EXISTENCE RESULTS FOR SOME NONLINEAR PARABOLIC EQUATIONS WITH DEGENERATE COERCIVITY AND SINGULAR LOWER-ORDER TERMS

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Abstract. In this paper, we study the existence results for some parabolic equations with degenerate coercivity, singular lower order term depending on the gradient, and positive initial data in L^1.

Keywords: singular equation; nonlinear parabolic equation; degenerate coercivity

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1. Introduction

We study the existence and regularity results for the following parabolic problem:

$$\text{(P)} \qquad \begin{cases} \partial_t u + Lu + B\dfrac{|\nabla u|^p}{u^\theta} = u^r & \text{in } Q_T = (0,T)\times\Omega,\\ u(0,x) = u_0(x) \geqslant 0 & \text{in } \Omega,\\ u = 0 & \text{on } \Gamma_T = (0,T)\times\partial\Omega, \end{cases}$$

where $T > 0$, $B > 0$ are real numbers, $u_0 \in L^1(\Omega)$, Ω is a bounded open subset of $\mathbb{R}^N$ ($N > 2$) with boundary denoted by $\partial\Omega$, p is a real number such that $p \geqslant 2$ and L is the operator given by

$$Lu = -\operatorname{div}(A(t,x,u)|\nabla u|^{p-2}\nabla u).$$

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Here, we suppose that $A\colon (0,T)\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and for almost every $(t,x)\in(0,T)\times\Omega$, for all $s\in\mathbb{R}$ satisfies

$$\frac{\beta}{(a(t,x)+|s|)^{\varrho}}\leqslant A(t,x,s)\leqslant\alpha, \tag{1.1}$$

where α, β are strictly positive real numbers and $\varrho\geqslant 0$, $a\colon (0,T)\times\mathbb{R}^N\to\mathbb{R}$ is a measurable nonnegative function verifying

$$a(t,x)\leqslant\delta, \tag{1.2}$$

where δ is a strictly positive real number. We furthermore suppose that

$$0<\theta<1,\quad 0<r<p-\theta. \tag{1.3}$$

If (1.1) holds true, the differential operator L is not coercive when u is large. Moreover, the lower order term is singular as u tends to zero. We overcome these two difficulties by approximation of (P) by a sequence of nondegenerate and nonsingular problems (in the case $u_0\in L^\infty(\Omega)$), and passing to the limit in the approximate problems we prove that (P) admits at least one solution $u\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(\Omega)$. In the case $u_0\in L^1(\Omega)$ we approach u_0 by $u_{0n}\in L^\infty(\Omega)$ and we use the results of the first case to achieve the passage to the limit in the approximate problems by proving the existence of the solution $u\in L^q(0,T;W_0^{1,q}(\Omega))$, $q=p-\theta N/(N+1)$.

For the case $p=2$, $\varrho=0$ and for positive initial data, the existence of solutions to problem (P) is proved in [3] under the operator $v\to-\operatorname{div}(M(t,x)\nabla v)$, where $M\colon Q_T\to\mathbb{R}^{N^2}$ is a measurable bounded and uniformly elliptic matrix.

If the nonlinear right-hand term is not present, i.e., in the evolutive case, problems as

$$\text{(P1)}\qquad \begin{cases} \partial_t u-\Delta_p u+B\dfrac{|\nabla u|^p}{u^\theta}=f & \text{in } Q_T=(0,T)\times\Omega,\\ u(0,x)=u_0(x) & \text{in } \Omega,\\ u=0 & \text{on } \Gamma_T=(0,T)\times\partial\Omega, \end{cases}$$

under various assumptions on the summability of the source f, have been considered in the case $p=2$ and $\theta<1$ in [8]. If $\theta=1$, the existence solution has been considered in [12] and [13] for smooth strictly positive data, while degenerate problems were studied in [15] in the one dimensional case and $p>2$.

Let us also mention that in [4] the authors proved the existence and nonexistence of solutions for a general class of singular homogeneous (i.e., $f\equiv 0$) parabolic problems as (P1) with $p\geqslant 2$.

In [11], the author showed the existence of positive solutions of elliptic equations with degenerate coercivity and singular quadratic lower-order terms

$$-\operatorname{div}(M(x,u)\nabla u) + b(x)\frac{|\nabla u|^2}{u^\theta} = u^r + f, \quad f \in L^1(\Omega).$$

The aim of this paper is to extend the results in [3] to the case of degenerate parabolic equations with $p \geqslant 2$ and establish the existence of weak solutions of problem (P) for nonnegative initial data $u_0 \in L^1(\Omega)$.

This paper is organized as follows. In Section 2, we define the weak solution and prove the existence of weak solutions u for the first case $u_0 \in L^\infty(\Omega)$. Section 3 is devoted to the study of (P) with an initial datum $u_0 \in L^1(\Omega)$. We give a better regularity result compared to [9] because if $\theta \in (0,1)$, we have

$$1 - \frac{\theta N}{N+1} > 1 - \frac{N}{N+1},$$

so Theorem 3.1 improves (see Theorem 1, [9]).

2. Bounded initial data ($u_0 \in L^\infty(\Omega)$)

In this section, we prove that there exists a weak solution of problem (P) for u_0 bounded. For this, we use the result in [1] and then an L^∞-estimate procedure introduced by [5]. Given a real positive number k, we define the functions

$$T_k(r) = \begin{cases} k & \text{if } r \geqslant k, \\ r & \text{if } |r| < k, \\ -k & \text{if } r \leqslant -k, \end{cases} \qquad r \in \mathbb{R}.$$

Its primitive $\Theta_k\colon \mathbb{R} \to \mathbb{R}^+$ is defined by

$$\Theta_k(r) = \int_0^r T_k(t)\,\mathrm{d}t = \begin{cases} \dfrac{r^2}{2} & \text{if } |r| \leqslant k, \\[2mm] k|r| - \dfrac{k^2}{2} & \text{if } |r| > k. \end{cases}$$

We then use the following results:

$$\int_0^T \langle \partial_t v, T_k(v)\rangle\,\mathrm{d}t = \int_\Omega \Theta_k(v(T))\,\mathrm{d}x - \int_\Omega \Theta_k(v(0))\,\mathrm{d}x \tag{2.1}$$

and

$$k|r| - \frac{k^2}{2} \leqslant \Theta_k(r) \leqslant k|r| \quad \forall\, r \in \mathbb{R}. \tag{2.2}$$

Definition 2.1. A function u is a weak solution of problem (P) if $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ such that $\partial_t u \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q_T)$, $u > 0$, $|\nabla u|^p/u^\theta$ belongs to $L^1(Q_T)$, and

$$\begin{aligned}\int_\Omega u(T)\varphi(T)\,\mathrm{d}x - \int_\Omega u(0)\varphi(0)\,\mathrm{d}x + \int_{Q_T} u\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{Q_T} A(t,x,u)|\nabla u|^{p-2}\nabla u\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\ + B\int_{Q_T}\frac{|\nabla u|^p}{u^\theta}\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} u^r\varphi\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for every $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ such that $\partial_t\varphi \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q_T)$.

Remark 2.2. Notice that because of the fact that $u \in C([0,T];L^2(\Omega))$ (see [7]), the functions $\varphi(T)$ and $\varphi(0)$ in the above definition have sense and the meaning of the initial condition $u(0) = u_0$ is clear.

Theorem 2.3. *Let* $p \geqslant 2$, $u_0 \in L^\infty(\Omega)$, *suppose that* (1.3) *holds true. Then problem* (P) *has at least one weak solution* $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T) \cap C([0,T];L^1(\Omega))$.

2.1. Proof of Theorem 2.3. We approximate problem (P) by following nonsingular problem:

$$(\mathrm{P}_n^*)\qquad \begin{cases} \partial_t u_n - \operatorname{div}(A(t,x,T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n) \\ \qquad + B\dfrac{u_n|\nabla u_n|^p}{(|u_n|+1/n)^{\theta+1}} = T_n(|u_n|^r) & \text{in } Q_T,\\ u_n(0,x) = u_0(x) & \text{in } \Omega,\\ u_n = 0 & \text{on } \Gamma_T.\end{cases}$$

Note that by (1.1) we have

$$A(t,x,T_n(u_n)) \geqslant \frac{\beta}{(a(t,x)+|T_n(u_n)|)^\varrho} \geqslant \frac{\beta}{(\delta+n)^\varrho},$$

so the operator $B\colon v \mapsto \operatorname{div}(A(t,x,T_n(v))|\nabla v|^{p-2}\nabla v)$ is coercive. Thus, the existence of the approximate solution is proved as in [7]. We begin by proving that $u_n \geqslant 0$, using $u_n^- = \min(0,u_n)$ as a test function in (P_n^*) and by (1.1), we have

$$\begin{aligned}(2.3)\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^T (u_n^-)^2\,\mathrm{d}t + \beta\int_{Q_T}\frac{|\nabla u_n^-|^p}{(a(t,x)+|T_n(u_n)|)^\varrho}\,\mathrm{d}x\,\mathrm{d}t \\ + B\int_{Q_T}\frac{u_n|\nabla u_n|^p}{(|u_n|+1/n)^{\theta+1}}u_n^-\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{Q_T}|u_n|^r u_n^-\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $u_0 \geqslant 0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^T (u_n^-)^2\,\mathrm{d}t = \frac{1}{2}\int_\Omega (u_n^-(T,x))^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega (u_n^-(0,x))^2\,\mathrm{d}x \geqslant 0.$$

The lower-order term has the same sign of the solution and dropping nonnegative terms, we get

$$\beta\int_{Q_T}\frac{|\nabla u_n^-|^p}{(a(t,x)+|T_n(u_n)|)^\varrho}\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{Q_T}|u_n|^r u_n^-\,\mathrm{d}x\,\mathrm{d}t \leqslant 0.$$

Thus, $u_n^- = 0$ and so $u_n \geqslant 0$. Therefore, u_n solves

$$(\mathrm{P}_n)\qquad \begin{cases} \partial_t u_n - \operatorname{div}(A(t,x,T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n) \\ \qquad\qquad + B\dfrac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}} = T_n(u_n^r) & \text{in } Q_T, \\ u_n(0,x) = u_0(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \Gamma_T. \end{cases}$$

Lemma 2.4. *Let $p \geqslant 2$ and u_n be the solutions to problems* (P_n). *Then we have for all $k > 0$*

$$\int_{Q_T}|\nabla T_k(u_n)|^p\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{k(\delta+k)^\varrho}{\beta}\Big(\int_{Q_T}u_n^r\,\mathrm{d}x\,\mathrm{d}t + \|u_0\|_{L^1(\Omega)}\Big). \tag{2.4}$$

Proof. Choosing $T_k(u_n)$ as test function in (P_n) and the fact that $T_n(u_n^r) \leqslant u_n^r$, we obtain

$$\begin{aligned} \int_\Omega \Theta_k(u_n)(T)\,\mathrm{d}x &+ \int_{Q_T}A(t,x,T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t \\ &+ B\int_{Q_T}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \int_{Q_T}u_n^r T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \Theta_k(u_n)(0)\,\mathrm{d}x. \end{aligned} \tag{2.5}$$

The first term is positive since we have $\Theta_k \geqslant 0$, so after dropping nonnegative terms and using (2.2), we obtain

$$\int_{Q_T}A(t,x,T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{Q_T}u_n^r|T_k(u_n)|\,\mathrm{d}x\,\mathrm{d}t + k\|u_0\|_{L^1(\Omega)}. \tag{2.6}$$

According to conditions (1.1) and for $n > k > 0$, we get

$$\frac{\beta}{(\delta+k)^\varrho}\int_{Q_T}|\nabla T_k(u_n)|^p\,\mathrm{d}x\,\mathrm{d}t \leqslant k\int_{Q_T}u_n^r\,\mathrm{d}x\,\mathrm{d}t + k\|u_0\|_{L^1(\Omega)}. \tag{2.7}$$

Therefore (2.4) is established. □

Lemma 2.5. *Let u_n be the solutions to problems* (P_n). *Then*

$$B\int_{Q_T}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t\leqslant\int_{Q_T}u_n^r\,\mathrm{d}x\,\mathrm{d}t+\|u_0\|_{L^1(\Omega)}. \tag{2.8}$$

Proof. Choosing $T_h(u_n)/h$ as a test function in (P_n), dropping the nonnegative terms, we obtain

$$B\int_{Q_T}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\frac{T_h(u_n)}{h}\,\mathrm{d}x\,\mathrm{d}t\leqslant\int_{Q_T}u_n^r\Big|\frac{T_h(u_n)}{h}\Big|\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{h}\int_\Omega|\Theta_h(u_n)(0)|\,\mathrm{d}x.$$

Using the fact that $|T_h(u_n)/h|\leqslant 1$ and (2.2), we have

$$B\int_{Q_T}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\frac{T_h(u_n)}{h}\,\mathrm{d}x\,\mathrm{d}t\leqslant\int_{Q_T}u_n^r\,\mathrm{d}x\,\mathrm{d}t+\|u_0\|_{L^1(\Omega)}.$$

Letting h tend to 0 and by Fatou's Lemma, we deduce (2.8). □

We shall denote by C or C_j various constants depending only on the structure of A, p, θ, r, T, u_0, $|\Omega|$ for $j\in\mathbb{N}$.

Lemma 2.6. *Let u_n be the solutions to problems* (P_n). *Then there exists a positive constant C such that*

$$\|u_n\|_{L^\infty(Q_T)}\leqslant C,\quad \|u_n\|_{L^p(0,T;W_0^{1,p}(\Omega))}\leqslant C\quad\forall\,n\in\mathbb{N}.$$

Proof. Choosing $\varphi=(u_n+\delta)^\nu-\delta^\nu$ as a test function in (P_n), where $\nu>0$, using (1.1), we obtain

$$\begin{aligned}\int_0^T\langle\partial_t u_n,(u_n+\delta)^\nu-\delta^\nu\rangle\,\mathrm{d}t&+\beta\nu\int_{Q_T}\frac{|\nabla u_n|^p}{(\delta+u_n)^\varrho}(u_n+\delta)^{\nu-1}\,\mathrm{d}x\,\mathrm{d}t\\&+B\int_{Q_T}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}(u_n+\delta)^\nu\,\mathrm{d}x\,\mathrm{d}t\\&\leqslant B\int_{Q_T}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\delta^\nu\,\mathrm{d}x\,\mathrm{d}t\\&+\int_{Q_T}u_n^r(u_n+\delta)^\nu-\int_{Q_T}u_n^r\delta^\nu\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

dropping the nonpositive term on the right-hand side and putting $\nu=1$, we get

$$\begin{aligned}\int_0^T\langle\partial_t u_n,u_n\rangle\,\mathrm{d}t+\int_{Q_T}|\nabla u_n|^p(u_n+\delta)^{1-\theta}\Big(\frac{\beta}{(\delta+u_n)^{\varrho-\theta+1}}+\frac{Bu_n(u_n+\delta)^\theta}{(u_n+1/n)^{\theta+1}}\Big)\,\mathrm{d}x\,\mathrm{d}t\\\leqslant B\delta\int_{Q_T}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}u_n^r(u_n+\delta)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $\varrho - \theta + 1 > 0$, there exists a positive constant C_0 such that

$$\frac{\beta}{(\delta + t)^{\varrho - \theta + 1}} + \frac{Bt(t+\delta)^{\theta}}{(t + 1/n)^{\theta+1}} \geqslant C_0 > 0 \quad \forall\, t \geqslant 0.$$

So, after dropping nonnegative terms, we obtain

$$C_0 \int_{Q_T} |\nabla u_n|^p (u_n + \delta)^{1-\theta} \,\mathrm{d}x\,\mathrm{d}t \leqslant B\delta \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} u_n^r (u_n + \delta) \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2} \int_{\Omega} u_0^2 \,\mathrm{d}x.$$

Using (2.8) and the fact that $u_n^r \leqslant (u_n + \delta)^r$, $u_0 \in L^\infty(\Omega)$, we get

$$\begin{aligned} C_0 \int_{Q_T} |\nabla u_n|^p (u_n + \delta)^{1-\theta} \,\mathrm{d}x\,\mathrm{d}t &\leqslant B\delta \int_{Q_T} (u_n + \delta)^r \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} (u_n + \delta)^{r+1} \,\mathrm{d}x\,\mathrm{d}t + C \\ &\leqslant C \int_{Q_T} (u_n + \delta)^{r+1} \,\mathrm{d}x\,\mathrm{d}t + C, \end{aligned}$$

which implies

$$\int_{Q_T} |\nabla (u_n + \delta)^{(p+1-\theta)/p}|^p \,\mathrm{d}x\,\mathrm{d}t \leqslant C \int_{Q_T} (u_n + \delta)^{r+1} \,\mathrm{d}x\,\mathrm{d}t + C.$$

Using Poincaré inequality, we have

$$\int_{Q_T} (u_n + \delta)^{p+1-\theta} \,\mathrm{d}x\,\mathrm{d}t \leqslant C \int_{Q_T} (u_n + \delta)^{r+1} \,\mathrm{d}x\,\mathrm{d}t + C,$$

since $r + 1 < p + 1 - \theta$, Young inequality yields

$$\int_{Q_T} (u_n + \delta)^{p+1-\theta} \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{2} \int_{Q_T} (u_n + \delta)^{p+1-\theta} \,\mathrm{d}x\,\mathrm{d}t + C,$$

which implies that $(u_n + \delta)_n$ is bounded in $L^{p+1-\theta}(Q_T)$, so $(u_n)_n$ is bounded in $L^{p+1-\theta}(Q_T)$. Now, we prove that the sequence $(u_n^r)_n$ is bounded in $L^m(Q_T)$ for some $m > \frac{1}{2}N + 1$. We choose u_n^η as a test function in (P_n), where $\eta > 1$, we find

$$\int_{Q_T} \frac{\beta\eta}{(u_n + \delta)^{\varrho}} |\nabla u_n|^p u_n^{\eta-1} \,\mathrm{d}x\,\mathrm{d}t + B \int_{Q_T} \frac{u_n^{\eta+1} |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} \,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{Q_T} u_n^{r+\eta} \,\mathrm{d}x\,\mathrm{d}t + C.$$

With the same previous calculations, we find

$$\int_{Q_T} |\nabla u_n|^p u_n^{\eta-\theta} \Big(\frac{\beta\eta}{u_n^{1-\theta} (\delta + u_n)^{\varrho}} + \frac{B u_n^{1+\theta}}{(u_n + 1/n)^{\theta+1}} \Big) \,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{Q_T} u_n^{r+\eta} \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\eta+1} \int_{\Omega} u_0^{\eta+1} \,\mathrm{d}x.$$

Since $1-\theta>0$, there exists a positive constant C_1 such that

$$\frac{\beta\eta}{t^{1-\theta}(\delta+t)^{\varrho}}+\frac{Bt^{1+\theta}}{(t+1/n)^{\theta+1}}\geqslant C_1>0\quad\forall t>0.$$

So,

$$\int_{Q_T}u_n^{p+\eta-\theta}\,\mathrm{d}x\,\mathrm{d}t\leqslant C\int_{Q_T}u_n^{r+\eta}\,\mathrm{d}x\,\mathrm{d}t+C. \tag{2.9}$$

We now choose $\eta+r=p+1-\theta$ (observe that $\eta>1$), so $p+\eta-\theta=2(p+1-\theta)-(r+1)$. Then by (1.3) and (2.9), we obtain that u_n is bounded in $L^{2(p+1-\theta)-(r+1)}(Q_T)$. Consequently, an iterating procedure gives us that (u_n) is bounded in $L^{\mu}(Q_T)$ for all $\mu<\infty$. Indeed, if we consider $\eta_1>1$ such that $r+\eta_1=2(p+1-\theta)-(r+1)$, (2.9) and the fact that (u_n) is bounded in $L^{2(p+1-\theta)-(r+1)}(Q_T)$, then it is bounded in $L^{3(p+1-\theta)-2(r+1)}$. Now consider $\eta_2>1$ such that $r+\eta_2=3(p+1-\theta)-2(r+1)$ and deduce that (u_n) is bounded in $L^{4(p+1-\theta)-3(r+1)}(Q_T)$. Hence, we can obtain that (u_n) is bounded in $L^{(q+1)(p+1-\theta)-q(r+1)}(Q_T)$ for all $q\in\mathbb{N}$. Since

$$(q+1)(p+1-\theta)-q(r+1)=q(p-r-\theta)+p+1-\theta\to\infty\quad\text{as } q\to\infty,$$

we deduce that (u_n) is bounded in $L^{\mu}(Q_T)$ for all $\mu<\infty$. Because there is $n'>0$ such that $(n'(p-r-\theta)+p+1-\theta)/r>(N/p)+1$,

$$(u_n^r)\text{ is bounded in } L^m(Q_T)\quad\text{for some } m>\frac{N}{p}+1.$$

Standard parabolic estimates, performed using only the principal part of the operator (see for example [5]), and taking advantage of the nonnegativity of the lower order gradient term, then imply that $(u_n)_n$ is bounded in $L^{\infty}(Q_T)$. Therefore by (2.7), we have

$$\|u_n\|_{L^p(0,T;W_0^{1,p}(\Omega))}\leqslant C.$$

Taking n large enough, we get $T_n(u_n^r)=u_n^r$ and $T_n(u_n)=u_n$, so we conclude that u_n is a weak solution of

$$\begin{cases}\partial_t u_n-\operatorname{div}(A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n)+B\dfrac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}=u_n^r & \text{in } Q_T,\\ u_n(0,x)=u_0(x) & \text{in } \Omega,\\ u_n=0 & \text{on } \Gamma_T.\end{cases} \tag{2.10}$$

□

Now, we are going to prove the strict positivity of the sequence of approximated solutions u_n.

Proposition 2.7. *Let ω be a compactly contained open subset of Ω. Then there exists a positive constant $C_{\omega,T}$ such that $u_n \geqslant C_{\omega,T}$ in $(0,T)\times\omega$.*

Proof. Following the ideas in [11], we define for $s \geqslant 0$,

$$H_n(s) = \int_0^s \frac{(\delta+\tau)^{\varrho}}{(\tau+1/n)^{\theta}}\,\mathrm{d}\tau, \quad \Phi_n(s) = \mathrm{e}^{-BH_n(s)/\beta},$$

where $0<\theta<1$ and $B>0$. Taking $\Phi_n(u_n)v$, with $v \in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q_T)$, $v \geqslant 0$, as test function in (2.10) and using (1.1)–(1.2) and that

$$\Phi_n'(s) = \frac{-B}{\beta}\frac{(\delta+s)^{\varrho}}{(s+1/n)^{\theta}}\Phi_n(s),$$

we obtain

$$\begin{aligned}\int_0^T \langle \partial_t u_n, \Phi_n(u_n)v\rangle\,\mathrm{d}t &+ \int_{Q_T} A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n \nabla v \Phi_n(u_n)\,\mathrm{d}x\,\mathrm{d}t\\ &\geqslant \int_{Q_T} \frac{B}{(u_n+1/n)^{\theta}}|\nabla u_n|^p\Phi_n(u_n)v\,\mathrm{d}x\,\mathrm{d}t\\ &\quad - B\int_{Q_T}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\Phi_n(u_n)v\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} u_n^r\Phi_n(u_n)v\,\mathrm{d}x\,\mathrm{d}t\\ &\geqslant 0.\end{aligned}$$

After dropping the nonnegative term, we derive

$$\begin{aligned}\int_0^t \langle \partial_t u_n, \Phi_n(u_n)v\rangle\,\mathrm{d}t &+ \int_{Q_T} A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n\nabla v\Phi_n(u_n)\,\mathrm{d}x\,\mathrm{d}t\\ &\geqslant \int_{Q_T} u_n^r\Phi_n(u_n)v\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.11}$$

Now, we consider the nonincreasing function ψ:

$$\psi(s) = \int_s^1 \Phi_n(t)\,\mathrm{d}t = \int_s^1 \mathrm{e}^{-BH_n(t)/\beta}\,\mathrm{d}t.$$

Then, inequality (2.11) implies that

$$\begin{aligned}-\int_0^T \langle \partial_t(\psi(u_n)), v\rangle\,\mathrm{d}t &- \int_{Q_T} A(t,x,u_n)|\nabla u_n|^{p-2}\nabla\psi(u_n)\nabla v\,\mathrm{d}x\,\mathrm{d}t\\ &\geqslant \int_{\{0\leqslant u_n\leqslant 1\}} \Phi_n(u_n)u_n^r v\,\mathrm{d}x\,\mathrm{d}t \geqslant \int_{\{0\leqslant u_n\leqslant 1\}} (\Phi_n(u_n)-1)u_n^r v\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.12}$$

We call

$$\widetilde{A}(t,x,s) = A(t,x,\psi^{-1}(s))|\nabla\psi^{-1}(s)|^{p-2},$$

and

$$H(s) = (1-\Phi_n(\psi^{-1}(s)))u_n^r\chi_{\{0\leqslant u_n\leqslant 1\}}.$$

Thus, see [2] for instance, we deduce that $\psi(u_n)$ is a sub-solution of

$$\partial_t z - \operatorname{div}(\widetilde{A}(t,x,z)\nabla z) = H(z) \quad \text{in } Q_T.$$

Since H is a nonnegative term and $u_0 > 0$ in Ω, we can apply Lemma 3.12 in [6] to the previous equation to obtain the existence of $c_{\omega,T} > 0$ such that

$$\psi(u_n) \leqslant c_{\omega,T} \quad \forall\,(t,x) \in (0,T)\times\omega \text{ and } \forall\, n > 1.$$

By the definition of ψ, there exists $C_{\omega,T} > 0$ (independent of n) such that

$$u_n \geqslant \psi^{-1}(c_{\omega,T}) = C_{\omega,T} \quad \text{in } (0,T)\times\omega.$$

□

2.1.1. Passage to the limit.

Lemma 2.8. *Let A be a function satisfying* (1.1) *and let $u_n \in L^p(0,T,W_0^{1,p}(\Omega))$ be a sequence of weak solutions to* (2.10). *Then there exists a subsequence of u_n (still denoted by u_n) converging to a measurable function u a.e. in Q_T, and*

$$\nabla u_n \to \nabla u \text{ a.e. in } Q_T. \tag{2.13}$$

P r o o f. Going back again to (2.10), the sequence $(\partial_t u_n)$ remains in a bounded set of the space

$$L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q_T), \quad p' = \frac{p}{p-1}.$$

Therefore, $(\partial_t u_n)$ is bounded in $L^1(0,T;W^{-1,s}(\Omega))$, for all $s < N/(N-1)$. So, we can use Corollary 4 of [10] to see that

$$u_n \text{ is relatively compact in } L^1(Q_T).$$

Summing up, there exists a function $u \in L^p(0,T;W_0^{1,p}(\Omega))$ and a subsequence, still denoted by (u_n), such that

$$u_n \rightharpoonup u \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{2.14}$$

$$u_n \to u \text{ strongly in } L^p(Q_T) \text{ and a.e. in } Q_T. \tag{2.15}$$

Now, we prove that

$$\nabla T_k(u_n) \to \nabla T_k(u) \text{ strongly in } (L^p(Q_T))^N \ \forall\, k \in \mathbb{N}. \tag{2.16}$$

We also introduce another time-regularization of truncations, we will use the sequence $(T_k(u))_\nu$ as approximation of $T_k(u)$. For $\nu > 0$, we define the regularization in time of the function $T_k(u)$ given by

$$(2.17)\qquad (T_k(u))_\nu(t,x) := \nu \int_{-\infty}^{t} \mathrm{e}^{\nu(s-t)} T_k(u(s,x))\,\mathrm{d}s + \mathrm{e}^{-\nu t} T_k(u_0),$$

where $T_k(u(s,x))$ is the zero extension of u for $s < 0$ (see [14]). Applying this regularization to the truncatures $T_k(u_m)$, we have the following properties:

▷ $((T_k(u_m))_\nu)_t = \nu(T_k(u_m) - (T_k(u_m))_\nu)$,
▷ $((T_k(u_m))_\nu)(0,x) = T_k(u_0)$,
▷ $|((T_k(u_m))_\nu)| \leqslant k$,
▷ $(T_k(u_m))_\nu \to T_k(u_m)$ strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ as $\nu \to \infty$.

Considering the function $\varphi_\lambda(s)$ defined by

$$\varphi_\lambda(s) = s\mathrm{e}^{\lambda s^2}, \quad \lambda > 0,$$

in what follows we use that for every $a, b > 0$ we have

$$(2.18)\qquad a\varphi'_\lambda(s) - b|\varphi_\lambda(s)| \geqslant \frac{a}{2} \quad \text{if } \lambda > \frac{b^2}{4a^2}.$$

We also denote by $\tau(m,n,\nu)$ any quantity I such that

$$\lim_{\nu\to\infty} \lim_{n\to\infty} \lim_{m\to\infty} I = 0,$$

likewise $\tau(n,\nu)$ denotes a quantity I such that $\lim\limits_{\nu\to\infty} \lim\limits_{n\to\infty} I = 0$. Let ϕ be a function in $C_c^\infty(\Omega)$ such that $\phi \geqslant 0$. By the same technique as in [1] we have that

$$(2.19)\qquad \int_0^T \langle \partial_t u_n, \varphi_\lambda(T_k(u_n) - (T_k(u_m))_\nu)\phi\rangle\,\mathrm{d}t \geqslant \tau(m,n,\nu).$$

Using (2.19) and taking $\psi_\lambda = \varphi_\lambda(T_k(u_n) - (T_k(u_m))_\nu)\phi$ as a test function in (2.10), we obtain

$$\begin{aligned}(2.20)\quad \tau(m,n,\nu) &+ \int_{Q_T} A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n \\ &\times \nabla(T_k(u_n) - (T_k(u_m))_\nu)\varphi'_\lambda(T_k(u_n) - (T_k(u_m))_\nu)\phi\,\mathrm{d}x\,\mathrm{d}t \\ &+ B\int_{Q_T} \frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\psi_\lambda\,\mathrm{d}x\,\mathrm{d}t \\ \leqslant &\int_{Q_T} u_n^r \psi_\lambda\,\mathrm{d}x\,\mathrm{d}t \\ &- \int_{Q_T} A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n \nabla\phi\varphi_\lambda(T_k(u_n) - (T_k(u_m))_\nu)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

By (2.15), $(T_k(u_m))_\nu \to (T_k(u))_\nu$ a.e. in Q_T and we have

$$|u_n^r \psi_\lambda| \leqslant \|u_n^r\|_{L^\infty(Q_T)} \varphi_\lambda(2k) \in L^1(Q_T),$$

by the Lebesgue dominated convergence theorem,

$$\lim_{\nu\to\infty}\left(\lim_{n\to\infty}\left(\lim_{m\to\infty}\int_{Q_T} u_n^r\psi_\lambda\right)\right) = 0.$$

By writing $Q_T = \{u_n \leqslant k\} \cup \{u_n > k\}$ and adopting the technique used in [1], we have

$$\lim_{\nu\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\int_{Q_T} A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n \nabla\phi\varphi_\lambda(T_k(u_n) - (T_k(u_m))_\nu) = 0.$$

Therefore

(2.21)
$$\int_{Q_T} u_n^r\psi_\lambda - \int_{Q_T} A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n\nabla\phi\varphi_\lambda(T_k(u_n) - (T_k(u_m))_\nu) = \tau(m,n,\nu).$$

We next turn to consider the last term on the left-hand side of (2.20). Choosing $\omega \subset\subset \Omega$ with $\operatorname{supp}\phi \subset \omega$, by the nonnegativity of $\varphi_\lambda(k - (T_k(u))_\nu)$, we have that

$$\begin{aligned}\text{(2.22)}\quad \lim_{m\to\infty}\int_{Q_T}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\psi_\lambda\,\mathrm{d}x\,\mathrm{d}t &\geqslant \int_{\{C_{\omega,T}\leqslant u_n\leqslant k\}}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\psi_\lambda\,\mathrm{d}x\,\mathrm{d}t\\ &\geqslant -C_{k,T}(\omega)\int_{Q_T}|\nabla T_k(u_n)|^p|\psi_\lambda|\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $C_{k,T}(\omega)$ is a positive constant such that

$$\text{(2.23)}\qquad \frac{u_n}{(u_n+1/n)^{\theta+1}} \leqslant \max_{u_n\in[C_{\omega,T},k]}\frac{1}{u_n^\theta} = C_{k,T}(\omega) \quad \forall\, n >> 1.$$

From the convergence

$$\nabla(T_k(u_m))_\nu \rightharpoonup \nabla(T_k(u))_\nu \text{ weakly in } (L^p(Q_T))^N \text{ as } m\to\infty$$

we get, by using (2.20)–(2.21) and (2.22), that

(2.24)
$$\begin{aligned}&\int_{Q_T} A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n\nabla(T_k(u_n)-(T_k(u))_\nu)\varphi_\lambda'(T_k(u_n)-(T_k(u))_\nu)\phi\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad - BC_{k,T}(\omega)\int_{Q_T}|\nabla T_k(u_n)|^p|\psi_\lambda|\,\mathrm{d}x\,\mathrm{d}t \leqslant \tau(\nu,n).\end{aligned}$$

Note that

$$\int_{Q_T} A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n$$
$$\times \nabla(T_k(u_n)-(T_k(u))_\nu)\varphi'_\lambda(T_k(u_n)-(T_k(u))_\nu)\phi\chi_{\{u_n\geqslant k\}}\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_{Q_T} A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n \nabla(T_k(u))_\nu \varphi'_\lambda(k-(T_k(u))_\nu)\phi\chi_{\{u_n\geqslant k\}}\,\mathrm{d}x\,\mathrm{d}t$$
$$= \tau(\nu,n),$$

so adding

$$-\int_{Q_T} A(t,x,u_n)|\nabla(T_k(u))_\nu|^{p-2}\nabla(T_k(u))_\nu$$
$$\times \nabla(T_k(u_n)-(T_k(u))_\nu)\varphi'_\lambda(T_k(u_n)-(T_k(u))_\nu)\phi\,\mathrm{d}x\,\mathrm{d}t = \tau(\nu,n).$$

On both sides of (2.24) and since

(2.25)
$$\int_{Q_T} |\nabla T_k(u_n)|^p|\psi_\lambda|\,\mathrm{d}x\,\mathrm{d}t \leqslant 2^{p-1}\int_{Q_T} |\nabla(T_k(u_n)-(T_k(u))_\nu)|^p|\psi_\lambda|\,\mathrm{d}x\,\mathrm{d}t$$
$$+ 2^{p-1}\int_{Q_T} |\nabla(T_k(u))_\nu|^p|\psi_\lambda|\,\mathrm{d}x\,\mathrm{d}t$$
$$= 2^{p-1}\int_{Q_T} |\nabla(T_k(u_n)-(T_k(u))_\nu)|^p|\psi_\lambda|\,\mathrm{d}x\,\mathrm{d}t + \tau(\nu,n),$$

and using the following well-known inequalities that hold for any two real vectors ξ, η and a real $p \geqslant 2$,

(2.26)
$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta) \geqslant 2^{2-p}|\xi-\eta|^p,$$

we find, by using also (1.1) and (2.25), for all $n > k > 0$,

$$2^{2-p}\frac{\beta}{(\delta+k)^\varrho}\int_{Q_T} |\nabla(T_k(u_n)-(T_k(u))_\nu)|^p\varphi'_\lambda(T_k(u_n)-(T_k(u))_\nu)\phi\,\mathrm{d}x\,\mathrm{d}t$$
$$- 2^{p-1}BC_{k,T}(\omega)\int_{Q_T} |\nabla(T_k(u_n)-(T_k(u))_\nu)|^p|\varphi_\lambda(T_k(u_n)-(T_k(u))_\nu)|\phi\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \tau(\nu,n).$$

Choosing λ such that (2.18) holds with $a = 2^{2-p}\beta/(\delta+k)^\varrho$ and $b = 2^{p-1}BC_{k,T}(\omega)$, we obtain (2.16) by setting $\nu \to \infty$. From this result we also deduce that (up to subsequences)

$$\nabla u_n \to \nabla u \ \text{ almost everywhere in } Q_T.$$

□

Lemma 2.9. *We have*

$$\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}} \to \frac{|\nabla u|^p}{u^\theta} \text{ strongly in } L^1(Q_T). \tag{2.27}$$

Proof. In view of (2.13) and (2.15), we have

$$\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}} \to \frac{|\nabla u|^p}{u^\theta} \text{ a.e. in } Q_T.$$

Now, we shall obtain local equi-integrability of $u_n|\nabla u_n|^p/(u_n+1/n)^{\theta+1}$ on Q_T. Observe that

$$\int_0^T\int_{u_n\geqslant k}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{k}\int_0^T\int_{u_n\geqslant k}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t.$$

We choose $\varphi = T_k(u_n)$ as a test function in problems (2.10), we find

$$\begin{aligned}\int_\Omega \mathrm{d}x\int_0^{u_n(T,x)} T_k(\sigma)\,\mathrm{d}\sigma + \int_{Q_T} A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t&\\ + \int_{Q_T}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t&\\ = \int_{Q_T} u_n^r T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \mathrm{d}x\int_0^{u_n(0,x)} T_k(\sigma)\,\mathrm{d}\sigma.&\end{aligned}$$

So, after dropping the nonnegative terms, we derive

$$\int_{Q_T}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{Q_T} u_n^r|T_k(u_n)|\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \mathrm{d}x\int_0^{u_n(0,x)}|T_k(\sigma)|\,\mathrm{d}\sigma.$$

Taking into account that for any $M>0$, $0\leqslant |T_k(s)|\leqslant M + k\mathbf{1}_{s>M}$, $s\in\mathbb{R}^+$, we have

$$\int_{Q_T} u_n^r|T_k(u_n)|\,\mathrm{d}x\,\mathrm{d}t \leqslant MC\|u_n\|^r_{L^p(Q_T)} + k\int_0^T\int_{u_n>M} u_n^r\,\mathrm{d}x\,\mathrm{d}t,$$

and

$$\int_\Omega \mathrm{d}x\int_0^{u_n(0,x)}|T_k(\sigma)|\,\mathrm{d}\sigma \leqslant M\|u_0\|_{L^1(\Omega)} + k\int_0^T\int_{u_n>M} u_{0n}\,\mathrm{d}x\,\mathrm{d}t.$$

Consequently, we have

$$\begin{aligned}&\int_0^T\int_{u_n\geqslant k}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \frac{1}{k}\Big(MC + k\int_0^T\int_{u_n>M} u_n^r\,\mathrm{d}x\,\mathrm{d}t + M\|u_0\|_{L^1(\Omega)} + k\int_0^T\int_{u_n>M} u_{0n}\,\mathrm{d}x\,\mathrm{d}t\Big)\\ &\leqslant C\frac{M}{k} + \int_0^T\int_\Omega \chi_{\{u_n>M\}}u_n^r\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \chi_{\{u_n>M\}}u_{0n}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We take $M=\sqrt{k}$, we obtain

$$\text{(2.28)}\qquad \int_0^T\int_{u_n\geqslant k}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t \to^{k\to\infty} 0 \text{ uniformly with respect to } n,$$

then, there exists $k_0>1$ such that

$$\text{(2.29)}\qquad \int_0^T\int_{u_n\geqslant k}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{\varepsilon}{2}\quad \forall\, k\geqslant k_0 \text{ and } \forall\, n\in\mathbb{N}.$$

Consequently, if $E\subset\subset\omega$, we have

(2.30)

$$\begin{aligned}\int_0^T\int_E &\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^T\int_{E\cap\{u_n\geqslant k\}}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{E\cap\{u_n\leqslant k\}}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant\int_0^T\int_{E\cap\{u_n\geqslant k\}}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t+C_{k,T}(\omega)\int_0^T\int_{E\cap\{u_n\leqslant k\}}|\nabla T_k(u_n)|^p\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

From (2.14) there exist n_ε and δ_ε such that for every $E\subset\subset\Omega$ with $\operatorname{meas}(E)<\delta_\varepsilon$ we have

$$\int_0^T\int_{E\cap\{u_n\leqslant k\}}|\nabla T_k(u_n)|^p\,\mathrm{d}x\,\mathrm{d}t<\frac{\varepsilon}{2C_{k,T}(\omega)}\qquad\forall\, n\geqslant n_\varepsilon.$$

By (2.29), (2.30), and taking $n\geqslant n_\varepsilon$, $k\geqslant k_0$, we see that $\operatorname{meas}(E)<\delta_\varepsilon$ implies

$$\int_0^T\int_E\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t<\varepsilon.$$

We deduce that $u_n|\nabla u_n|^p/(u_n+1/n)^{\theta+1}$ is equi-integrable in Q_T, then by Vitali's theorem convergence, we have (2.27) and $|\nabla u|^p/u^\theta\in L^1(Q_T)$. □

Lemma 2.10. *The sequence* (u_n) *is a Cauchy sequence in* $C([0,T];L^1(\Omega))$, *hence* u_n *converges to* $u\in C([0,T];L^1(\Omega))$.

P r o o f. To do this, fix $t\in[0,T]$. Taking $T_k(u_n-u_m)$ as a test function in (2.10) for u_n and u_m, subtracting up both identities, we deduce that

$$\begin{aligned}&\int_\Omega\Theta_k(u_n(t)-u_m(t))\,\mathrm{d}x\\ &\quad+\int_{Q_t}(A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n-A(t,x,u_m)|\nabla u_m|^{p-2}\nabla u_m)\nabla T_k(u_n-u_m)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+B\int_{Q_t}\Big(\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}-\frac{u_m|\nabla u_m|^p}{(u_m+1/m)^{\theta+1}}\Big)T_k(u_n-u_m)\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant\int_{Q_t}|u_n^r-u_m^r||T_k(u_n-u_m)|\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega|\Theta_k(u_n(0)-u_m(0))|\,\mathrm{d}x.\end{aligned}$$

So, by (2.2) we obtain

$$
\begin{aligned}
&\int_\Omega \Theta_k(u_n(t)-u_m(t))\,\mathrm{d}x\\
&\quad\leqslant \int_{Q_t} |A(t,x,u_m)|\nabla u_m|^{p-2}\nabla u_m - A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n||\nabla T_k(u_n-u_m)|\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad + Bk\int_{Q_t}\Big|\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}} - \frac{u_m|\nabla u_m|^p}{(u_m+1/m)^{\theta+1}}\Big|\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad + k\int_{Q_t}|u_n^r-u_m^r|\,\mathrm{d}x\,\mathrm{d}t + k\int_\Omega |u_n(0)-u_m(0)|\,\mathrm{d}x.
\end{aligned}
$$

Using (2.2) and dividing this inequality by k, we obtain

$$
\begin{aligned}
&\sup_{t\in[0,T]}\int_\Omega |u_n(t)-u_m(t)|\,\mathrm{d}x\\
&\leqslant \frac{1}{k}\int_{Q_T} |A(t,x,u_m)|\nabla u_m|^{p-2}\nabla u_m - A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n||\nabla T_k(u_n-u_m)|\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + B\int_{Q_T}\Big|\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}} - \frac{u_m|\nabla u_m|^p}{(u_m+1/m)^{\theta+1}}\Big|\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_{Q_T}|u_n^r-u_m^r|\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega |u_n(0)-u_m(0)|\,\mathrm{d}x + \frac{k}{2}.
\end{aligned}
$$

By (1.1), (2.13), (2.14) and (2.15), we have

$$
|A(t,x,u_m)|\nabla u_m|^{p-2}\nabla u_m - A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n| \rightharpoonup 0 \quad \text{in } L^{p'}(Q_T). \tag{2.31}
$$

Taking into account (2.27) and letting $k\to 0$, we deduce that (u_n) is a Cauchy sequence in $C([0,T];L^1(\Omega))$. Consequently, $u_n\to u$ in $C([0,T],L^1(\Omega))$. This ends the proof of Lemma 2.10. □

2.2. The end of the proof of Theorem 2.3. Let $\varphi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q_T)$. We have

$$
\begin{aligned}
-\int_\Omega u_n(0)\varphi(0)\,\mathrm{d}x + \int_{Q_T} u_n\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t&\\
+ B\int_{Q_T}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} u_n^r\varphi\,\mathrm{d}x\,\mathrm{d}t.&
\end{aligned}
\tag{2.32}
$$

Arguing as in (2.31), we have

$$
\lim_{n\to\infty}\int_{Q_T} A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} A(t,x,u)|\nabla u|^{p-2}\nabla u\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t.
$$

Therefore by (2.27), we can easily pass to the limit in (2.32). Theorem 2.3 is proved. □

3. L^1 Initial data

Theorem 3.1. *Given $u_0 \in L^1(\Omega)$, suppose that* (1.3) *holds true. Then problem* (P) *has at least a weak solution u, i.e., a function u belonging to $L^q(0,T;W_0^{1,q}(\Omega)) \cap C([0,T];L^1(\Omega))$, $u > 0$, $|\nabla u|^p/u^\theta \in L^1(Q_T)$, such that*

$$
(3.1) \quad -\int_\Omega u(0)\varphi(0)\,\mathrm{d}x + \int_{Q_T} u\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} A(t,x,u)|\nabla u|^{p-2}\nabla u\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\
+ B\int_{Q_T} \frac{|\nabla u|^p}{u^\theta}\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} u^r\varphi\,\mathrm{d}x\,\mathrm{d}t, \quad q = p - \frac{\theta N}{N+1}
$$

for every $\varphi \in W^{1,\infty}(0,T;L^\infty(\Omega))$ and such that $\varphi(T) = 0$ in Ω.

3.1. Proof of Theorem 3.1. Let (u_{0n}), $u_{0n} = T_n(u_0) \geqslant 0$ be a sequence of bounded functions defined in Ω, which converges to u_0 in $L^1(\Omega)$, such that

$$
\begin{cases} \|u_{0n}\|_{L^1(\Omega)} \leqslant \|u_0\|_{L^1(\Omega)}, \\ u_{0n} \leqslant n. \end{cases}
$$

A nonnegative weak solution $u_n \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ to problem (P) with $u_n(0,x) = u_{0n}(x)$ does exist by Theorem 2.3. Therefore, u_n satisfies

$$
(3.2) \quad \int_0^T \langle \partial_t u_n, \varphi\rangle\,\mathrm{d}t + \int_{Q_T} A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\
+ B\int_{Q_T} \frac{|\nabla u_n|^p}{u_n^\theta}\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} u_n^r\varphi\,\mathrm{d}x\,\mathrm{d}t
$$

for all $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$. Using the same technique as in Proposition 2.7, there exists a positive constant $C_{\omega,T}$ such that

$$
(3.3) \quad u_n \geqslant C_{\omega,T} \quad \text{in } (0,T)\times\omega,
$$

where ω is a compactly contained open subset of Ω.

Lemma 3.2. *Assume that* (1.3) *hold with $p \geqslant 2$ and u_n is the solution to problems* (3.2)*. Then there exists a positive constant C such that*

$$
(3.4) \quad \int_{Q_T} u_n^r\,\mathrm{d}x\,\mathrm{d}t \leqslant C,
$$

$$
(3.5) \quad \int_{Q_T} |\nabla u_n|^q\,\mathrm{d}x\,\mathrm{d}t \leqslant C, \quad q = p - \frac{\theta N}{N+1}.
$$

P r o o f. Take $\varphi = T_1(u_n)$ as a test function in the weak formulation (3.2). By (1.1), we have

(3.6)
$$\int_\Omega \Theta_1(u_n)(T)\,\mathrm{d}x + \int_{Q_T} \frac{\beta}{(\delta+u_n)^\varrho}|\nabla T_1(u_n)|^p\,\mathrm{d}x\,\mathrm{d}t + B\int_{Q_T}\frac{|\nabla u_n|^p}{u_n^\theta}T_1(u_n)\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{Q_T} u_n^r T_1(u_n)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \Theta_1(u_n)(0)\,\mathrm{d}x.$$

Since

$$\int_{Q_T}\frac{|\nabla u_n|^p}{u_n^\theta}T_1(u_n)\,\mathrm{d}x\,\mathrm{d}t \geqslant \int_{\{u_n>1\}\cap Q_T}\frac{|\nabla u_n|^p}{u_n^\theta}\,\mathrm{d}x\,\mathrm{d}t,$$

dropping nonnegative terms in (3.6), it follows that

$$\begin{aligned}(3.7)\qquad B\int_{\{u_n>1\}\cap Q_T}\frac{|\nabla u_n|^p}{u_n^\theta}\,\mathrm{d}x\,\mathrm{d}t &\leqslant \int_{Q_T} u_n^r T_1(u_n)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \Theta_1(u_n)(0)\,\mathrm{d}x\\ &\leqslant \int_{\{u_n\leqslant 1\}\cap Q_T} u_n^{r+1}\,\mathrm{d}x\,\mathrm{d}t + \int_{\{u_n>1\}\cap Q_T} u_n^r\,\mathrm{d}x\,\mathrm{d}t + \|u_0\|_{L^1(\Omega)}\\ &\leqslant |Q_T| + C_1 + C\int_{\{u_n>1\}\cap Q_T}(u_n-1)^r\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Consequently, denoting $G_1(r) = r - T_1(r)$, we get the inequality

$$\int_{\{u_n>1\}\cap Q_T}|\nabla u_n|^p u_n^{-\theta}\,\mathrm{d}x\,\mathrm{d}t \leqslant C_2 + C_2\int_{\{u_n>1\}\cap Q_T}(G_1(u_n))^r\,\mathrm{d}x\,\mathrm{d}t,$$

so

$$\int_{\{u_n>1\}\cap Q_T}|\nabla u_n|^p(G_1(u_n)+1)^{-\theta}\,\mathrm{d}x\,\mathrm{d}t \leqslant C_2 + C_2\int_{\{u_n>1\}\cap Q_T}(G_1(u_n))^r\,\mathrm{d}x\,\mathrm{d}t,$$

which yields

$$\Bigl(1-\frac{\theta}{p}\Bigr)^{-p}\int_{\{u_n>1\}\cap Q_T}|\nabla(G_1(u_n)+1)^{(1-\theta/p)}|^p\,\mathrm{d}x\,\mathrm{d}t \leqslant C_2 + C_2\int_{\{u_n>1\}\cap Q_T}(G_1(u_n))^r\,\mathrm{d}x\,\mathrm{d}t.$$

Now, the Poincaré inequality implies

$$\int_{\{u_n>1\}\cap Q_T}(G_1(u_n)+1)^{p-\theta}\,\mathrm{d}x\,\mathrm{d}t \leqslant C_3 + C_3\int_{\{u_n>1\}\cap Q_T}(G_1(u_n))^r\,\mathrm{d}x\,\mathrm{d}t.$$

Observe that $r < p - \theta$. By Young's inequality we obtain

$$(3.8)\qquad \int_{\{u_n>1\}\cap Q_T}|G_1(u_n)|^{p-\theta}\,\mathrm{d}x\,\mathrm{d}t \leqslant C_4.$$

Therefore

$$\int_{\{u_n>1\}\cap Q_T} u_n^r \,\mathrm{d}x\,\mathrm{d}t = \int_{\{u_n>1\}\cap Q_T} (G_1(u_n)+1)^r \,\mathrm{d}x\,\mathrm{d}t \leqslant C_5.$$

So (3.4) is proved. To prove (3.5) we have by (3.4) and (3.6)

$$\begin{aligned}(3.9)\qquad \int_{Q_T} |\nabla T_1(u_n)|^p \,\mathrm{d}x\,\mathrm{d}t &= \int_{\{u_n\leqslant 1\}\cap Q_T} \frac{|\nabla T_1(u_n)|^p}{(\delta+u_n)^\varrho}(\delta+u_n)^\varrho \,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant (1+\delta)^\varrho \int_{\{u_n\leqslant 1\}\cap Q_T} \frac{|\nabla T_1(u_n)|^p}{(\delta+u_n)^\varrho} \,\mathrm{d}x\,\mathrm{d}t \leqslant C.\end{aligned}$$

From (3.4), (3.7) and $q = p - \theta N/(N+1)$, we write

$$\begin{aligned}(3.10)\quad &\int_{\{u_n>1\}\cap Q_T} |\nabla u_n|^q \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\{u_n>1\}\cap Q_T} \frac{|\nabla u_n|^q}{u_n^{\theta q/p}} u_n^{\theta q/p} \,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \Big(\int_{\{u_n>1\}\cap Q_T} \frac{|\nabla u_n|^p}{u_n^\theta} \,\mathrm{d}x\,\mathrm{d}t\Big)^{q/p} \Big(\int_{\{u_n>1\}\cap Q_T} u_n^{\theta q/(p-q)} \,\mathrm{d}x\,\mathrm{d}t\Big)^{1-q/p} \\ &\leqslant C\Big(\int_{Q_T} u_n^s \,\mathrm{d}x\,\mathrm{d}t\Big)^{1-q/p}, \quad s = \frac{q(N+1)}{N}.\end{aligned}$$

By (2.2) and (3.6), we have

$$(3.11)\qquad \sup_{t\in[0,T]} \int_\Omega u_n(t,x) \,\mathrm{d}x \leqslant C.$$

Use the following interpolation argument: $\|u_n\|_{L^s(\Omega)} \leqslant \|u_n\|_{L^1(\Omega)}^\tau \|u_n\|_{L^{q^*}(\Omega)}^{1-\tau}$ with $1-\tau = ((1-s)/(1-q^*))(q^*/s)$, where $q^* = Nq/(N-q)$ if $q < N$ and $q^* > 1$ satisfying $(1-\tau)s = q$ otherwise. Using (3.11) and the Sobolev inequality we obtain

$$\int_0^T \|u_n\|_{L^s(\Omega)}^s \,\mathrm{d}t \leqslant C \int_0^T \|\nabla u_n\|_{L^q(\Omega)}^{(1-\tau)s} \,\mathrm{d}t.$$

By this last inequality, (3.10), $q < p$, and (3.9) we have

$$\begin{aligned}(3.12)\qquad \int_{\{u_n>1\}\cap Q_T} |\nabla u_n|^q \,\mathrm{d}x\,\mathrm{d}t &\leqslant C\Big(\int_{Q_T} |\nabla u_n|^q \,\mathrm{d}x\,\mathrm{d}t\Big)^{1-q/p} \\ &\leqslant C + C\Big(\int_{\{u_n>1\}\cap Q_T} |\nabla u_n|^q \,\mathrm{d}x\,\mathrm{d}t\Big)^{1-q/p},\end{aligned}$$

which implies that

$$\int_{\{u_n>1\}\cap Q_T} |\nabla u_n|^q \,\mathrm{d}x\,\mathrm{d}t \leqslant C.$$

Furthermore, (3.9) implies estimate (3.5) and Lemma 3.2 is prooved. □

3.2. Passage to the limit and finishing the proof of Theorem 3.1. Arguing as in Lemma 2.8, we obtain a subsequence (u_n) and a mesurable function $u \in L^q(0,T,W_0^{1,q}(\Omega))$ such that

$$u_n \rightharpoonup u \text{ weakly in } L^q(0,T;W_0^{1,q}(\Omega)), \tag{3.13}$$

$$u_n \to u \text{ strongly in } L^q(Q_T) \text{ and a.e. in } Q_T, \tag{3.14}$$

$$\nabla u_n \to \nabla u \text{ a.e. in } Q_T. \tag{3.15}$$

From (3.14), (3.15), (1.1), and $q/(p-1) > 1$, we obtain

$$A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup A(t,x,u)|\nabla u|^{p-2}\nabla u \text{ in } (L^{q/(p-1)}(Q_T))^N. \tag{3.16}$$

By the technique used in the proof of Lemma 2.9,

$$\frac{|\nabla u_n|^p}{u_n^\theta} \to \frac{|\nabla u|^p}{u^\theta} \text{ strongly in } L^1(Q_T). \tag{3.17}$$

We also deduce that

$$u_n^r \to u^r \text{ strongly in } L^1(Q_T). \tag{3.18}$$

Indeed, thanks to (3.14), we just have to show that the sequence (u_n^r) is equi-integrable, but this is straightforward taking into account (3.4), (3.8), $r < p - \theta$, and Hölder's inequality. Finally, for $\varphi \in W^{1,\infty}(0,T;L^\infty(\Omega))$,

$$\begin{aligned} -\int_\Omega u_n(0)\varphi(0)\,\mathrm{d}x + \int_{Q_T} u_n\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} A(t,x,u_n)|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\ + B\int_{Q_T}\frac{|\nabla u_n|^p}{u_n^\theta}\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} u_n^r\varphi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.19}$$

Using (3.16), (3.17) and (3.18), we can easily pass to the limit in (3.19). Taking into account (3.3) and Lemma 2.10, Theorem 3.1 is proved. □

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