STABILITY RESULT FOR A THERMOELASTIC BRESSE SYSTEM WITH DELAY TERM IN THE INTERNAL FEEDBACK

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Abstract. The studies considered here are concerend with a linear thermoelastic Bresse system with delay term in the feedback. The heat conduction is also given by Cattaneo's law. Under an appropriate assumption between the weight of the delay and the weight of the damping, we prove the well-posedness of the problem using the semigroup method. Furthermore, based on the energy method, we establish an exponential stability result depending of a condition on the constants of the system that was first considered by A. Keddi, T. Apalara, S. A. Messaoudi in 2018.

Keywords: Bresse system; delay; decay rate; energy method; semigroup method; thermoelastic

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1. Introduction and related results

In the present paper the following problem is considered:

$$\begin{cases} \varrho_1\varphi_{tt} - k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) & \\ \qquad +\mu_1\varphi_t(x,t) + \mu_2\varphi_t(x,t-\tau) = 0 & \text{in } (0,1)\times(0,\infty), \\ \varrho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + lw + \psi) + \gamma\theta_x = 0 & \text{in } (0,1)\times(0,\infty), \\ \varrho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + lw + \psi) = 0 & \text{in } (0,1)\times(0,\infty), \\ \varrho_3\theta_t + \kappa q_x + \gamma\psi_{tx} = 0 & \text{in } (0,1)\times(0,\infty), \\ \alpha q_t + \beta q + \kappa\theta_x = 0 & \text{in } (0,1)\times(0,\infty) \end{cases} \tag{1.1}$$

with the initial-boundary conditions

$$\begin{cases} \varphi(0,t) = \varphi_x(1,t) = \psi_x(0,t) = \psi(1,t), \\ w_x(0,t) = w(1,t) = \theta(0,t) = q(1,t) = 0, \quad t \geqslant 0, \end{cases} \tag{1.2}$$

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and

$$
(1.3) \qquad \begin{cases}
\varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x), & x\in(0,1),\\
\psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x), & x\in(0,1),\\
w(x,0)=w_0(x),\ w_t(x,0)=w_1(x), & x\in(0,1),\\
\theta(x,0)=\theta_0(x),\ q(x,0)=q_0(x), & x\in(0,1),\\
\varphi_t(x,t-\tau)=f_0(x,t-\tau), & t\geqslant 0,\ x\in(0,1),
\end{cases}
$$

where $\tau>0$ is a time delay, μ_1 and μ_2 are positive real numbers. The functions w, φ, ψ, θ and q represent the longitudinal, vertical displacement, shear angle displacement, the temperature difference and the heat flux, respectively. The parameters ϱ_1, ϱ_2, ϱ_3, k, l, k_0, b, γ, κ, α, β are positive constants. We have proved the well-posedness and established stability results under the following condition on the parameters that was first considered in [14],

$$
(1.4) \qquad \eta=\Big(1-\frac{\alpha k\varrho_3}{\varrho_1}\Big)\Big(\frac{\varrho_1}{k}-\frac{\varrho_2}{b}\Big)-\frac{\gamma^2\alpha}{b} \quad\text{and}\quad k=k_0.
$$

It is well-known that, in the single wave equation, if $\mu_2=0$, that is, in the absence of delay, the energy of system exponentially decays (see [7], [8], [16], [24]). On the contrary, if $\mu_1=0$, that is, if there exists only the delay part in the interior, the system becomes unstable (see [9]). It is shown that a small delay in a boundary control can turn such a well-behaved hyperbolic system into a wild one and, therefore, the delay becomes a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms are necessary (see [25], [26], [35]).

Originally the Bresse system consists of three wave equations where the main variables describe the longitudinal, vertical and shear angle displacements, which can be represented as (see [6]):

$$
(1.5) \qquad \begin{cases}
\varrho_1\varphi_{tt}=Q_x+lN+F_1,\\
\varrho_2\psi_{tt}=M_x-Q+F_2,\\
\varrho_1 w_{tt}=N_x-IQ+F_3,
\end{cases}
$$

where

$$
(1.6) \qquad N=k_0(w_x-l\varphi),\quad Q=k(\varphi_x+lw+\psi),\quad M=b\psi_x.
$$

We use N, Q and M to denote the axial force, the shear force and the bending moment. Here $\varrho_1=\varrho A$, $\varrho_2=\varrho I$, $b=EI$, $k_0=EA$, $k=k'GA$ and $l=R^{-1}$. Concerning material properties, we use ϱ for density, E for the modulus of elasticity,

G for the shear modulus, K for the shear factor, A for the cross-sectional area, I for the second moment of area of the cross-section and R for the radius of curvature and we assume that all these quantities are positive. By F_i we denote external forces. The Bresse system (1.5) is more general than the well-known Timoshenko system where the longitudinal displacement ω is not considered, i.e., $l = 0$. There are a number of publications concerning the stabilization of the Timoshenko system with different kinds of damping; in this regard, we note further references (see [11], [15], [19], [20], [21], [22], [23], [27], [28], [30], [33]).

System (1.5) is an undamped system and its associated energy remains constant when the time t evolves. To stabilize the system (1.5), many damping terms have been considered by several authors (see [1], [3], [4], [5], [13], [17], [32], [36]).

In the following text, we present some works, which studied the stability of the dissipative Bresse system. The paper [2] was concerned with the asymptotic stability of a Bresse system with two frictional dissipations

$$
\begin{cases}
\varrho_1\varphi_{tt} - k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) = -\gamma_1\varphi_t, \\
\varrho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + lw + \psi) = -\gamma_2\psi_t, \\
\varrho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk(\varphi_x + lw + \psi) = 0.
\end{cases} \tag{1.7}
$$

Under the condition of equal speeds of wave propagation, the authors prove that the system is exponentially stable. Otherwise, they show that the Bresse system is not exponentially stable. Then, they prove that the solution decays polynomially to zero with the optimal decay rate, depending on the regularity of the initial data.

There are several works dedicated to the mathematical analysis of the Bresse system. They are mainly concerned with decay rates of solutions of the linear system. This is done by adding suitable damping effects that can be of thermal, viscous or viscoelastic nature, see for instance ([10], [31], [34]), among others.

Concerning the thermoelastic Bresse system, [17] considered

$$
\begin{cases}
\varrho_1\varphi_{tt} - k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) + l\gamma\theta_1 = 0, \\
\varrho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + lw + \psi) + \gamma\theta_x = 0, \\
\varrho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + lw + \psi) + \gamma\theta_{1x} = 0, \\
\varrho_3\theta_t - \theta_{xx} + \gamma\psi_{tx} = 0, \\
\varrho_3\theta_{1x} - \theta_{1xx} + \gamma(w_{tx} - l\varphi_t) = 0
\end{cases} \tag{1.8}
$$

together with initial and specific boundary conditions and proved an exponential and only polynomial-type decay stability results.

Taking in particular $\mu_1 = \mu_2 = 0$ in (1.1), then the system is reduced to the next one, which is studied in [14] under the initial and boundary conditions (1.2)–(1.3)

$$
(1.9) \qquad \begin{cases} \varrho_1\varphi_{tt} - k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) = 0 & \text{in } (0,1)\times(0,\infty), \\ \varrho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + lw + \psi) + \gamma\theta_x = 0 & \text{in } (0,1)\times(0,\infty), \\ \varrho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + lw + \psi) = 0 & \text{in } (0,1)\times(0,\infty), \\ \varrho_3\theta_t + q_x + \gamma\psi_{tx} = 0 & \text{in } (0,1)\times(0,\infty), \\ \alpha q_t + \beta q + \theta_x = 0 & \text{in } (0,1)\times(0,\infty). \end{cases}
$$

In [14] it is obtained the exponential stability of the energy solution (see Theorem 3.9) by assuming the condition (1.6) therein that corresponds to (1.4) of the present article. Moreover, in the sequel the authors prove the lack of exponential stability of (1.9) when (1.6) fails (see [14], Theorem 4.2). In all these results the only dissipation is given by the heat flux q, see for instance the equation (3.2) in this reference. On the other hand, in this article, we deal with (1.1) by taking the assumption (2.6) into account with respect to parameters μ_1, μ_2. This leads to, at least, two damping mechanisms as one can see from the inequality (3.2) in Lemma 3. Then, the main result on exponential stability stated by Theorem 2 is obtained under the assumption (1.4) that corresponds to (1.6) in [14]. Therefore, the same result is obtained, under basically the same assumption on coefficients by using more damping as given in the inequality (3.2), where one can easily see that more damping was added to system. This means that the result on the exponential stability is weaker than the one provided in [14] from the stability point of view by clarifying that delay spoils the energy estimates.

In [12], the authors considered two Cauchy problems related to the Bresse model with two dissipative mechanisms corresponding to the heat conduction coupled to the system. The first of them is the Bresse system with thermoelasticity of Type I:

$$
(1.10) \qquad \begin{aligned} &\varrho_1\varphi_{tt} - k(\varphi_x - \psi - l\omega)_x - k_0 l(\omega_x - l\varphi) + l\gamma\theta_1 = 0 && \text{in } \mathbb{R}\times(0,\infty), \\ &\varrho_2\psi_{tt} - b\psi_{xx} - k(\varphi_x - \psi - l\omega) + \gamma\theta_{2x} = 0 && \text{in } \mathbb{R}\times(0,\infty), \\ &\varrho_1\omega_{tt} - k_0(\omega_x - l\varphi)_x - kl(\varphi_x - \psi - l\omega) + \gamma\theta_{1x} = 0 && \text{in } \mathbb{R}\times(0,\infty), \\ &\theta_{1t} - k_1\theta_{1xx} + m_1(\omega_x - l\varphi)_t = 0 && \text{in } \mathbb{R}\times(0,\infty), \\ &\theta_{2t} - k_2\theta_{2xx} + m_2\psi_{xt} = 0 && \text{in } \mathbb{R}\times(0,\infty) \end{aligned}
$$

with the initial data

$$
(\varphi, \varphi_t, \psi, \psi_t, \omega, \omega_t, \theta_1, \theta_2)(x,0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, \theta_{10}, \theta_{20})(x).
$$

The second one is the Bresse system with thermoelasticity of Type III:

$$
(1.11)\quad \begin{aligned}
&\varrho_1\varphi_{tt} - k(\varphi_x - \psi - l\omega)_x - k_0 l(\omega_x - l\varphi) + l\gamma\theta_{1t} = 0 && \text{in } \mathbb{R}\times(0,\infty),\\
&\varrho_2\psi_{tt} - b\psi_{xx} - k(\varphi_x - \psi - l\omega) + \gamma\theta_{2xt} = 0 && \text{in } \mathbb{R}\times(0,\infty),\\
&\varrho_1\omega_{tt} - k_0(\omega_x - l\varphi)_x - kl(\varphi_x - \psi - l\omega) + \gamma\theta_{1xt} = 0 && \text{in } \mathbb{R}\times(0,\infty),\\
&\theta_{1tt} - k_1\theta_{1xx} - \alpha_1\theta_{1xxt} + m_1(\omega_x - l\varphi)_t = 0 && \text{in } \mathbb{R}\times(0,\infty),\\
&\theta_{2tt} - k_2\theta_{2xx} - \alpha_2\theta_{2xxt} + m_2\psi_{xt} = 0 && \text{in } \mathbb{R}\times(0,\infty)
\end{aligned}
$$

with the initial data

$$
(\varphi,\varphi_t,\psi,\psi_t,\omega,\omega_t,\theta_1,\theta_2,\theta_{1t},\theta_{2t})(x,0) = (\varphi_0,\varphi_1,\psi_0,\psi_1,\omega_0,\omega_1,\theta_{10},\theta_{20},\theta_{11},\theta_{21})(x),
$$

where α_1, α_2, ϱ_1, ϱ_2, γ, b, k, k_0, k_1, k_2, l, m_1 and m_2 are positive constants. The authors proved that the decay rates of the solutions are very slow in the whole line, where they show that the solutions decay with the rate of $(1+t)^{-1/8}$ in the L^2-norm, whenever the initial data belong to $L^1(\mathbb{R})\cap H^s(\mathbb{R})$ for a suitable s. The main tool used to prove results is the energy method in the Fourier space.

In [3], the authors considered the Bresse system in a bounded domain with delay terms in the internal feedbacks,

$$
(1.12)\quad \begin{cases}
\varrho_1\varphi_{tt} - Gh(\varphi_x + lw + \psi)_x - Ehl(w_x - l\varphi) + \mu_1\varphi_t + \mu_2\varphi_t(x,t-\tau_1) = 0,\\
\varrho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + lw + \psi) + \widetilde{\mu_1}\psi_t + \widetilde{\mu_2}\psi_t(x,t-\tau_2) = 0,\\
\varrho_1 w_{tt} - Eh(w_x - l\varphi)_x + lGh(\varphi_x + lw + \psi) + \widetilde{\widetilde{\mu_1}} w_t + \widetilde{\widetilde{\mu_2}} w_t(x,t-\tau_3) = 0,
\end{cases}
$$

where $(x,t)\in(0,L)\times(0,\infty)$, $\tau_i > 0$ $(i = 1,2,3)$ is a time delay, μ_1,μ_2, $\widetilde{\mu_1}$, $\widetilde{\mu_2}$, $\widetilde{\widetilde{\mu_1}}$, $\widetilde{\widetilde{\mu_2}}$ are positive real numbers. This system is subjected to the Dirichlet boundary conditions and to the initial conditions which belong to a suitable Sobolev space. First, the authors proved the global existence of its solutions in Sobolev spaces by means of semigroup theory under a condition between the weight of the delay terms in the feedbacks and the weight of the terms without delay. Furthermore, they studied the asymptotic behavior of solutions using the multiplier method.

2. Preliminaries and well-posedness

First, let us assume the hypothesis

$$
|\mu_2| < \mu_1.
$$

We will prove that the systems (1.1)–(1.3) are well posed using semigroup theory by introducing the following new variable (see [26])

$$
(2.1)\qquad z(x,\varrho,t) = \varphi_t(x,t-\tau\varrho),\quad x\in(0,1),\ \varrho\in(0,1),\ t>0.
$$

Then, we have

$$\tau z_t(x,\varrho,t) + z_\varrho(x,\varrho,t) = 0 \quad \text{in } (0,1)\times(0,1)\times(0,\infty). \tag{2.2}$$

Therefore, the problem (1.1) takes the form

$$\begin{cases} \varrho_1\varphi_{tt} - k(\varphi_x + lw + \psi)_x - lk_0(w_x - l\varphi) + \mu_1\varphi_t(x,t) + \mu_2 z(x,1,t) = 0,\\ \tau z_t(x,\varrho,t) + z_\varrho(x,\varrho,t) = 0,\\ \varrho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + lw + \psi) + \gamma\theta_x = 0,\\ \varrho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk(\varphi_x + lw + \psi) = 0,\\ \varrho_3\theta_t + q_x + \gamma\psi_{tx} = 0,\\ \alpha q_t + \beta q + \theta_x = 0 \end{cases} \tag{2.3}$$

with the boundary conditions

$$\begin{aligned} \varphi(0,t) &= \varphi_x(1,t) = \psi_x(0,t) = \psi(1,t) = w_x(0,t) = w(1,t)\\ &= \theta(0,t) = q(1,t) = 0, \quad t \geqslant 0, \end{aligned} \tag{2.4}$$

and the initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ \psi(x,0) = \psi_0(x), & x\in(0,1),\\ \psi_t(x,0) = \psi_1(x),\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), & x\in(0,1),\\ \theta(x,0) = \theta_0(x),\ q(x,0) = q_0(x), & x\in(0,1),\\ \varphi_t(x,-t) = f_0(x,t) & \text{in } (0,1)\times(0,\tau). \end{cases} \tag{2.5}$$

Let ξ be a positive constant such that

$$\tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|), \tag{2.6}$$

where τ is a real number with $0<\tau$ and μ_1, μ_2 are positive constants, and the initial data (φ_0, φ_1, ψ_0, ψ_1, w_0, w_1, f, θ_0, q_0) belong to a suitable space (see below).

If we set

$$U = (\varphi,\varphi_t,z,\psi,\psi_t,w,w_t,\theta,q)^\top,$$

then

$$U' = (\varphi_t,\varphi_{tt},z_t,\psi_t,\psi_{tt},w_t,w_{tt},\theta_t,q_t)^\top.$$

Therefore, the problem (2.3)–(2.5) can be written as

$$\begin{cases} U'(t) - AU(t) = 0,\\ U(0) = (\varphi_0,\varphi_1,f_1(\cdot,\tau),\psi_0,\psi_1,w_0,w_1,\theta_0,q_0), \end{cases} \tag{2.7}$$

where the operator A is defined by

$$
(2.8)\qquad A\begin{pmatrix} c\varphi \\ u \\ z \\ \psi \\ v \\ w \\ \varpi \\ \theta \\ q \end{pmatrix} = \begin{pmatrix} cu \\ \dfrac{k}{\varrho_1}(\varphi_x + lw + \psi)_x + \dfrac{k_0 l}{\varrho_1}(w_x - l\varphi) - \dfrac{\mu_1}{\varrho_1}u - \dfrac{\mu_2}{\varrho_1}z(\cdot,1) \\ -\Big(\dfrac{1}{\tau}\Big)z_\varrho \\ v \\ \dfrac{b}{\varrho_2}\psi_{xx} - \dfrac{k}{\varrho_2}(\varphi_x + lw + \psi) - \dfrac{\gamma}{\varrho_2}\theta_x \\ \varpi \\ \dfrac{k_0}{\varrho_1}(w_x - l\varphi)_x - \dfrac{kl}{\varrho_1}(\varphi_x + lw + \psi) \\ -\dfrac{1}{\varrho_3}q_x - \dfrac{\gamma}{\varrho_3}u_x \\ -\dfrac{\beta}{\alpha}q - \dfrac{1}{\alpha}\theta_x \end{pmatrix}.
$$

We consider the spaces

$$
H^1_*(0,1) = \{h \in H^1(0,1)\colon\ h(0) = 0\}, \quad \widetilde{H}^1_*(0,1) = \{h \in H^1(0,1)\colon\ h(1) = 0\},
$$
$$
H^2_*(0,1) = H^2(0,1) \cap H^1_*(0,1), \quad \widetilde{H}^2_*(0,1) = H^2(0,1) \cap \widetilde{H}^1_*(0,1),
$$

and

$$
\begin{aligned}
\mathcal{H} = {} & H^1_*(0,1) \times L^2(0,1) \times L^2((0,1), H^1_0(0,1)) \times \widetilde{H}^1_*(0,1) \\
& \times L^2(0,1) \times \widetilde{H}^1_*(0,1) \times L^2(0,1) \times L^2(0,1) \times L^2(0,1),
\end{aligned}
$$

where $L^2((0,1), H^1_0(0,1))$ is the space of square summable functions of $H^1_0(0,1)$.

We show that A generates a C_0 semigroup on $\mathcal{H}$ under the assumption (2.6). Let us define on the Hilbert space $\mathcal{H}$ the inner product, for $U = (\varphi, u, z, \psi, v, w, \varpi, \theta, q)^\top$, $\overline{U} = (\overline{\varphi}, \overline{u}, \overline{z}, \overline{\psi}, \overline{v}, \overline{w}, \overline{\varpi}, \overline{\theta}, \overline{q})^\top$,

$$
\begin{aligned}
(2.9)\quad \langle U, \overline{U}\rangle_{\mathcal{H}} = {} & k\int_0^1 (\varphi_x + \psi + lw)(\overline{\varphi}_x + \overline{\psi} + l\overline{w})\,\mathrm{d}x + k_0\int_0^1 (w_x - l\varphi)(\overline{w}_x - l\overline{\varphi})\,\mathrm{d}x \\
& + \varrho_1\int_0^1 u\overline{u}\,\mathrm{d}x + \varrho_2\int_0^1 v\overline{v}\,\mathrm{d}x + \varrho_1\int_0^1 \varpi\overline{\varpi}\,\mathrm{d}x + b\int_0^1 \psi_x\overline{\psi_x}\,\mathrm{d}x \\
& + \xi\int_0^1\int_0^1 z\overline{z}d\varrho\,\mathrm{d}x + \varrho_3\int_0^1 \theta\overline{\theta}\,\mathrm{d}x + \alpha\int_0^1 q\overline{q}\,\mathrm{d}x.
\end{aligned}
$$

Note that $\mathcal{H}$ equipped with the inner product (2.9) is an Hilbert space. The domain of A is given by

$$
(2.10)\qquad D(A) = \begin{cases} U \in \mathcal{H}/\varphi \in H^2_*(0,1);\ \psi, w \in \widetilde{H}^2_*(0,1), \\ u, \theta \in H^1_*(0,1);\ v, \varpi, q \in \widetilde{H}^1_*(0,1),\ u = z(\cdot,0), \\ z_\varrho \in L^2((0,1);\ L^2(0,1)),\ \varphi_x(1) = 0,\ w_x(0) = \psi_x(0) = 0. \end{cases}
$$

We now prove that A is a maximal monotone operator. For this purpose we need the following two lemmas.

Lemma 1. *The operator A is dissipative and satisfies, for any $U \in D(A)$,*

$$\langle AU, U\rangle_{\mathcal{H}} = -\beta \int_0^1 q^2 \,\mathrm{d}x + \Big(-\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2\tau}\Big) \int_0^1 u^2 \,\mathrm{d}x + \Big(\frac{\mu_2}{2} - \frac{\xi}{2\tau}\Big) \int_0^1 z^2(x,1) \,\mathrm{d}x \leqslant 0. \tag{2.11}$$

P r o o f. For any $U \in D(A)$, using the inner product,

$$\langle AU, U\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} u \\ \dfrac{k}{\varrho_1}(\varphi_x + lw + \psi)_x + \dfrac{k_0 l}{\varrho_1}(w_x - l\varphi) - \dfrac{\mu_1}{\varrho_1}u - \dfrac{\mu_2}{\varrho_1}z(\cdot,1) \\ -\Big(\dfrac{1}{\tau}\Big)z_\varrho \\ v \\ \dfrac{b}{\varrho_2}\psi_{xx} - \dfrac{k}{\varrho_2}(\varphi_x + lw + \psi) - \dfrac{\gamma}{\varrho_2}\theta_x \\ -\varpi \\ \dfrac{k_0}{\varrho_1}(w_x - l\varphi)_x - \dfrac{kl}{\varrho_1}(\varphi_x + lw + \psi) \\ -\dfrac{1}{\varrho_3}q_x - \dfrac{\gamma}{\varrho_3}u_x \\ -\dfrac{\beta}{\alpha}q - \dfrac{1}{\alpha}\theta_x \end{pmatrix}, \begin{pmatrix} c\varphi \\ u \\ z \\ \psi \\ v \\ w \\ \varpi \\ \theta \\ q \end{pmatrix} \right\rangle_{\mathcal{H}}.$$

Then

$$\begin{aligned} \langle AU, U\rangle_{\mathcal{H}} = {} & k\int_0^1 (u_x + v + l\varpi)(\varphi_x + lw + \psi)\,\mathrm{d}x \\ & + k_0\int_0^1 (\varpi_x - lu)(w_x - l\varphi)\,\mathrm{d}x + k\int_0^1 (\varphi_x + lw + \psi)u\,\mathrm{d}x \\ & + k_0 l\int_0^1 (w_x - l\varphi)u\,\mathrm{d}x - \mu_1\int_0^1 u^2\,\mathrm{d}x - \mu_2\int_0^1 z(x,1)u\,\mathrm{d}x \\ & + b\int_0^1 \psi_{xx}v\,\mathrm{d}x - k\int_0^1 (\varphi_x + lw + \psi)v\,\mathrm{d}x - \gamma\int_0^1 \theta_x v\,\mathrm{d}x \\ & + k_0\int_0^1 (w_x - l\varphi)\varpi\,\mathrm{d}x - kl\int_0^1 (\varphi_x + lw + \psi)\varpi\,\mathrm{d}x \\ & + b\int_0^1 v_x\psi_x\,\mathrm{d}x - \int_0^1 q_x\theta\,\mathrm{d}x - \gamma\int_0^1 u_x\theta\,\mathrm{d}x \\ & - \beta\int_0^1 q^2\,\mathrm{d}x - \int_0^1 \theta_x q\,\mathrm{d}x - \frac{\xi}{\tau}\int_0^1\int_0^1 zz_\varrho\,\mathrm{d}\varrho\,\mathrm{d}x. \end{aligned} \tag{2.12}$$

By the fact that

$$
\begin{aligned}
-\beta \int_0^1 q^2 \,\mathrm{d}x - \mu_1 \int_0^1 u^2 \,\mathrm{d}x - \mu_2 \int_0^1 z(x,1)u \,\mathrm{d}x - \frac{\xi}{\tau} \int_0^1 \int_0^1 z(x,\varrho) z_\varrho(x,\varrho) \,\mathrm{d}\varrho \,\mathrm{d}x \\
= & -\beta \int_0^1 q^2 \,\mathrm{d}x - \mu_1 \int_0^1 u^2 \,\mathrm{d}x - \mu_2 \int_0^1 z(x,1)u \,\mathrm{d}x \\
& - \frac{\xi}{2\tau} \int_0^1 \int_0^1 \frac{\partial}{\partial \varrho} z^2(x,\varrho) \,\mathrm{d}\varrho \,\mathrm{d}x \\
= & -\beta \int_0^1 q^2 \,\mathrm{d}x - \mu_1 \int_0^1 u^2 \,\mathrm{d}x - \mu_2 \int_0^1 z(x,1)u \,\mathrm{d}x \\
& - \frac{\xi}{2\tau} \int_0^1 (z^2(x,1) - z^2(x,0)) \,\mathrm{d}x \\
= & -\beta \int_0^1 q^2 \,\mathrm{d}x - \mu_1 \int_0^1 u^2 \,\mathrm{d}x - \mu_2 \int_0^1 z(x,1)u \,\mathrm{d}x \\
& - \frac{\xi}{2\tau} \int_0^1 z^2(x,1) \,\mathrm{d}x + \frac{\xi}{2\tau} \int_0^1 u^2 \,\mathrm{d}x
\end{aligned}
$$

and by using Young's inequality, we obtain

$$
\langle AU, U \rangle_H \leqslant -\beta \int_0^1 q^2 \,\mathrm{d}x + \Big(-\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2\tau}\Big) \int_0^1 u^2 \,\mathrm{d}x + \Big(\frac{\mu_2}{2} - \frac{\xi}{2\tau}\Big) \int_0^1 z^2(x,1) \,\mathrm{d}x.
$$

Keeping in mind the condition (2.6), the desired result follows. □

Lemma 2. *The operator $I - A$ is surjective.*

Proof. We need to show that for all $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5, f_6, f_7 f_8, f_9)^\top \in \mathcal{H}$, there exists $U \in D(A)$ such that

$$
U - AU = \mathcal{F}, \tag{2.13}
$$

that is,

$$
\begin{gathered}
-u + \varphi = f_1 \in H_*^1(0,1), \\
-k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) + \varrho_1 u + \mu_1 u + \mu_2 z(\cdot,1) = \varrho_1 f_2 \in L^2(0,1), \\
z + \tau^{-1} z_\varrho = f_3 \in L^2((0,1), \quad H^1(0,1)), \\
-v + \psi = f_4 \in \widetilde{H}_*^1(0,1), \\
-b\psi_{xx} + k(\varphi_x + lw + \psi) + \varrho_2 v + \gamma \theta_x = \varrho_2 f_5 \in L^2(0,1), \\
-\varpi + w = f_6 \in \widetilde{H}_*^1(0,1), \\
-k_0(w_x - l\varphi)_x + kl(\varphi_x + lw + \psi) + \varrho_1 \varpi = \varrho_1 f_7 \in L^2(0,1), \\
q_x + \gamma v_x + \varrho_3 \theta = \varrho_3 f_8 \in L^2(0,1), \\
(\beta + \alpha) q + \theta_x = \alpha f_9 \in L^2(0,1).
\end{gathered} \tag{2.14}
$$

From (2.14), we put

$$\theta = \alpha \int_0^x f_9(y)\,\mathrm{d}y - (\beta+\alpha)\int_0^x q(y)\,\mathrm{d}y. \tag{2.15}$$

Then $\theta(0,t)=0$. Inserting $u=\varphi-f_1$, $v=\psi-f_4$, $\varpi=w-f_6$ and (2.14) into (2.15), we get

$$\begin{aligned} -k(\varphi_x+lw+\psi)_x - k_0 l(w_x-l\varphi)+\mu_3\varphi+\mu_2 z(\cdot,1) &= h_1\in L^2(0,1),\\ -b\psi_{xx}+k(\varphi_x+lw+\psi)+\varrho_2\psi-\gamma(\beta+\alpha)q &= h_2\in L^2(0,1),\\ -k_0(w_x-l\varphi)_x+kl(\varphi_x+lw+\psi)+\varrho_1 w &= h_3\in L^2(0,1),\\ q_x+\varrho_3(\beta+\alpha)\int_0^x q(y)\,\mathrm{d}y+\gamma\psi_x &= h_4\in L^2(0,1),\\ z+\tau^{-1}z_\varrho &= h_5\in L^2(0,1), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} &h_1=\varrho_1(f_1+f_2)+\mu_1 f_1,\quad h_2=\varrho_2(f_4+f_5)-\alpha\gamma f_9,\quad h_3=\varrho_1(f_6+f_7),\\ &h_4=\gamma f_{4x}+\varrho_3\Big(f_8-\alpha\int_0^x f_9(y)\,\mathrm{d}y\Big),\quad h_5=\tau f_3,\quad \mu_3=\varrho_1+\mu_1. \end{aligned} \tag{2.17}$$

Furthermore, from $(2.16)_5$, we can easily obtain

$$z(x,\varrho)=\varphi(x)\mathrm{e}^{-\tau\varrho}-f_1\mathrm{e}^{\tau\varrho}+\tau\mathrm{e}^{-\tau\varrho}\int_0^\varrho f_3(x,s)\mathrm{e}^{\tau\varrho s}\,\mathrm{d}s$$

and we observe that

$$z_0(x)=-f_1\mathrm{e}^{-\tau}+\tau\mathrm{e}^{-\tau}\int_0^\varrho f_3(x,s)\mathrm{e}^{\tau s}\,\mathrm{d}s,\quad z(x,1)=\varphi(x)\mathrm{e}^{-\tau}+z_0(x)$$

for all $x\in(0,1)$.

To solve (2.16), we consider the variational formulation

$$a((\varphi,\psi,w,q),(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}))=L(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}), \tag{2.18}$$

where

$$a\colon\ [H^1_*(0,1)\times\widetilde{H}^1_*(0,1)\times\widetilde{H}^1_*(0,1)\times L^2(0,1)]^2\to\mathbb{R},$$

is the bilinear form given by

$$
\begin{aligned}
(2.19)\qquad & a((\varphi,\psi,w,q),(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}))\\
&= k\int_0^1(\varphi_x+lw+\psi)(\widetilde{\varphi}_x+l\widetilde{w}+\widetilde{\psi})\,\mathrm{d}x+(\beta+\alpha)\int_0^1 q\widetilde{q}\,\mathrm{d}x\\
&\quad+b\int_0^1\psi_x\widetilde{\psi}_x\,\mathrm{d}x+\varrho_2\int_0^1\psi\widetilde{\psi}\,\mathrm{d}x-\gamma(\beta+\alpha)\int_0^1 q\widetilde{\psi}\,\mathrm{d}x\\
&\quad+\varrho_1\int_0^1\psi\widetilde{\psi}\,\mathrm{d}x+\gamma(\beta+\alpha)\int_0^1\psi\widetilde{q}\,\mathrm{d}x+\varrho_1\int_0^1 w\widetilde{w}\,\mathrm{d}x\\
&\quad+k_0\int_0^1(w_x-l\varphi)(\widetilde{w}_x-l\widetilde{\varphi})\,\mathrm{d}x+(\mu_3+\mu_2\mathrm{e}^{-\tau})\int_0^1\varphi\widetilde{\varphi}\,\mathrm{d}x\\
&\quad+\varrho_3(\beta+\alpha)\int_0^1\Big(\int_0^x q(y)\,\mathrm{d}y\int_0^x\widetilde{q}(y)\,\mathrm{d}y\Big)\,\mathrm{d}x
\end{aligned}
$$

and

$$L\colon\ [H_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times L^2(0,1)]\to\mathbb{R}$$

is the linear form defined by

$$
\begin{aligned}
(2.20)\qquad L(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q})&=\int_0^1(h_1-\mu_2z_0)\widetilde{\varphi}\,\mathrm{d}x+\int_0^1 h_2\widetilde{\psi}\,\mathrm{d}x+\int_0^1 h_3\widetilde{w}\,\mathrm{d}x\\
&\quad+(\alpha+\beta)\int_0^1 h_4\int_0^x\widetilde{q}(y)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}
$$

As in [1], it is easy to verify that a is continuous and coercive, and L is continuous. So, by applying the Lax-Milgram theorem, we deduce that for all $(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q})\in H_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times L^2(0,1)$ the problem (2.18) admits the unique solution $(\varphi,\psi,w,q)\in H_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times L^2(0,1)$. Substituting φ, ψ, w and q into $(2.14)_1$, $(2.14)_4$, $(2.14)_6$, and $(2.14)_9$, respectively, we get

$$u\in H_*^1(0,1),\quad v\in\widetilde{H}_*^1(0,1),\quad \varpi\in\widetilde{H}_*^1(0,1),\quad \theta\in H_*^1(0,1).$$

Now if $(\widetilde{\psi},\widetilde{w},\widetilde{q})\equiv(0,0,0)\in\widetilde{H}_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times L^2(0,1)$ then (2.18) reduces to

$$
\begin{aligned}
(2.21)\qquad & k\int_0^1(\varphi_x+lw+\psi)\widetilde{\varphi}_x-k_0l\int_0^1(w_x-l\varphi)\widetilde{\varphi}\,\mathrm{d}x\\
&\quad+(\mu_3+\mu_2\mathrm{e}^{-\tau})\int_0^1\varphi\widetilde{\varphi}\,\mathrm{d}x=\int_0^1(h_1-\mu_2z_0)\widetilde{\varphi}\,\mathrm{d}x
\end{aligned}
$$

for all $\widetilde{\varphi}\in H_*^1(0,1)$, which implies

$$(2.22)\quad -k\varphi_{xx}=k\varphi_x+l(k+k_0)w_x-(l^2k_0+\mu_2\mathrm{e}^{-\tau}+\mu_3)\varphi+h_1-\mu_2z_0\in L^2(0,1).$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$\varphi\in H_*^2(0,1).$$

Moreover, (2.21) is also true for any $\phi \in C^1([0,1])$, $\phi(0) = 0$, which is in $H^1_*(0,1)$. Hence, for all $\phi \in C^1([0,1])$, $\phi(0) = 0$, we have

$$k\int_0^1 \varphi_x \phi_x \,\mathrm{d}x - \int_0^1 (k\varphi_x + l(k+k_0)w_x - (l^2 k_0 + \mu_2 \mathrm{e}^{-\tau} + \mu_3)\varphi + h_1 - \mu_2 z_0)\phi \,\mathrm{d}x = 0.$$

Thus, using integration by parts and bearing in mind (2.22), we get

$$\varphi_x(1)\phi(1) = 0 \quad \forall\, \phi \in C^1([0,1]),\ \phi(0) = 0,$$

since ϕ is arbitrary. Therefore,

$$\varphi_x(1) = 0.$$

Similarly, we get

$$\begin{aligned} -b\psi_{xx} + k(\varphi_x + lw + \psi) + \varrho_2\psi - \gamma(\beta+\alpha)q &= h_2 \in L^2(0,1),\\ -k_0(w_x - l\varphi)_x + kl(\varphi_x + lw + \psi) + \varrho_1 w &= h_3 \in L^2(0,1),\\ q_x + \varrho_3(\beta+\alpha)\int_0^x q(y)\,\mathrm{d}y + \gamma\psi_x &= h_4 \in L^2(0,1), \end{aligned}$$

thus, we have $\psi, w \in \widetilde{H}^2_*(0,1)$ and $q \in \widetilde{H}^1_*(0,1)$, $\psi_x(1) = w_x(1) = 0$.

Finally, the application of the regularity theory for the linear elliptic equations guarantees the existence of unique $U \in D(A)$ such that (2.7) is satisfied. Consequently, using Lemmas 2.1 and 2.2, we conclude that A is a maximal monotone operator. Hence, by the Lumer-Philips theorem (see [18] and [29]) we have the following well-posedness result. □

Theorem 1. *Let $U_0 \in \mathcal{H}$, then there exists a unique weak solution $U \in C(\mathbb{R}^+, \mathcal{H})$ of the problem* (2.3)–(2.5). *Moreover, if $U_0 \in D(A)$, then*

$$U \in C(\mathbb{R}^+, D(A)) \cap C^1(\mathbb{R}^+, \mathcal{H}).$$

3. Exponential stability

In this section, we state and prove our stability result for the energy of the solution of the system (2.3)–(2.5) by using the multiplier technique. So

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^1 (\varrho_1\varphi_t^2 + \varrho_2\psi_t^2 + \varrho_1 w_t^2 + b\psi_x^2 + \varrho_3\theta^2 + \alpha q^2 + k(\varphi_x + \psi + lw)^2 \\ + k_0(w_x - l\varphi)^2)\,\mathrm{d}x + \frac{\xi}{2\tau}\int_0^1\int_0^1 z^2(x,\varrho,t)\,\mathrm{d}\varrho\,\mathrm{d}x. \end{aligned} \tag{3.1}$$

To achieve our goal, we need the following lemmas.

Lemma 3. *Let* $(\varphi,\psi,w,\theta,q,z)$ *be a solution of* (2.3)–(2.5). *Then the energy functional, defined by* (3.1), *satisfies*

$$\text{(3.2)}\quad \mathrm{e}'(t) \leqslant -\beta\int_0^1 q^2\,\mathrm{d}x - \Big(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\Big)\,\|\varphi_t\|_2^2 - \Big(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\Big)\|z(x,1,t)\|_2^2.$$

Proof. Multiply $(2.3)_1$–$(2.3)_6$ by φ_t, $z\xi/\tau$, ψ_t, w_t, θ, and q, respectively. Integrate over $(0,1)$ and over $(0,1)\times(0,1)$ as well as use integration by parts with the boundary conditions. By summing the results, we obtain (3.2). □

Lemma 4. *Let* $(\varphi,\psi,w,\theta,q,z)$ *be a solution of* (2.3–(2.5). *Then the functional*

$$\text{(3.3)}\qquad F_1(t) := \alpha\varrho_3\int_0^1 \theta\int_0^x q(y)\,\mathrm{d}y\,\mathrm{d}x$$

satisfies, for any $\varepsilon_1>0$, *the estimate*

$$\text{(3.4)}\qquad F_1'(t) \leqslant -\frac{\varrho_3}{2}\int_0^1 \theta^2\,\mathrm{d}x + \varepsilon_1\int_0^1 \psi_t^2\,\mathrm{d}x + c\Big(1+\frac{1}{\varepsilon_1}\Big)\int_0^1 q^2\,\mathrm{d}x.$$

Proof. Differentiating the functional F_1, using $(2.3)_5$ and $(2.3)_6$ and integrating by parts, we get

$$\text{(3.5)}\quad F_1'(t) = -\varrho_3\int_0^1 \theta^2\,\mathrm{d}x + \alpha\int_0^1 q^2\,\mathrm{d}x + \alpha\gamma\int_0^1 q\psi_t\,\mathrm{d}x - \beta\varrho_3\int_0^1 \theta\int_0^x q(y)\,\mathrm{d}y\,\mathrm{d}x.$$

The Cauchy-Schwarz and Young's inequalities lead to (3.4) with $\varepsilon_1>0$. □

Lemma 5. *Let* $(\varphi,\psi,w,\theta,q,z)$ *be a solution of* (2.3)–(2.5). *Then the functional*

$$\text{(3.6)}\qquad F_2(t) := -\frac{\varrho_2\varrho_3}{\gamma}\int_0^1 \theta\int_0^x \psi_t(y)\,\mathrm{d}y\,\mathrm{d}x$$

satisfies, for any $\varepsilon_1,\varepsilon_2>0$, *the estimate*

$$\begin{aligned}\text{(3.7)}\qquad F_2'(t) := &-\frac{\varrho_2}{\gamma}\int_0^1 \psi_t^2\,\mathrm{d}x + \varepsilon_2\int_0^1 (\varphi_x+\psi+lw)^2\,\mathrm{d}x\\ &+\varepsilon_3\int_0^1 \psi_x^2\,\mathrm{d}x + c\Big(1+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}\Big)\int_0^1 \theta^2\,\mathrm{d}x + c\int_0^1 q^2\,\mathrm{d}x.\end{aligned}$$

P r o o f. By differentiating F_2, using $(2.3)_3$ and $(2.3)_5$, and integrating by parts, we get

$$(3.8)\quad F_2'(t) = -\varrho_2\int_0^1 \psi_t^2\,\mathrm{d}x - \frac{\varrho_2}{\gamma}\int_0^1 q\psi_t\,\mathrm{d}x + \varrho_3\int_0^1 \theta^2\,\mathrm{d}x - \frac{b\varrho_3}{\gamma}\int_0^1 \theta\psi_x\,\mathrm{d}x + \frac{k\varrho_3}{\gamma}\int_0^1 (\varphi_x+\psi+lw)\int_0^x \theta(y)\,\mathrm{d}y\,\mathrm{d}x.$$

The estimate (3.7) follows by using the Cauchy-Schwarz and Young's inequalities. □

Lemma 6. *Let* $(\varphi,\psi,w,\theta,q,z)$ *be a solution of* (2.3)–(2.5). *Then the functional*

$$(3.9)\qquad F_3(t) := -\varrho_1\int_0^1 (\varphi\varphi_t + ww_t)\,\mathrm{d}x$$

satisfies the estimate

$$(3.10)\quad F_3'(t) \leqslant -\Big(\varrho_1 - \frac{1}{4\varepsilon_4}\Big)\int_0^1 \varphi_t^2\,\mathrm{d}x + c\int_0^1 \psi_x^2\,\mathrm{d}x + k_0\int_0^1 (w_x - l\varphi)^2\,\mathrm{d}x + c\int_0^1 (\varphi_x+\psi+lw)^2\,\mathrm{d}x - \varrho_1\int_0^1 w_t^2\,\mathrm{d}x + (\varepsilon_5\mu_2 + \mu_1\varepsilon_4)\int_0^1 \varphi^2\,\mathrm{d}x + \frac{\mu_2}{4\varepsilon_5}\int_0^1 z^2(x,1,t)\,\mathrm{d}x.$$

P r o o f. Taking the derivative of F_3, and exploiting $(2.3)_1$ and $(2.3)_4$, we get

$$(3.11)\quad F_3'(t) = -\varrho_1\int_0^1 \varphi_t^2\,\mathrm{d}x + k\int_0^1 (\varphi_x+\psi+lw)^2\,\mathrm{d}x - \varrho_1\int_0^1 w_t^2\,\mathrm{d}x - k\int_0^1 (\varphi_x+\psi+lw)\psi\,\mathrm{d}x + k_0\int_0^1 (w_x - l\varphi)^2\,\mathrm{d}x + \mu_1\int_0^1 \varphi\varphi_t\,\mathrm{d}x + \mu_2\int_0^1 \varphi z(x,1,t)\,\mathrm{d}x.$$

The estimate (3.10) is established by using Young's and Poincaré's inequalities. □

Lemma 7. *Let* $(\varphi,\psi,w,\theta,q,z)$ *be a solution of* (2.3)–(2.5). *Then the functional*

$$(3.12)\qquad F_4(t) := \varrho_2\int_0^1 \psi\psi_t\,\mathrm{d}x$$

satisfies the estimate

$$(3.13)\quad F_4'(t) \leqslant \frac{b}{2}\int_0^1 \psi_x^2\,\mathrm{d}x + \varrho_2\int_0^1 \psi_t^2\,\mathrm{d}x + \frac{k^2}{b}\int_0^1 (\varphi_x+\psi+lw)^2\,\mathrm{d}x + c\int_0^1 \theta^2\,\mathrm{d}x.$$

P r o o f. Differentiating F_4 and using $(2.3)_3$, we arrive at

$$(3.14)\quad F_4'(t) = -b\int_0^1 \psi_x^2\,\mathrm{d}x + \varrho_2\int_0^1 \psi_t^2\,\mathrm{d}x + \gamma\int_0^1 \psi_x\theta\,\mathrm{d}x - k\int_0^1 (\varphi_x+\psi+lw)\,\mathrm{d}x.$$

Using Young's and Poincaré's inequalities, we obtain the estimate (3.13). □

Lemma 8. *Let* $(\varphi,\psi,w,\theta,q,z)$ *be a solution of* (2.3)–(2.5). *Then the functional*

$$(3.15)\qquad F_5(t) := -\varrho_1\int_0^1 \varphi_t(w_x-l\varphi)\,\mathrm{d}x - \varrho_1\int_0^1 w_t(\varphi_x+\psi+lw)\,\mathrm{d}x$$

satisfies, for any $\varepsilon_6,\varepsilon_7>0$, *the estimate*

$$(3.16)\qquad \begin{aligned} F_5'(t) \leqslant{}& -\Big(lk_0 - \frac{\mu_1}{4\varepsilon_6} - \frac{\mu_2}{4\varepsilon_7}\Big)\int_0^1 (w_x-l\varphi)^2\,\mathrm{d}x - \frac{l\varrho_1}{2}\int_0^1 w_t^2\,\mathrm{d}x \\ &+ (l\varrho_1+\varepsilon_6\mu_1)\int_0^1 \varphi_t^2\,\mathrm{d}x + c\int_0^1 \psi_t^2\,\mathrm{d}x \\ &+ lk\int_0^1 (\varphi_x+\psi+lw)^2\,\mathrm{d}x + \varepsilon_7\mu_2\int_0^1 z^2(x,1,t)\,\mathrm{d}x. \end{aligned}$$

P r o o f. By differentiating F_5, and using $(2.3)_1$ and $(2.3)_4$, we have

$$(3.17)\qquad \begin{aligned} F_5'(t) ={}& -lk_0\int_0^1 (w_x-l\varphi)^2\,\mathrm{d}x - l\varrho_1\int_0^1 w_t^2\,\mathrm{d}x + l\varrho_1\int_0^1 \varphi_t^2\,\mathrm{d}x \\ &+ lk\int_0^1 (\varphi_x+\psi+lw)^2\,\mathrm{d}x - \varrho_1\int_0^1 \psi_t w_t\,\mathrm{d}x \\ &+ \mu_1\int_0^1 \varphi_t(w_x-l\varphi)\,\mathrm{d}x + \mu_2\int_0^1 z(x,1,t)(w_x-l\varphi)\,\mathrm{d}x. \end{aligned}$$

Thanks to Young's inequality, we can easily obtain the estimate (3.16). □

Lemma 9. *Let* $(\varphi,\psi,w,\theta,q,z)$ *be a solution of* (2.3)–(2.5) *and let* $k=k_0$. *Then the functional*

$$(3.18)\quad F_6(t) := -\varrho_1\int_0^1 (w_x-l\varphi)\int_0^x w_t(y)\,\mathrm{d}y\,\mathrm{d}x - \varrho_1\int_0^1 \varphi_t\int_0^x (\varphi_x+\psi+lw)\,\mathrm{d}y\,\mathrm{d}x$$

satisfies the estimate

$$(3.19)\qquad \begin{aligned} F_6'(t) \leqslant{}& -\frac{\varrho_1}{2}\int_0^1 \varphi_t^2\,\mathrm{d}x - k_0\int_0^1 (w_x-l\varphi)^2\,\mathrm{d}x + \varrho_1\int_0^1 w_t^2\,\mathrm{d}x \\ &+ k\int_0^1 (\varphi_x+\psi+lw)^2\,\mathrm{d}x + \frac{\varrho_1}{2}\int_0^1 \psi_t^2\,\mathrm{d}x. \end{aligned}$$

P r o o f. A simple differentiation of F_6, using $(2.3)_1$, $(2.3)_4$ and integrating by parts lead to

$$
\begin{aligned}
(3.20) \qquad F_6'(t) = & -\varrho_1 \int_0^1 \varphi_t^2 \,\mathrm{d}x - k_0 \int_0^1 (w_x - l\varphi)^2 \,\mathrm{d}x + \varrho_1 \int_0^1 w_t^2 \,\mathrm{d}x \\
& - \varrho_1 \int_0^1 \varphi_t \int_0^x \psi_t(y) \,\mathrm{d}y + k \int_0^1 (\varphi_x + \psi + lw)^2 \,\mathrm{d}x \\
& + l(k - k_0) \int_0^1 (w_x - l\varphi) \int_0^x (\varphi_x + \psi + lw) \,\mathrm{d}y \,\mathrm{d}x.
\end{aligned}
$$

Using the fact that $k = k_0$, Young's and the Cauchy-Schwarz inequalities, we get (3.19). □

Lemma 10. *Let* $(\varphi, \psi, w, \theta, q, z)$ *be a solution of* (2.3)–(2.5) *and let* (1.4) *hold. Then the functional*

$$
\begin{aligned}
(3.21) \qquad F_7(t) := & \varrho_2 \int_0^1 \psi_t(\varphi_x + \psi + lw) \,\mathrm{d}x + \frac{b\varrho_1}{k} \int_0^1 \varphi_t \psi_x \,\mathrm{d}x \\
& + \frac{b\varrho_3}{\gamma}\Big(\frac{\varrho_1}{k} - \frac{\varrho_2}{b}\Big) \int_0^1 \theta \varphi_t \,\mathrm{d}x - \frac{bl^2\varrho_2}{k_0} \int_0^1 \psi\psi_t \,\mathrm{d}x \\
& - \frac{b}{\gamma}\Big(\frac{\varrho_1}{k} - \frac{\varrho_2}{b}\Big) \int_0^1 q(\varphi_x + \psi + lw) \,\mathrm{d}x + \frac{bl\varrho_1}{k_0} \int_0^1 \psi w_t \,\mathrm{d}x
\end{aligned}
$$

satisfies, for any $\varepsilon_4, \varepsilon_5 > 0$, *the estimate*

$$
\begin{aligned}
(3.22) \quad F_7'(t) \leqslant & -\Big(\frac{k}{2} - \frac{b\eta}{\gamma\alpha\varepsilon_{10}}\Big) \int_0^1 (\varphi_x + \psi + lw)^2 \,\mathrm{d}x + \varepsilon_8 \int_0^1 w_t^2 \,\mathrm{d}x \\
& + \frac{b^2l^2}{k} \int_0^1 \psi_x^2 \,\mathrm{d}x + \varepsilon_9 \int_0^1 (w_x - l\varphi)^2 \,\mathrm{d}x + c\Big(1 + \frac{1}{\varepsilon_8}\Big) \int_0^1 \psi_t^2 \,\mathrm{d}x \\
& + c\Big(1 + \frac{1}{\varepsilon_8}\Big) \int_0^1 q^2 \,\mathrm{d}x + c\Big(1 + \frac{1}{\varepsilon_9}\Big) \int_0^1 \theta^2 \,\mathrm{d}x + \frac{b\eta}{\gamma\alpha}\varepsilon_{10} \int_0^1 \theta_x^2 \,\mathrm{d}x.
\end{aligned}
$$

P r o o f. By differentiating F_7, we have

$$
\begin{aligned}
(3.23) \quad F_7'(t) = & \varrho_2 \int_0^1 \psi_{tt}(\varphi_x + \psi + lw) \,\mathrm{d}x + \varrho_2 \int_0^1 \psi_t(\varphi_x + \psi + lw)_t \,\mathrm{d}x \\
& + \frac{b\varrho_1}{k} \int_0^1 \psi_{tt}\psi_x \,\mathrm{d}x - \frac{b\varrho_1}{k} \int_0^1 \psi_t \psi_{xt} \,\mathrm{d}x - \frac{bl^2\varrho_2}{k_0} \int_0^1 \psi_t^2 \,\mathrm{d}x \\
& - \frac{bl^2\varrho_2}{k_0} \int_0^1 \psi_{tt}\psi \,\mathrm{d}x + \frac{bl\varrho_1}{k_0} \int_0^1 w_{tt}\psi \,\mathrm{d}x + \frac{bl\varrho_1}{k_0} \int_0^1 w_t\psi \,\mathrm{d}x \\
& + \frac{b\varrho_3}{\gamma}\Big(\frac{\varrho_1}{k} - \frac{\varrho_2}{b}\Big) \int_0^1 \theta_t \varphi_t \,\mathrm{d}x + \frac{b\varrho_3}{\gamma}\Big(\frac{\varrho_1}{k} - \frac{\varrho_2}{b}\Big) \int_0^1 \theta \varphi_{tt} \,\mathrm{d}x
\end{aligned}
$$

$$-\frac{b}{\gamma}\Big(\frac{\varrho_1}{k}-\frac{\varrho_2}{b}\Big)\int_0^1 q_t(\varphi_x+\psi+lw)\,\mathrm{d}x$$
$$-\frac{b}{\gamma}\Big(\frac{\varrho_1}{k}-\frac{\varrho_2}{b}\Big)\int_0^1 q(\varphi_x+\psi+lw)_t\,\mathrm{d}x.$$

Now, by using the equations in (2.3) and integrating by parts,

$$\begin{aligned}
\varrho_2\int_0^1 \psi_{tt}(\varphi_x+\psi+lw)\,\mathrm{d}x
&= -k\int_0^1(\varphi_x+\psi+lw)^2\,\mathrm{d}x-\gamma\int_0^1\theta_x(\varphi_x+\psi+lw)\,\mathrm{d}x\\
&\quad -b\int_0^1\psi_x(\varphi_x+\psi+lw)_x\,\mathrm{d}x,
\end{aligned}\tag{3.24}$$

$$\varrho_1\int_0^1\varphi_{tt}\psi_x\,\mathrm{d}x=k\int_0^1\psi_x(\varphi_x+\psi+lw)_x\,\mathrm{d}x+k_0l\int_0^1(w_x-l\varphi)\,\mathrm{d}x,$$
$$\varrho_3\int_0^1\theta_t\varphi_t\,\mathrm{d}x=\int_0^1 q\varphi_{xt}\,\mathrm{d}x+\gamma\int_0^1\psi_t\varphi_{xt}\,\mathrm{d}x,$$
$$\int_0^1\theta\varphi_{tt}\,\mathrm{d}x=-\frac{k}{\varrho_1}\int_0^1\theta_x(\varphi_x+\psi+lw)\,\mathrm{d}x+\frac{lk_0}{\varrho_1}\int_0^1\theta(w_x-l\varphi)\,\mathrm{d}x,$$
$$-\int_0^1 q_t(\varphi_x+\psi+lw)\,\mathrm{d}x=\frac{\beta}{\alpha}\int_0^1 q(\varphi_x+\psi+lw)\,\mathrm{d}x+\frac{1}{\alpha}\int_0^1\theta_x(\varphi_x+\psi+lw)\,\mathrm{d}x,$$
$$-\varrho_2\int_0^1\psi_{tt}\psi\,\mathrm{d}x=b\int_0^1\psi_x^2\,\mathrm{d}x+k\int_0^1\psi(\varphi_x+\psi+lw)\,\mathrm{d}x-\gamma\int_0^1\theta\psi_x\,\mathrm{d}x,$$
$$\varrho_1\int_0^1 w_{tt}\psi\,\mathrm{d}x=-k_0\int_0^1\psi_x(w_x-l\varphi)\,\mathrm{d}x-kl\int_0^1\psi(\varphi_x+\psi+lw)\,\mathrm{d}x.\tag{3.25}$$

Substituting (3.24)–(3.25) into (3.23) and bearing in mind (1.4), we obtain

$$\begin{aligned}
F_7'(t)=&-k\int_0^1(\varphi_x+\psi+lw)^2\,\mathrm{d}x+\Big(\varrho_2-\frac{bl^2\varrho_2}{k_0}\Big)\int_0^1\psi_t^2\,\mathrm{d}x\\
&+\Big(l\varrho_2+\frac{bl\varrho_1}{k_0}\Big)\int_0^1\psi_tw_t\,\mathrm{d}x+\frac{b\eta}{\alpha\gamma}\int_0^1\theta_x(\varphi_x+\psi+lw)\,\mathrm{d}x\\
&-\frac{b}{\gamma}\Big(\frac{\varrho_1}{k}-\frac{\varrho_2}{b}\Big)\int_0^1 q\psi_t\,\mathrm{d}x-\frac{bl}{\gamma}\Big(\frac{\varrho_1}{k}-\frac{\varrho_2}{b}\Big)\int_0^1 qw_t\,\mathrm{d}x\\
&+\frac{blk_0\varrho_3}{\gamma\varrho_1}\Big(\frac{\varrho_1}{k}-\frac{\varrho_2}{b}\Big)\int_0^1\theta(w_x-l\varphi)\,\mathrm{d}x\\
&+\frac{b\beta}{\alpha\gamma}\Big(\frac{\varrho_1}{k}-\frac{\varrho_2}{b}\Big)\int_0^1 q(\varphi_x+\psi+lw)\,\mathrm{d}x+\frac{b^2l^2}{k_0}\int_0^1\psi_x^2\,\mathrm{d}x\\
&-\frac{\gamma bl^2}{k_0}\int_0^1\theta\psi_x\,\mathrm{d}x+bl\Big(\frac{k_0}{k}-1\Big)\int_0^1\psi_x(w_x-l\varphi)\,\mathrm{d}x.
\end{aligned}\tag{3.26}$$

The estimate (3.22) follows by using Young's inequality and the fact that $k=k_0$. □

Lemma 11. *Let* $(\varphi,\psi,w,\theta,q,z)$ *be a solution of* (2.3)–(2.5)*. Then the functional*

$$F_8(t) := \int_0^1 \varrho_1\varphi_t\varphi\,\mathrm{d}x + \frac{\mu_1}{2}\int_0^1 \varphi^2\,\mathrm{d}x \tag{3.27}$$

satisfies, for any $\varepsilon_{11} > 0$, *the estimate*

$$\begin{aligned} F_8'(t) \leqslant \Big(-K + \varepsilon_{11}\Big(\frac{K}{2} + \frac{\mu_2 c}{2}\Big)\Big)\int_0^1 \varphi_x^2\,\mathrm{d}x + \frac{K}{2\varepsilon_{11}}\int_0^1 \psi_x^2\,\mathrm{d}x \\ + \frac{\mu_2}{2\varepsilon_{11}}\int_0^1 z^2(x,1,t)\,\mathrm{d}x + \varrho_1\int_0^1 \varphi_t^2\,\mathrm{d}x, \end{aligned} \tag{3.28}$$

where $c = \pi^{-2}$ *is the Poincaré constant.*

P r o o f. Differentiating (3.27), we obtain

$$F_8'(t) = \varrho_1\int_0^1 \varphi_{tt}\varphi\,\mathrm{d}x + \varrho_1\int_0^1 \varphi_t^2\,\mathrm{d}x + \mu_1\int_0^1 \varphi_t\varphi\,\mathrm{d}x. \tag{3.29}$$

Then, by using $(2.3)_1$, we find

$$F_8'(t) = k\int_0^1 (\varphi_x + \psi + lw)_x\varphi\,\mathrm{d}x - \mu_2\int_0^1 \varphi z(x,1,t)\,\mathrm{d}x + \varrho_1\int_0^1 \varphi_t^2\,\mathrm{d}x. \tag{3.30}$$

Consequently,

$$F_8'(t) = -k\int_0^1 (\varphi_x + \psi + lw)\varphi_x\,\mathrm{d}x - \mu_2\int_0^1 \varphi z(x,1,t)\,\mathrm{d}x + \varrho_1\int_0^1 \varphi_t^2\,\mathrm{d}x. \tag{3.31}$$

The estimate (3.27) follows by applying Young's and Poincaré's inequalities. □

Lemma 12. *Let* $(\varphi,\psi,w,\theta,q,z)$ *be a solution of* (2.3)–(2.5)*. Then, we introduce the functional*

$$F_9(t) := \int_0^1\int_0^1 \mathrm{e}^{-2\tau\varrho} z^2(x,\varrho,t)\,\mathrm{d}\varrho\,\mathrm{d}x, \tag{3.32}$$

that satisfies

$$F_9'(t) \leqslant -F_9(t) - \frac{c_1}{2\tau}\int_0^1 z^2(x,1,t)\,\mathrm{d}x + \frac{1}{2\tau}\int_0^1 \psi_t^2(x,t)\,\mathrm{d}x, \tag{3.33}$$

where c_1 *is a positive constant.*

P r o o f. Differentiating (3.32) and using the equation (2.2), we have

$$(3.34)\quad \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^1\int_0^1 \mathrm{e}^{-2\tau\varrho}z^2(x,\varrho,t)\,\mathrm{d}\varrho\,\mathrm{d}x\Big) = -\frac{1}{\tau}\int_0^1\int_0^1 \mathrm{e}^{-2\tau\varrho}zz_\varrho(x,\varrho,t)\,\mathrm{d}\varrho\,\mathrm{d}x$$
$$= -\int_0^1\int_0^1 \mathrm{e}^{-2\tau\varrho}z^2(x,\varrho,t)\,\mathrm{d}\varrho\,\mathrm{d}x - \frac{1}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\varrho}(\mathrm{e}^{-2\tau\varrho}z^2(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x.$$

The above estimate implies that there exists a positive constant c_1 such that (3.33) holds. □

Now, we are ready to state and prove the main result of this section.

Theorem 2. *Let* $(\varphi,\psi,w,\theta,q,z)$ *be a solution of* (2.3)–(2.5) *and assume that* $\eta=0$, $k=k_0$. *Then the energy functional* (3.1) *satisfies*

$$(3.35)\qquad E(t)\leqslant \lambda_0\mathrm{e}^{-\lambda_1 t},\quad t\geqslant 0,$$

where the positive constant λ_0 *directly depends on the initial data and the uniform constant* λ_1 *depends only on the coefficients of the system.*

P r o o f. For $N, N_i>0$, let

$$(3.36)\qquad \mathcal{L}(t):=NE(t)+\sum_{i=1}^{i=9}N_iF_i(t).$$

By taking the derivative of $\mathcal{L}$, we get

$$(3.37)\quad \mathcal{L}'(t)\leqslant -\beta N\int_0^1 q^2\,\mathrm{d}x - N\Big(\mu_1-\frac{\xi}{2}-\frac{|\mu_2|}{2}\Big)\int_0^1\varphi_t^2\,\mathrm{d}x$$
$$- N\Big(\frac{\xi}{2}-\frac{|\mu_2|}{2}\Big)\int_0^1 z^2(x,1,t)\,\mathrm{d}x - \frac{N_1\varrho_3}{2}\int_0^1\theta^2\,\mathrm{d}x$$
$$+\varepsilon_1N_1\int_0^1\psi_t^2\,\mathrm{d}x + cN_1\Big(1+\frac{1}{\varepsilon_1}\Big)\int_0^1 q^2\,\mathrm{d}x - N_2\frac{\varrho_2}{\gamma}\int_0^1\psi_t^2\,\mathrm{d}x$$
$$+\varepsilon_2N_2\int_0^1(\varphi_x+\psi+lw)^2\,\mathrm{d}x+\varepsilon_3N_2\int_0^1\psi_x^2\,\mathrm{d}x + cN_2\int_0^1 q^2\,\mathrm{d}x$$
$$+cN_2\Big(1+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}\Big)\int_0^1\theta^2\,\mathrm{d}x - \Big(\varrho_1-\frac{1}{4\varepsilon_4}\Big)N_3\int_0^1\varphi_t^2\,\mathrm{d}x$$
$$-\varrho_1N_3\int_0^1 w_t^2\,\mathrm{d}x + cN_3\int_0^1\psi_x^2\,\mathrm{d}x + k_0N_3\int_0^1(w_x-l\varphi)^2\,\mathrm{d}x$$
$$+cN_3\int_0^1(\varphi_x+\psi+lw)^2\,\mathrm{d}x + \frac{\mu_2}{4\varepsilon_5}N_3\int_0^1 z^2(x,1,t)\,\mathrm{d}x$$

$$
\begin{aligned}
&+ (\varepsilon_5\mu_2 + \varepsilon_4\mu_1)N_3 \int_0^1 \varphi^2\,\mathrm{d}x + \frac{b}{2}N_4 \int_0^1 \psi_x^2\,\mathrm{d}x + \varrho_2 N_4 \int_0^1 \psi_t^2\,\mathrm{d}x\\
&+ \frac{k^2}{b}N_4 \int_0^1 (\varphi_x + \psi + lw)^2\,\mathrm{d}x + cN_4 \int_0^1 \theta^2\,\mathrm{d}x - \frac{l\varrho_1}{2}N_5 \int_0^1 w_t^2\,\mathrm{d}x\\
&- l\Big(lk_0 - \frac{\mu_1}{4\varepsilon_6} - \frac{\mu_2}{4\varepsilon_7}\Big)N_5 \int_0^1 (w_x - l\varphi)^2\,\mathrm{d}x + cN_5 \int_0^1 \psi_t^2\,\mathrm{d}x\\
&+ (l\varrho_1 + \varepsilon_6\mu_1)N_5 \int_0^1 \varphi_t^2\,\mathrm{d}x + lkN_5 \int_0^1 (\varphi_x + \psi + lw)^2\,\mathrm{d}x\\
&+ \varepsilon_7\mu_2 N_5 \int_0^1 z^2(x,1,t)\,\mathrm{d}x - \frac{\varrho_1}{2}N_6 \int_0^1 \varphi_t^2\,\mathrm{d}x + \varrho_1 N_6 \int_0^1 w_t^2\,\mathrm{d}x\\
&- k_0 N_6 \int_0^1 (w_x - l\varphi)^2\,\mathrm{d}x + kN_6 \int_0^1 (\varphi_x + \psi + lw)^2\,\mathrm{d}x\\
&+ \frac{\varrho_1}{2}N_6 \int_0^1 \psi_t^2\,\mathrm{d}x - \Big(\frac{k}{2} - \frac{b\eta}{\alpha\gamma\varepsilon_{10}}\Big)N_7 \int_0^1 (\varphi_x + \psi + lw)^2\,\mathrm{d}x\\
&+ \varepsilon_8 N_7 \int_0^1 w_t^2\,\mathrm{d}x + \frac{b^2 l^2}{k}N_7 \int_0^1 \psi_x^2\,\mathrm{d}x + \varepsilon_9 N_7 \int_0^1 (w_x - l\varphi)^2\,\mathrm{d}x\\
&+ c\Big(1 + \frac{1}{\varepsilon_8}\Big)N_7 \int_0^1 \psi_t^2\,\mathrm{d}x + c\Big(1 + \frac{1}{\varepsilon_8}\Big)N_7 \int_0^1 q^2\,\mathrm{d}x\\
&+ c\Big(1 + \frac{1}{\varepsilon_9}\Big)N_7 \int_0^1 \theta^2\,\mathrm{d}x + \frac{b\varepsilon_{10}}{\gamma\alpha}\eta N_7 \int_0^1 \theta_x^2\,\mathrm{d}x\\
&+ N_8\Big(-K + \varepsilon_{11}\Big(\frac{K}{2} + \frac{\mu_2 c}{2}\Big)\Big) \int_0^1 \varphi_x^2\,\mathrm{d}x + \frac{K}{2\varepsilon_{11}}N_8 \int_0^1 \psi_x^2\,\mathrm{d}x\\
&+ \frac{\mu_2}{2\varepsilon_{11}}N_8 \int_0^1 z^2(x,1,t)\,\mathrm{d}x + \varrho_1 N_8 \int_0^1 \varphi_t^2\,\mathrm{d}x - N_9 F_9(t)\\
&- \frac{c_1}{2\tau}N_9 \int_0^1 z^2(x,1,t)\,\mathrm{d}x + \frac{1}{2\tau}N_9 \int_0^1 \psi_t^2(x,t)\,\mathrm{d}x.
\end{aligned}
$$

Direct computations and the use of (3.2), (3.6), (3.10), (3.13), (3.16), (3.19), (3.22), (3.28), and (3.33) give

$$
\begin{aligned}
\mathcal{L}'(t) \leqslant{}& \Big(-\beta N + c_1\Big(1 + \frac{1}{\varepsilon_1}\Big) + cN_2 + c\Big(1 + \frac{1}{\varepsilon_8}\Big)N_7\Big) \int_0^1 q^2\,\mathrm{d}x\\
&- N\Big(\mu_1 + \frac{\xi}{2} - \frac{|\mu_2|}{2}\Big) \int_0^1 \varphi_t^2\,\mathrm{d}x - N\Big(\frac{\xi}{2} - \frac{|\mu_2|}{2}\Big) \int_0^1 z^2(x,1,t)\,\mathrm{d}x\\
&- \Big(\frac{N_1\varrho_3}{2} - cN_2\Big(1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\Big) - cN_4 - c\Big(1 + \frac{1}{\varepsilon_9}\Big)N_7\Big) \int_0^1 \theta^2\,\mathrm{d}x\\
&+ \Big(\varepsilon_1 N_1 - N_2\frac{\varrho_2}{\gamma} + \varrho_2 N_4 + cN_5 + \frac{\varrho_1}{2}N_6\\
&+ c\Big(1 + \frac{1}{\varepsilon_8}\Big)N_7 + \varrho_1 N_8 + \frac{1}{2\tau}N_9\Big) \int_0^1 \psi_t^2\,\mathrm{d}x
\end{aligned}
\tag{3.38}
$$

$$
\begin{aligned}
&+\Big(\varepsilon_2 N_2 + cN_3 + \frac{k^2}{b}N_4 + lkN_5 + kN_6 - \Big(\frac{k}{2} - \frac{b\eta}{\alpha\gamma\varepsilon_{10}}\Big)N_7\Big) \\
&\times \int_0^1 (\varphi_x + lw + \psi)^2 \,\mathrm{d}x \\
&+\Big(\varepsilon_3 N_2 + cN_3 + \frac{b}{2}N_4 + \frac{b^2 l^2}{k}N_7 + \frac{k}{2\varepsilon_{11}}N_8 - kN_8 \\
&+\varepsilon_{11}\Big(\frac{k}{2} + \frac{\mu_2 c}{2}\Big)N_8\Big)\int_0^1 \psi_x^2 \,\mathrm{d}x \\
&+\Big(-\varrho_1 N_3 - \frac{l\varrho_1}{2}N_5 + \varrho_1 N_6 + \varepsilon_8 N_7\Big)\int_0^1 w_t^2 \,\mathrm{d}x \\
&+\Big(k_0 N_3 - \Big(lk_0 - \frac{\mu_1}{4\varepsilon_6} - \frac{\mu_2}{4\varepsilon_7}\Big)N_5 - k_0 N_6 + \varepsilon_9 N_7\Big) \\
&\times \int_0^1 (w_x - l\varphi)^2 \,\mathrm{d}x \\
&+\Big(\frac{\mu_2}{4\varepsilon_5}N_3 + \varepsilon_7\mu_2 N_5 + \frac{\mu_2}{2\varepsilon_{11}}N_8 - \frac{c_1}{2\tau}N_9\Big)\int_0^1 z^2(x,1,t)\,\mathrm{d}x \\
&+\Big(\varepsilon_5\mu_2 + \varepsilon_4\mu_1\Big)N_3 \int_0^1 \varphi^2 \,\mathrm{d}x \\
&-\Big(\Big(1 - \frac{1}{4\varepsilon_4}\Big)N_3 - (l\varrho_1 + \varepsilon_6\mu_1)N_5 + \frac{\varrho_1}{2}N_6 - \varrho_1 N_8\Big)\int_0^1 \varphi_t^2 \,\mathrm{d}x \\
&+\Big(\Big(-k + \varepsilon_{11}\Big(\frac{k}{2} + \frac{\mu_2 c}{2}\Big)\Big)N_8\Big)\int_0^1 \varphi_x^2 \,\mathrm{d}x \\
&+\Big(\frac{b\eta\varepsilon_{10}}{\alpha\gamma}N_7\Big)\int_0^1 \theta_x^2 \,\mathrm{d}x - N_9 F_9(t).
\end{aligned}
$$

At this point, we have to choose our constants very carefully. First, we choose ε_i, $i = 1, \ldots, 10$, small enough such that

$$\varepsilon_1 \leqslant \frac{1}{N_1}\Big(N_2\frac{\varrho_2}{\gamma} + \varrho_2 N_4 + cN_5 + \frac{\varrho_1}{2}N_6\Big).$$

After that, we can choose N large enough such that

$$N \geqslant \frac{1}{\beta}\Big(c_1\Big(1 + \frac{1}{\varepsilon_1}\Big) + cN_2 + c\Big(1 + \frac{1}{\varepsilon_8}\Big)N_7\Big).$$

Moreover, we pick N_9 large enough so that

$$\frac{\mu_2}{4\varepsilon_5}N_3 + \varepsilon_7\mu_2 N_5 + \frac{\mu_2}{2\varepsilon_{11}}N_8 - \frac{c_1}{2\tau}N_9 < 0$$

and

$$N_9 \geqslant \frac{c_1}{2\tau}\Big(\frac{\mu_2}{4\varepsilon_5}N_3 + \varepsilon_7\mu_2 N_5 + \frac{\mu_2}{2\varepsilon_{11}}N_8\Big),$$

and we take ε_{11} small enough such that

$$\varepsilon_{11} \leqslant \frac{k}{N_8}\Big(\frac{k}{2} + \frac{\mu_2 c}{2}\Big)^{-1}.$$

Next, let N_5 be large enough such that

$$\frac{N_5 \varrho_3 \kappa}{4} \geqslant N_4\Big(\gamma\varrho_3 + \frac{\varrho_3}{2\varepsilon_4'}(b + 2\kappa)\Big).$$

Consequently, there exists a positive constant η_1 such that (3.37) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \leqslant & -\eta_1 \int_0^1 (\psi_t^2 + \psi_x^2 + \varphi_t^2 + (\varphi_x + lw + \psi)^2 + \theta^2 + q^2)\,\mathrm{d}x \\ & -\eta_1 \int_0^1 \int_0^1 z^2(x,\varrho,t)\,\mathrm{d}\varrho\,\mathrm{d}x,\end{aligned} \tag{3.39}$$

which implies by (3.1) that there exists also η_2, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \leqslant -\eta_2 E(t) \quad \forall\, t \geqslant 0. \tag{3.40}$$

Consequently, there exists a positive constant η_1 such that (3.38) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \leqslant & -\eta_1 \int_0^1 (\psi_t^2 + \psi_x^2 + \varphi_t^2 + (\varphi_x + lw + \psi)^2 + \theta^2 + q^2)\,\mathrm{d}x \\ & -\eta_1 \int_0^1 \int_0^1 z^2(x,\varrho,t)\,\mathrm{d}\varrho\,\mathrm{d}x,\end{aligned} \tag{3.41}$$

which implies by (3.1) that there exists also η_2, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \leqslant -\eta_2 E(t) \quad \forall\, t \geqslant 0. \tag{3.42}$$

Moreover, we have the following lemma. □

Lemma 13. *For N large enough, there exist two positive constants β_1 and β_2 depending on N_i, $i = 1, \ldots, 9$, and ε_i, $i = 1, \ldots, 11$, such that*

$$\beta_1 E(t) \leqslant \mathcal{L}(t) \leqslant \beta_2 E(t) \quad \forall\, t \geqslant 0. \tag{3.43}$$

P r o o f. We consider the functional

$$H(t) = \sum_{i=1}^{i=9} N_i F_i(t)$$

and show that

$$|H(t)| \leqslant C E(t), \quad C > 0.$$

From (3.3), (3.6), (3.9), (3.12), (3.15), (3.18), (3.21), (3.27) and (3.32) we obtain

$$
\begin{aligned}
|H(t)| \leqslant N_1\Big|\alpha\varrho_3\int_0^1\theta\int_0^x q(y)\,\mathrm{d}y\,\mathrm{d}x\Big| &+ N_2\Big|-\frac{\varrho_2\varrho_3}{\gamma}\int_0^1\theta\,\mathrm{d}x\int_0^x\psi_t(y)\,\mathrm{d}y\,\mathrm{d}x\Big|\\
&+ N_3\Big|\varrho_1\int_0^1(\varphi\varphi_t+ww_t)\,\mathrm{d}x\Big| + N_4\Big|\varrho_2\int_0^1\int_0^x\psi\psi_t(t,x)\,\mathrm{d}x\Big|\\
&+ N_5\Big|-\varrho_1\int_0^1\varphi_t(w_x-l\varphi)\,\mathrm{d}x-\varrho_1\int_0^1 w_t(\varphi_x+\psi+lw)\,\mathrm{d}x\Big|\\
&+ N_6\Big|-\varrho_1\int_0^1(w_x-l\varphi)\int_0^x w_t(y)\,\mathrm{d}y\,\mathrm{d}x-\varrho_1\int_0^1\varphi_t\int_0^x(\varphi_x+\psi+lw)\,\mathrm{d}y\,\mathrm{d}x\Big|\\
&+ N_7\Big|\varrho_2\int_0^1(\varphi_x+\psi+lw)\,\mathrm{d}x+\frac{b\varrho_1}{k}\int_0^1\varphi_t\psi_x\,\mathrm{d}x\Big|\\
&+ N_8\Big|\int_0^1\varrho_1\varphi\varphi_t\,\mathrm{d}x+\frac{\mu_1}{2}\int_0^1\varphi^2\,\mathrm{d}x\Big| + N_9\int_0^1\int_0^1\mathrm{e}^{-2\tau\varrho}z^2(x,\varrho,t)\,\mathrm{d}\varrho\,\mathrm{d}x.
\end{aligned}
$$

Using the trivial relation

$$
\int_0^1(\varphi+lw)^2\,\mathrm{d}x \leqslant 2c\int_0^1(\varphi_x+lw+\psi)^2\,\mathrm{d}x+2c\int_0^1\psi_x^2\,\mathrm{d}x,
$$

and Young's and Poincaré's inequalities, we get

$$
\begin{aligned}
(3.44)\quad |H(t)| \leqslant \alpha_1\int_0^1\varphi_t^2\,\mathrm{d}x &+ \alpha_2\int_0^1\psi_t^2\,\mathrm{d}x+\alpha_3\int_0^1 w_t^2+\alpha_4\int_0^1\psi_x^2+\alpha_5\int_0^1\theta^2\,\mathrm{d}x\\
&+\alpha_6\int_0^1 q^2\,\mathrm{d}x+\alpha_7\int_0^1((\varphi_x+lw+\psi)^2+(w_x-l\varphi)^2)\,\mathrm{d}x\\
&+\int_0^1\int_0^1 z^2(x,\varrho,t)\,\mathrm{d}\varrho\,\mathrm{d}x,
\end{aligned}
$$

where the positive constants $\alpha_1,\ldots,\alpha_6$ are determined as

$$
\begin{gathered}
\alpha_1 := \frac{1}{2}(N_3\varrho_1+N_8\varrho_1),\quad \alpha_2 := \frac{1}{2}\Big(N_4\varrho_2+N_2\frac{\varrho_2\varrho_3}{\gamma}\Big),\\
\alpha_3 = \frac{1}{2}(N_3\varrho_1+N_6\varrho_1),\quad \alpha_4 := \frac{b\varrho_1}{2k},\quad \alpha_5 := \frac{1}{2}\Big(N_1\varrho_3+\frac{\varrho_2\varrho_3}{\gamma}\Big),\\
\alpha_6 := \frac{1}{2}(N_1\varrho_3+N_5\tau_0\varrho_3),\quad \alpha_7 := \frac{1}{2}(N_7\varrho_2+3\varrho_1).
\end{gathered}
$$

According to (3.44), we have

$$
|H(t)| \leqslant \widehat{C}E(t)\quad\text{for } \widehat{C} = \frac{\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}}{\min\{\varrho_1,\varrho_2,\varrho_3,k,b,\kappa,\gamma,\delta,\tau_0\}}.
$$

Therefore we obtain $|\mathcal{L}(t)-NE(t)| \leqslant \widehat{C}E(t)$. So, we can choose N large enough so that $\beta_1 = N-\widehat{C} > 0$. Then (3.43) holds true for $\beta_2 = N+\widehat{C} > 0$ and this concludes the proof of Lemma 13. □

Combining now (3.42) and (3.43), we conclude that there exists some $\Lambda > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \leqslant -\Lambda\mathcal{L}(t) \quad \forall\, t \geqslant 0. \tag{3.45}$$

A simple integration of (3.45) leads to

$$\mathcal{L}(t) \leqslant \mathcal{L}(0)\mathrm{e}^{-\Lambda t} \quad \forall\, t \geqslant 0. \tag{3.46}$$

Again, the use of (3.43) and (3.46) yields the desired result (3.35). This completes the proof of Theorem 2. □

Remark 1. At the end of these studies, we remark that one can propose a more interesting subject which is to consider the case of different wave speeds and analyze the decay rate. The major question will be: Does the system decay exponentially? If not, what is the expected decay? And how about its rate of decay? This subject may be our next work.

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