta-3T/4}^{\delta}\int_{T/4}^{3T/4}(u^2(t,a,x)+v^2(t,a,x))\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\quad+\widehat{C}\frac{2\mathrm{e}^{\kappa_1 T}}{T}\int_0^1\int_0^{\delta}(u_T^2(a,x)+v_T^2(a,x))\,\mathrm{d}a\,\mathrm{d}x.\end{aligned}$$

Subsequently, with the help of the Hardy-Poincaré inequality and the definitions of φ_i, $i=1,2$, stated in (19) we arrive to

$$\begin{aligned}\int_0^1\int_0^A &(u^2(0,a,x)+v^2(0,a,x))\,\mathrm{d}a\,\mathrm{d}x \\ &\leqslant \widehat{C}\frac{2\mathrm{e}^{\kappa_1 T}}{T}\int_0^1\int_0^{\delta}(u_T^2(a,x)+v_T^2(a,x))\,\mathrm{d}a\,\mathrm{d}x \\ &\quad+\widetilde{M}_7\frac{2\mathrm{e}^{\kappa_1 T}}{T}\bigg(\int_0^1\int_{\delta-3T/4}^{\delta}\int_{T/4}^{3T/4}s\Theta k_1(x)u_x^2(t,a,x)\mathrm{e}^{2s\varphi_1}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\quad+\int_0^1\int_{\delta-3T/4}^{\delta}\int_{T/4}^{3T/4}s\Theta k_2(x)v_x^2(t,a,x)\mathrm{e}^{2s\varphi_2}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\bigg).\end{aligned}$$

The proof of observability inequality (73) is finished thanks to (115) and applying the Carleman estimate (71) stated in Theorem 3.12 in the second term on the right-hand side of the last inequality. □

R e m a r k 4.2. The proofs of (86), (113) and (114) can be adopted to demonstrate that for any bounded function R_1 in Q depending on time t, age a and gene type x, the functions $\overline{u} := R_1 u$ and $\overline{v} := R_1 v$ realize the inequalities

$$(115)\qquad \int_\omega\int_0^A\int_0^T|\overline{u}(t,a,x)|^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \leqslant \widetilde{M}_{14}\int_0^1\int_0^{\delta}|u_T(a,x)|^2\,\mathrm{d}a\,\mathrm{d}x,$$

and

$$\int_\omega \int_0^A \int_0^T |\overline{v}(t,a,x)|^2 \,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \leqslant \widetilde{M}_{15} \int_0^1 \int_0^\delta |u_T(a,x)|^2 \,\mathrm{d}a\,\mathrm{d}x, \tag{116}$$

where (u,v) is the solution of (41), ω is the region where the control ϑ acts and $\widetilde{M}_{14}$ and $\widetilde{M}_{15}$ are given positive constants which do not depend on δ.

With the density dilemma provided in the third point of Proposition 3.8, one can extend Proposition 4.1 to:

Proposition 4.3. *Assume that* (5) *(or* (25)*) and* (6) *hold. Let* $A,T>0$ *be given and fixed such that* $T<\delta$*, where* δ *verifies* (3)*. Then there exists a positive constant* $C_{\mathrm{obs},\delta}$ *such that for every solution* (u,v) *of* (11) *(or with Newmann conditions on* $x=0$*) the following observability inequality is satisfied:*

$$\begin{aligned}\int_0^1 \int_0^A (u^2(0,a,x)+v^2(0,a,x))\,\mathrm{d}a\,\mathrm{d}x \\ \leqslant C_{\mathrm{obs},\delta}\Big(\int_q u^2 \,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \int_0^1 \int_0^\delta (u_T^2(a,x)+v_T^2(a,x))\,\mathrm{d}a\,\mathrm{d}x\Big).\end{aligned} \tag{117}$$

With the aid of the observability inequality (117) we are now able to show the result of null controllability (4) related to the Lotka-Volterra model (1).

4.2. Null controllability result. This paragraph deals with the null controllability property (4) of the Holling type I functional predator response system (1) (or with Newmann conditions on $x=0$). It is stipulated as follows:

Theorem 4.4. *Assume that* (5) *(or* (25)*) and* (6) *hold. Let* $A,T>0$ *be given and fixed such that* $T<\delta$*, where* δ *verifies* (3)*. Then for all* $(y_0,p_0)\in L^2(Q_A)\times L^2(Q_A)$ *there exists a control* $\vartheta\in L^2(q)$ *depending on* δ *such that the associated solution* (y,p) *of* (1) *(or with Newmann conditions on* $x=0$*) verifies*

$$\begin{cases} y(T,a,x)=0, & \text{a.e. in } (\delta,A)\times(0,1),\\ p(T,a,x)=0, & \text{a.e. in } (\delta,A)\times(0,1).\end{cases} \tag{118}$$

Recall that $q=(0,T)\times(0,A)\times\omega$.

Proof. Let $\varepsilon>0$ and consider the cost function

$$J_\varepsilon = \frac{1}{2\varepsilon}\int_0^1 \int_\delta^A (y^2(T,a,x)+p^2(T,a,x))\,\mathrm{d}a\,\mathrm{d}x + \frac{1}{2}\int_q \vartheta^2(t,a,x)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x.$$

We can prove that J_ε is continuous, convex and coercive. Then it admits at least one minimizer ϑ_ε and we have

$$\vartheta_\varepsilon = -u_\varepsilon(t,a,x)\chi_\omega(x) \quad \text{in } Q \tag{119}$$

with u_ε being the solution of the following system:

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} + \frac{\partial u_\varepsilon}{\partial a} + (k_1(x)(u_\varepsilon)_x)_x - \mu_1(t,a,x)u_\varepsilon - b(t,a,x)u_\varepsilon p_\varepsilon & \\ = -\beta_1 u_\varepsilon(t,0,x) & \qquad \text{in } Q, \\ u_\varepsilon(t,a,1) = u_\varepsilon(t,a,0) = 0 & \qquad \text{on } (0,T)\times(0,A), \\ u_\varepsilon(T,a,x) = \frac{1}{\varepsilon} y_\varepsilon(T,a,x)\chi_{(\delta,A)} & \qquad \text{in } (0,A)\times(0,1), \\ u_\varepsilon(t,A,x) = 0 & \qquad \text{in } Q_T. \end{aligned} \tag{120}$$

Consider also v_ε the solution of the system

$$\begin{aligned} \frac{\partial v_\varepsilon}{\partial t} + \frac{\partial v_\varepsilon}{\partial a} + (k_2(x)(v_\varepsilon)_x)_x - \mu_2(t,a,x)v_\varepsilon + \mu_3(t,a,x)v_\varepsilon y_\varepsilon & \\ = -\beta_2 v_\varepsilon(t,0,x) & \qquad \text{in } Q, \\ v_\varepsilon(t,a,1) = v_\varepsilon(t,a,0) = 0 & \qquad \text{on } (0,T)\times(0,A), \\ v_\varepsilon(T,a,x) = \frac{1}{\varepsilon} p_\varepsilon(T,a,x)\chi_{(\delta,A)} & \qquad \text{in } Q_A, \\ v_\varepsilon(t,A,x) = 0 & \qquad \text{in } Q_T, \end{aligned} \tag{121}$$

where $(y_\varepsilon, p_\varepsilon)$ is the solution of (1) associated to the control ϑ_ε. Multiplying the first equation of (120) by y_ε and integrating over Q we obtain

$$\begin{aligned} &\int_Q y_\varepsilon\Big(\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial u_\varepsilon}{\partial a} + (k_1(x)(u_\varepsilon)_x)_x - \mu_1(t,a,x)u_\varepsilon - b(t,a,x)u_\varepsilon p_\varepsilon\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\quad = -\int_Q u_\varepsilon\Big(\frac{\partial y_\varepsilon}{\partial t} + \frac{\partial y_\varepsilon}{\partial a} - (k_1(x)(y_\varepsilon)_x)_x + \mu_1(t,a,x)y_\varepsilon + b(t,a,x)y_\varepsilon p_\varepsilon\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \\ &\qquad + \int_0^1\int_0^A y_\varepsilon(T,a,x)u_\varepsilon(T,a,x)\,\mathrm{d}a\,\mathrm{d}x \\ &\qquad - \int_0^1\int_0^A y_\varepsilon(0,a,x)u_\varepsilon(0,a,x)\,\mathrm{d}a\,\mathrm{d}x \\ &\qquad - \int_0^1\int_0^T y_\varepsilon(t,0,x)u_\varepsilon(t,0,x)\,\mathrm{d}t\,\mathrm{d}x. \end{aligned}$$

Thus,

$$(122)\quad \int_Q y_\varepsilon\Big(\frac{\partial u_\varepsilon}{\partial t}+\frac{\partial u_\varepsilon}{\partial a}+(k_1(x)(u_\varepsilon)_x)_x-\mu_1(t,a,x)u_\varepsilon-b(t,a,x)u_\varepsilon p_\varepsilon\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x$$
$$=-\int_Q u_\varepsilon\vartheta_\varepsilon\chi_\omega\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x+\int_0^1\int_\delta^A\frac{1}{\varepsilon}y_\varepsilon^2(T,a,x)\,\mathrm{d}a\,\mathrm{d}x$$
$$-\int_0^1\int_0^A y_\varepsilon(0,a,x)u_\varepsilon(0,a,x)\,\mathrm{d}a\,\mathrm{d}x$$
$$-\int_0^1\int_0^A\int_0^T\beta_1 y_\varepsilon(t,a,x)u_\varepsilon(t,0,x)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x.$$

Also, one can see that

$$(123)\quad \int_Q y_\varepsilon\Big(\frac{\partial u_\varepsilon}{\partial t}+\frac{\partial u_\varepsilon}{\partial a}+(k_1(x)(u_\varepsilon)_x)_x-\mu_1(t,a,x)u_\varepsilon-b(t,a,x)u_\varepsilon p_\varepsilon\Big)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x$$
$$=-\int_0^1\int_0^A\int_0^T\beta_1 y_\varepsilon(t,a,x)u_\varepsilon(t,0,x)\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x.$$

Consequently, (122) and (123) lead to

$$(124)\quad \int_q\vartheta_\varepsilon^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x+\int_0^1\int_\delta^A\frac{1}{\varepsilon}y_\varepsilon^2(T,a,x)\,\mathrm{d}a\,\mathrm{d}x=\int_0^1\int_0^A y_\varepsilon(0,a,x)u_\varepsilon(0,a,x)\,\mathrm{d}a\,\mathrm{d}x.$$

Similarly, multiplying the first equation of (121) by p_ε and integrating over Q, we can conclude that

$$(125)\quad \int_0^1\int_\delta^A\frac{1}{\varepsilon}p_\varepsilon^2(T,a,x)\,\mathrm{d}a\,\mathrm{d}x=\int_0^1\int_0^A p_\varepsilon(0,a,x)v_\varepsilon(0,a,x)\,\mathrm{d}a\,\mathrm{d}x.$$

Summing (124) and (125) side by side and applying Young's inequality we have

$$\int_q\vartheta_\varepsilon^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x+\frac{1}{\varepsilon}\int_0^1\int_\delta^A(y_\varepsilon^2(T,a,x)+p_\varepsilon^2(T,a,x))\,\mathrm{d}a\,\mathrm{d}x$$
$$=\int_0^1\int_0^A y_\varepsilon(0,a,x)u_\varepsilon(0,a,x)\,\mathrm{d}a\,\mathrm{d}x$$
$$+\int_0^1\int_0^A p_\varepsilon(0,a,x)v_\varepsilon(0,a,x)\,\mathrm{d}a\,\mathrm{d}x$$
$$\leqslant\frac{1}{4C_{\mathrm{obs},\delta}}\int_0^1\int_0^A(u_\varepsilon^2(0,a,x)+v_\varepsilon^2(0,a,x))\,\mathrm{d}a\,\mathrm{d}x$$
$$+C_{\mathrm{obs},\delta}\int_0^1\int_0^A(y_0^2(a,x)+p_0^2(a,x))\,\mathrm{d}a\,\mathrm{d}x$$

with $C_{\mathrm{obs},\delta}$ being the constant of observability inequality (117).

This together with observability inequality (117) allow us to say that

$$\int_q \vartheta_\varepsilon^2 \,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \frac{1}{\varepsilon}\int_0^1\int_\delta^A (y_\varepsilon^2(T,a,x) + p_\varepsilon^2(T,a,x))\,\mathrm{d}a\,\mathrm{d}x$$
$$\leqslant \frac{1}{4}\Big(\int_q u_\varepsilon^2 \,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \int_0^1\int_0^\delta (u_{\varepsilon,T}^2(a,x) + v_{\varepsilon,T}^2(a,x))\,\mathrm{d}a\,\mathrm{d}x\Big)$$
$$+ C_{\mathrm{obs},\delta}\int_0^1\int_0^A (y_0^2(a,x) + p_0^2(a,x))\,\mathrm{d}a\,\mathrm{d}x.$$

Replacing by the expressions of $u_{\varepsilon,T}$ and $v_{\varepsilon,T}$ in the last inequality and keeping in mind relation (119), the last inequality reads as

$$\frac{3}{4}\int_q \vartheta_\varepsilon^2 \,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \frac{1}{\varepsilon}\int_0^1\int_\delta^A (y_\varepsilon^2(T,a,x) + p_\varepsilon^2(T,a,x))\,\mathrm{d}a\,\mathrm{d}x$$
$$\leqslant C_{\mathrm{obs},\delta}\int_0^1\int_0^A (y_0^2(a,x) + p_0^2(a,x))\,\mathrm{d}a\,\mathrm{d}x.$$

Hence, it follows that

$$\left\{\begin{aligned} &\int_q \vartheta_\varepsilon^2 \,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \leqslant \frac{4C_{\mathrm{obs},\delta}}{3}\int_0^1\int_0^A (y_0^2(a,x) + p_0^2(a,x))\,\mathrm{d}a\,\mathrm{d}x,\\ &\int_0^1\int_\delta^A y_\varepsilon^2(T,a,x)\,\mathrm{d}a\,\mathrm{d}x \leqslant \varepsilon C_{\mathrm{obs},\delta}\int_0^1\int_0^A (y_0^2(a,x) + p_0^2(a,x))\,\mathrm{d}a\,\mathrm{d}x,\\ &\int_0^1\int_\delta^A p_\varepsilon^2(T,a,x)\,\mathrm{d}a\,\mathrm{d}x \leqslant \varepsilon C_{\mathrm{obs},\delta}\int_0^1\int_0^A (y_0^2(a,x) + p_0^2(a,x))\,\mathrm{d}a\,\mathrm{d}x. \end{aligned}\right. \tag{126}$$

Then, we can extract two subsequences of $(y_\varepsilon, p_\varepsilon)$ and ϑ_ε denoted also by $(y_\varepsilon, p_\varepsilon)$ and ϑ_ε that converge weakly towards (y,p) and ϑ in $L^2((0,T)\times(0,A), H^1_{k_1}(0,1)\times H^1_{k_2}(0,1))$ and $L^2(q)$, respectively.

Now, by a variational technic, we prove that (y,p) is a solution of (1) corresponding to the control ϑ and, by the second and the third estimates of (126), (y,p) satisfies (4). Another deduction from (126), specially the first inequality, is that the researched control ϑ depends on δ. □

R e m a r k 4.5. Theorem 4.4 is important since it amounts to saying that we can control with one control force a very wide age classes of the two coupled populations (prey and predator) in a minimum time of control and then with a minimum cost control $C_{\mathrm{obs},\delta}$.

5. Appendix

As is mentioned in the introduction, this section is devoted to the proofs of some intermediate results useful to show the full ω-Carleman estimate associated to system (11). Firstly, we begin by Caccioppoli's inequality stated in the following lemma.

Lemma 5.1. *Let ω' be a subset of ω such that $\omega' \subset\subset \omega$. Let (u,v) be a solution of* (42) *(or with Newmann conditions on $x=0$). Then there exists a positive constant $C_{\tilde{\varepsilon}}$ such that*

$$(126)\quad \int_{\omega'}\int_0^A\int_0^T (u_x^2+v_x^2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \leqslant C_{\tilde{\varepsilon}}\Big(\int_q s^2\Theta^2(u^2+v^2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x + \int_q (h_1^2+h_2^2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\Big)$$

with φ_i, $i=1,2$, defined by (19).

P r o o f. The proof of this result is similar to the one of [26], Lemma 5.1. Indeed, consider the smooth cut-off function ζ defined by

$$(127)\quad \begin{cases} 0\leqslant \zeta(x)\leqslant 1, & x\in\mathbb{R},\\ \zeta(x)=0, & x<x_1 \text{ and } x>x_2,\\ \zeta(x)=1, & x\in\omega'. \end{cases}$$

Put $(\cdot)_l = \frac{\partial\cdot}{\partial l}$, where $l=t,a,x$. For (u,v), the solution of (42), one has

$$\begin{aligned}(128)\quad 0 &= \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^1\int_0^A \zeta^2\mathrm{e}^{2s\varphi_i}(u^2+v^2)\,\mathrm{d}a\,\mathrm{d}x\Big)\,\mathrm{d}t\\
&= 2s\int_0^1\int_0^A\int_0^T \zeta^2(\varphi_i)_t(u^2+v^2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\
&\quad + 2\int_0^1\int_0^A\int_0^T \zeta^2 uu_t\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\
&\quad + 2\int_0^1\int_0^A\int_0^T \zeta^2 vv_t\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\
&= 2s\int_0^1\int_0^A\int_0^T \zeta^2(\varphi_i)_t(u^2+v^2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\
&\quad + 2\int_0^1\int_0^A\int_0^T \zeta^2 u(h_1-u_a-(k_1u_x)_x+\mu_1u+bup_{\tilde{\varepsilon}})\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\
&\quad + 2\int_0^1\int_0^A\int_0^T \zeta^2 v(h_2-v_a-(k_2v_x)_x+\mu_2v-\mu_3vy_{\tilde{\varepsilon}})\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x.\end{aligned}$$

Then integrating by parts we obtain

$$
\begin{aligned}
2\int_Q \zeta^2(k_1u_x^2+k_2v_x^2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x
&= -2s\int_Q \zeta^2(u^2+v^2)\psi_i(\Theta_a+\Theta_t)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\
&\quad -2\int_Q \zeta^2(uh_1+vh_2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x-2\int_Q \zeta^2(\mu_1u^2+\mu_2v^2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\
&\quad +\int_Q (k_1(\zeta\mathrm{e}^{2s\varphi_i})_x)_xu^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x+\int_Q (k_2(\zeta\mathrm{e}^{2s\varphi_i})_x)_xv^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\
&\quad -2\int_Q \zeta^2bu^2p_{\tilde{\varepsilon}}\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x+2\int_Q \zeta^2\mu_3v^2y_{\tilde{\varepsilon}}\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x.
\end{aligned}
$$

On the other hand, by the definitions of ζ given in (128), ψ_i, $i=1,2$, and Θ given in (19), using Young's inequality and taking s quite large, one can prove the existence of a positive constant c such that

$$
\begin{aligned}
&2\int_Q \zeta^2(k_1u_x^2+k_2v_x^2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\
&\qquad \geqslant 2\min\big(\min_{x\in\omega'}k_1(x),\min_{x\in\omega'}k_2(x)\big)\int_{\omega'}\int_0^A\int_0^T (u_x^2+v_x^2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x,\\
&\int_Q (k_1(\zeta\mathrm{e}^{2s\varphi_i})_x)_xu^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\leqslant c\int_\omega\int_0^A\int_0^T s^2\Theta^2u^2\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x,\\
&\int_Q (k_2(\zeta\mathrm{e}^{2s\varphi_i})_x)_xv^2\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\leqslant c\int_\omega\int_0^A\int_0^T s^2\Theta^2v^2\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x,\\
&-2s\int_Q \zeta^2(u^2+v^2)\psi_i(\Theta_a+\Theta_t)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\
&\qquad \leqslant c\int_\omega\int_0^A\int_0^T s^2\Theta^2(u^2+v^2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x,\\
&-2\int_Q \zeta^2(uh_1+vh_2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\
&\qquad \leqslant c\int_\omega\int_0^A\int_0^T s^2\Theta^2(u^2+v^2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\\
&\qquad\quad +c\int_\omega\int_0^A\int_0^T (h_1^2+h_2^2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x,\\
&-2\int_Q \zeta^2(\mu_1u^2+\mu_2v^2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x\leqslant c\int_\omega\int_0^A\int_0^T s^2\Theta^2(u^2+v^2)\mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x.
\end{aligned}
$$

Now, keeping in mind the first hypotheses in (6) on b and μ_3, since $y_{\tilde{\varepsilon}},p_{\tilde{\varepsilon}}\in C^\infty(Q)$ and taking again s quite large, we can infer that there exist another positive constants c_1 and c_2 such that

$$-2\int_Q \zeta^2 b u^2 p_{\tilde\varepsilon} \mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \leqslant c_1 \int_\omega\int_0^A\int_0^T s^2\Theta^2 u^2 \mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x,$$
$$2\int_Q \zeta^2 \mu_3 v^2 y_{\tilde\varepsilon} \mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x \leqslant c_2 \int_\omega\int_0^A\int_0^T s^2\Theta^2 v^2 \mathrm{e}^{2s\varphi_i}\,\mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x,$$

where c_1 and c_2 depend, respectively, on the semi-norms of $p_{\tilde\varepsilon}$ and $y_{\tilde\varepsilon}$ defined by (38). Combining all the previous inequalities we reach finally estimate (127). □

Remark 5.2.

(1) In Lemma 5.1, the set ω' is chosen so that 0, which is exactly the point of degeneracy of the dispersion coefficients k_i, $i=1,2$, does not belong to $\overline{\omega'}$. More generally, if the degeneracy occurs at a point $x_0\in(0,1)$, one must take x_0 out of $\overline{\omega'}$ in the case of interior degeneracy to establish a Caccioppoli's type inequality (see [34] for more details in this context).

(2) Lemma 5.1 remains true in $L^2(Q)$ using the density argument cited in the third point of Proposition 3.8 since $y_{\tilde\varepsilon}$ and $p_{\tilde\varepsilon}$ approximate, respectively, y and p, the solutions of (1) in $L^2(Q)$ when $\tilde\varepsilon$ goes to 0.

We close this section by the following result.

Lemma 5.3. *Assume that conditions* (21) *hold. Then the interval*

$$I=\Big[\frac{k_2(1)(2-\gamma)(\mathrm{e}^{2\kappa\|\sigma\|_\infty}-1)}{d_2k_2(1)(2-\gamma)-1},\frac{4(\mathrm{e}^{2\kappa\|\sigma\|_\infty}-\mathrm{e}^{\kappa\|\sigma\|_\infty})}{3d_2}\Big)$$

is not empty.

Proof. Indeed one has

$$\begin{aligned}\frac{4(\mathrm{e}^{2\kappa\|\sigma\|_\infty}-\mathrm{e}^{\kappa\|\sigma\|_\infty})}{3d_2}&-\frac{k_2(1)(2-\gamma)(\mathrm{e}^{2\kappa\|\sigma\|_\infty}-1)}{d_2k_2(1)(2-\gamma)-1}\\
&=\frac{4(\mathrm{e}^{2\kappa\|\sigma\|_\infty}-\mathrm{e}^{\kappa\|\sigma\|_\infty})(d_2k_2(1)(2-\gamma)-1)-3d_2k_2(1)(2-\gamma)(\mathrm{e}^{2\kappa\|\sigma\|_\infty}-1)}{3d_2(d_2k_2(1)(2-\gamma)-1)}\\
&=\frac{\mathrm{e}^{2\kappa\|\sigma\|_\infty}(d_2k_2(1)(2-\gamma)-4)-4\mathrm{e}^{\kappa\|\sigma\|_\infty}(d_2k_2(1)(2-\gamma)-1)}{3d_2(d_2k_2(1)(2-\gamma)-1)}\\
&\quad+\frac{k_2(1)(2-\gamma)}{d_2k_2(1)(2-\gamma)-1}\\
&=\frac{\mathrm{e}^{\kappa\|\sigma\|_\infty}(\mathrm{e}^{\kappa\|\sigma\|_\infty}(d_2k_2(1)(2-\gamma)-4)-4(d_2k_2(1)(2-\gamma)-1))}{3d_2(d_2k_2(1)(2-\gamma)-1)}\\
&\quad+\frac{k_2(1)(2-\gamma)}{d_2k_2(1)(2-\gamma)-1}.\end{aligned}$$

Using the fact that $d_2 \geqslant 5/(k_2(1)(2-\gamma))$, we can conclude that

$$\frac{4(d_2k_2(1)(2-\gamma)-1)}{(d_2k_2(1)(2-\gamma)-4)} \leqslant 16.$$

Since $\kappa \geqslant 4\ln(2)/\|\sigma\|_\infty$, we have $e^{\kappa\|\sigma\|_\infty} \geqslant 16$. Therefore, the previous difference is positive and subsequently $I \neq \emptyset$. □

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References

[1] *B. Ainseba*: Exact and approximate controllability of the age and space population dynamics structured model. J. Math. Anal. Appl. *275* (2002), 562–574. zbl MR doi

[2] *B. Ainseba*: Corrigendum to "Exact and approximate controllability of the age and space population dynamics structured model" (J. Math. Anal. Appl. 275 (2) (2002), 562-574). J. Math. Anal. Appl. *393* (2012), 328. zbl MR doi

[3] *B. Ainseba, S. Aniţa*: Local exact controllability of the age-dependent population dynamics with diffusion. Abstr. Appl. Anal. *6* (2001), 357–368. zbl MR doi

[4] *B. Ainseba, S. Aniţa*: Internal exact controllability of the linear population dynamics with diffusion. Electron. J. Differ. Equ. *2004* (2004), Article ID 112, 11 pages. zbl MR

[5] *B. Ainseba, S. Aniţa*: Internal stabilizability for a reaction-diffusion problem modeling a predator-prey system. Nonlinear Anal., Theory Methods Appl., Ser. A *61* (2005), 491–501. zbl MR doi

[6] *B. Ainseba, Y. Echarroudi, L. Maniar*: Null controllability of a population dynamics with degenerate diffusion. Differ. Integral Equ. *26* (2013), 1397–1410. zbl MR

[7] *B. Ainseba, M. Langlais*: On a population dynamics control problem with age dependence and spatial structure. J. Math. Anal. Appl. *248* (2000), 455–474. zbl MR doi

[8] *E. M. Ait Ben Hassi, F. Ammar Khodja, A. Hajjaj, L. Maniar*: Null controllability of degenerate parabolic cascade systems. Port. Math. (N.S.) *68* (2011), 345–367. zbl MR doi

[9] *F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli*: Carleman estimates for degenerate parabolic operators with applications to null controllability. J. Evol. Equ. *6* (2006), 161–204. zbl MR doi

[10] *S. Aniţa*: Analysis and Control of Age-Dependent Population Dynamics. Mathematical Modelling: Theory and Applications 11. Kluwer Acadamic, Dordrecht, 2000. zbl MR doi

[11] *N. Apreutesei, G. Dimitriu*: On a prey-predator reaction-diffusion system with Holling type III functional response. J. Comput. Appl. Math. *235* (2010), 366–379. zbl MR doi

[12] *V. Barbu, M. Iannelli, M. Martcheva*: On the controllability of the Lotka-McKendrick model of population dynamics. J. Math. Anal. Appl. *253* (2001), 142–165. zbl MR doi

[13] *I. Boutaayamou, Y. Echarroudi*: Null controllability of population dynamics with interior degeneracy. Electron. J. Differ. Equ. *2017* (2017), Article ID 131, 21 pages. zbl MR

[14] *I. Boutaayamou, G. Fragnelli*: A degenerate population system: Carleman estimates and controllability. Nonlinear Anal., Theory Methods Appl., Ser. A *195* (2020), Article ID 111742, 29 pages. zbl MR doi

[15] *I. Boutaayamou, J. Salhi*: Null controllability for linear parabolic cascade systems with interior degeneracy. Electron. J. Differ. Equ. *2016* (2016), Article ID 305, 22 pages. zbl MR

[16] *T. Cabello, M. Gámez, Z. Varga*: An improvement of the Holling type III functional response in entomophagous species model. J. Biol. Syst. *15* (2007), 515–524. zbl doi
[17] *M. Campiti, G. Metafune, D. Pallara*: Degenerate self-adjoint evolution equations on the unit interval. Semigroup Forum *57* (1998), 1–36. zbl MR doi
[18] *P. Cannarsa, G. Fragnelli*: Null controllability of semilinear degenerate parabolic equations in bounded domains. Electron. J. Differ. Equ. *2006* (2006), Article ID 136, 20 pages. zbl MR
[19] *P. Cannarsa, G. Fragnelli, D. Rocchetti*: Null controllability of degenerate parabolic with drift. Netw. Heterog. Media *2* (2007), 695–715. zbl MR doi
[20] *P. Cannarsa, G. Fragnelli, D. Rocchetti*: Controllability results for a class of one-dimensional degenerate parabolic problems in nondivergence form. J. Evol. Equ. *8* (2008), 583–616. zbl MR doi
[21] *P. Cannarsa, G. Fragnelli, J. Vancostenoble*: Linear degenerate parabolic equations in bounded domains: Controllability and observability. Systems, Control, Modeling and Optimization. IFIP International Federation for Information Processing 202. Springer, New York, 2006, pp. 163–173. zbl MR doi
[22] *P. Cannarsa, G. Fragnelli, J. Vancostenoble*: Regional controllability of semilinear degenerate parabolic equations in bounded domains. J. Math. Anal. Appl. *320* (2006), 804–818. zbl MR doi
[23] *P. Cannarsa, P. Martinez, J. Vancostenoble*: Persistent regional null controllability for a class of degenerate parabolic equations. Commun. Pure Appl. Anal. *3* (2004), 607–635. zbl MR doi
[24] *P. Cannarsa, P. Martinez, J. Vancostenoble*: Null controllability of degenerate heat equations. Adv. Differ. Equ. *10* (2005), 153–190. zbl MR
[25] *J. H. P. Dawes, M. O. Souza*: A derivation of Holling's type I, II and III functional responses in predator-prey systems. J. Theor. Biol. *327* (2013), 11–22. zbl MR doi
[26] *Y. Echarroudi, L. Maniar*: Null controllability of a model in population dynamics. Electron. J. Differ. Equ. *2014* (2014), Article ID 240, 20 pages. zbl MR
[27] *Y. Echarroudi, L. Maniar*: Null controllability of a degenerate cascade model in population dynamics. Studies in Evolution Equations and Related Topics. STEAM-H: Science, Technology, Engineering, Agriculture, Mathematics & Health. Springer, Cham, 2021, pp. 211–268. zbl doi
[28] *G. Fragnelli*: An age-dependent population equation with diffusion and delayed birth process. Int. J. Math. Math. Sci. *2005* (2005), 3273–3289. zbl MR doi
[29] *G. Fragnelli*: Null controllability of degenerate parabolic equations in non divergence form via Carleman estimates. Discrete Contin. Dyn. Syst., Ser. S *6* (2013), 687–701. zbl MR doi
[30] *G. Fragnelli*: Carleman estimates and null controllability for a degenerate population model. J. Math. Pures Appl. (9) *115* (2018), 74–126. zbl MR doi
[31] *G. Fragnelli*: Null controllability for a degenerate population model in divergence form via Carleman estimates. Adv. Nonlinear Anal. *9* (2020), 1102–1129. zbl MR doi
[32] *G. Fragnelli, A. Idrissi, L. Maniar*: The asymptotic behavior of a population equation with diffusion and delayed birth process. Discrete Contin. Dyn. Syst., Ser. B *7* (2007), 735–754. zbl MR doi
[33] *G. Fragnelli, P. Martinez, J. Vancostenoble*: Qualitative properties of a population dynamics system describing pregnancy. Math. Models Methods Appl. Sci. *15* (2005), 507–554. zbl MR doi
[34] *G. Fragnelli, D. Mugnai*: Carleman estimates and observability inequalities for parabolic equations with interior degeneracy. Adv. Nonlinear Anal. *2* (2013), 339–378. zbl MR doi
[35] *G. Fragnelli, D. Mugnai*: Carleman estimates, observability inequalities and null controllability for interior degenerate non smooth parabolic equations. Mem. Am. Math. Soc. *1146* (2016), 88 pages. zbl MR doi

[36] *G. Fragnelli, L. Tonetto*: A population equation with diffusion. J. Math. Anal. Appl. *289* (2004), 90–99. zbl MR doi
[37] *A. V. Fursikov, O. Y. Imanuvilov*: Controllability of Evolutions Equations. Lecture Notes Series, Seoul 34. Seoul National University, Seoul, 1996. zbl MR
[38] *A. Hajjaj, L. Maniar, J. Salhi*: Carleman estimates and null controllability of degenerate/singular parabolic systems. Electron. J. Differ. Equ. *2016* (2016), Article ID 292, 25 pages. zbl MR
[39] *N. Hegoburu, M. Tucsnak*: Null controllability of the Lotka-Mckendrick system with spatial diffusion. Math. Control Relat. Fields *8* (2018), 707–720. zbl MR doi
[40] *Y. Jia, J. Wu, H.-K. Xu*: Positive solutions of a Lotka-Volterra competition model with cross-diffusion. Comput. Math. Appl. *68* (2014), 1220–1228. zbl MR doi
[41] *A. Juska, L. Gouveia, J. Gabriel, S. Koneck*: Negotiating bacteriological meat contamination standards in the US: The case of *E. Coli* O157:H7. Sociologia Ruralis *40* (2000), 249–271. doi
[42] *R. E. Kooij, A. Zegeling*: A predator-prey model with Ivlev's functional response. J. Math. Anal. Appl. *198* (1996), 473–489. zbl MR doi
[43] *M. Langlais*: A nonlinear problem in age-dependent population diffusion. SIAM J. Math. Anal. *16* (1985), 510–529. zbl MR doi
[44] *B. Liu, Y. Zhang, L. Chen*: Dynamics complexities of a Holling I predator-prey model concerning periodic biological and chemical control. Chaos Solitons Fractals *22* (2004), 123–134. zbl MR doi
[45] *X. Liu, Q. Huang*: The dynamics of a harvested predator-prey system with Holling type IV functional response. Biosystems *169-170* (2018), 26–39. doi
[46] *A. Y. Mozorov*: Emergence of Holling type III zooplankton functional response: Bringing together field evidence and mathematical modelling. J. Theor. Biol. *265* (2010), 45–54. zbl MR doi
[47] *L. Pavel*: Classical solutions in Sobolev spaces for a class of hyperbolic Lotka-Volterra systems. SIAM J. Control Optim. *51* (2013), 2132–2151. zbl MR doi
[48] *R. Peng, J. Shi*: Non-existence of non-constant positive steady states of two Holling type-II predator-prey systems: Strong interaction case. J. Differ. Equations *247* (2009), 866–886. zbl MR doi
[49] *S. Piazzera*: An age-dependent population equation with delayed birth process. Math. Methods Appl. Sci. *27* (2004), 427–439. zbl MR doi
[50] *M. A. Pozio, A. Tesei*: Degenerate parabolic problems in population dynamics. Japan J. Appl. Math. *2* (1985), 351–380. MR doi
[51] *A. Pugliese, L. Tonetto*: Well-posedness of an infinite system of partial differential equations modelling parasitic infection in age-structured host. J. Math. Anal. Appl. *284* (2003), 144–164. zbl MR doi
[52] *A. Rhandi, R. Schnaubelt*: Asymptotic behaviour of a non-autonomous population equation with diffusion in L^1. Discrete Contin. Dyn. Syst. *5* (1999), 663–683. zbl MR doi
[53] *J. Salhi*: Null controllability for a coupled system of degenerate/singular parabolic equations in nondivergence form. Electron. J. Qual. Theory Differ. Equ. *2018* (2018), Article ID 31, 28 pages. zbl MR doi
[54] *G. Seo, D. L. DeAngelis*: A predator-prey model with a Holling type I functional response including a predator mutual interference. J. Nonlinear Sci. *21* (2011), 811–833. zbl MR doi
[55] *G. T. Skalski, J. F. Gilliam*: Functional responses with predator interference: Viable alternatives to the Holling type II model. Ecology *82* (2001), 3083–3092. doi
[56] *O. Traore*: Null controllability of a nonlinear population dynamics problem. Int. J. Math. Math. Sci. *2006* (2006), Article ID 49279, 20 pages. zbl MR doi
[57] *W. Wang, L. Zhang, H. Wang, Z. Li*: Pattern formation of a predator-prey system with Ivlev-type functional response. Ecological Modelling *221* (2010), 131–140. doi

[58] *G. F. Webb*: Population models structured by age, size, and spatial position. Structured Population Models in Biology and Epidemiology. Lecture Notes in Mathematics 1936. Springer, Berlin, 2008, pp. 1–49. MR doi
[59] *Y. Zhang, Z. Xu, B. Liu, L. Chen*: Dynamic analysis of a Holling I predator-prey system with mutual interference concerning pest control. J. Biol. Syst. *13* (2005), 45–58. zbl doi
[60] *C. Zhao, M. Wang, P. Zhao*: Optimal control of harvesting for age-dependent predator-prey system. Math. Comput. Modelling *42* (2005), 573–584. zbl MR doi

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