POSITIVE SOLUTIONS OF A FOURTH-ORDER DIFFERENTIAL EQUATION WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. We study the existence of positive solutions to the fourth-order two-point boundary value problem

$$\begin{cases} u''''(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u'(0) = u'(1) = u''(0) = 0, & u(0) = \alpha[u], \end{cases}$$

where $\alpha[u] = \int_0^1 u(t) \mathrm{d}A(t)$ is a Riemann-Stieltjes integral with $A \geqslant 0$ being a nondecreasing function of bounded variation and $f \in \mathcal{C}([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$. The sufficient conditions obtained are new and easy to apply. Their approach is based on Krasnoselskii's fixed point theorem and the Avery-Peterson fixed point theorem.

Keywords: boundary value problem; fixed point; positive solution; cone; existence theorem

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1. Introduction

The study of fourth-order boundary value problems (BVPs) has been a major focus among researchers in recent years due to their applications to problems in heat conduction, thermoelasticity, plasma physics, control theory, and many applied sciences. The study of fourth-order differential equations becomes more important when one uses nonlocal boundary conditions as is evident from the works in [1], [3], [4], [6], [7], [9], [10], [11], [13]–[22]. As can be seen from the above cited articles, the main approaches to the study of nonlocal fourth-order BVPs are the use of Krasnoselskii's fixed point theorem, the Leggett-William fixed point theorem, the upper-lower solution method, the fixed point index property, and the method of successive iterations.

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In a recent work, Benaicha and Haddouchi (see [3]) studied the existence of positive solutions to the fourth-order two-point BVP

$$(1.1)\qquad \begin{cases} u''''(t) + f(u(t)) = 0, & t \in (0,1), \\ u'(0) = u'(1) = u''(0) = 0, & u(0) = \int_0^1 a(s)u(s)\,\mathrm{d}s, \end{cases}$$

where a is a positive continuous function. Later, Haddouchi et al. in [5] extended the results for (1.1) to the problem

$$(1.2)\qquad \begin{cases} u''''(t) + f(u(t)) = 0, & t \in (0,1), \\ u'(0) = u'(1) = u''(0) = 0, & u(0) = \alpha \displaystyle\int_0^1 u(s)\,\mathrm{d}s + \sum_{i=1}^{n} \beta_i u(\eta_i), \end{cases}$$

where $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$, $\alpha \geqslant 0$, $\beta_i \geqslant 0$, $1 \leqslant i \leqslant n$, $0 < \eta_1 < \eta_2 < \ldots < \eta_n < 1$, and $\alpha + \sum\limits_{i=1}^{n} \beta_i < 1$.

In this work, we propose to study the existence of a positive solution to the nonlinear fourth order two point boundary value problem (BVP)

$$(1.3)\qquad \begin{cases} u''''(t) + f(t, u(t)) = 0, & t \in [0,1], \\ u'(0) = u'(1) = u''(0) = 0, & u(0) = \alpha[u], \end{cases}$$

where $\alpha[u] = \int_0^1 u(t)\,\mathrm{d}A(t)$ is the Riemann-Stieltjes integral, A is a function of bounded variation, and $f \in \mathcal{C}([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$. By a *solution* of problem (1.3) we mean a function $u \in \mathcal{C}^{(4)}([0,1], \mathbb{R}_+)$ that satisfies the equation and the boundary conditions. Throughout this paper, we assume that

(H1) $0 < \alpha[1] < 1$ and $A(t) \geqslant 0$ is nondecreasing.

Riemann-Stieltjes integrals of the form $\alpha[u]$ play an important role in the literature covering a variety of nonlocal boundary conditions including the cases:

$$\alpha[u] = \lambda u(\eta), \quad \lambda \geqslant 0,\ \eta \in (0,1);$$

$$\alpha[u] = \sum_{j=1}^{l} \lambda_j u(\mu_j), \quad \lambda_i \in \mathbb{R},\ j = 1, 2, \ldots, l,\ 0 < \eta_1 < \eta_2 < \ldots < \eta_l < 1;$$

$$\alpha[u] = \int_0^1 u(t)h(t)\,\mathrm{d}t, \quad h \in C((0,1), \mathbb{R}).$$

Some important features of $\alpha[u]$ are:

(i) If $\alpha[u] = \sum\limits_{i=1}^{l} \alpha_i u(\eta_i)$, $0 < \eta_i < 1$, then assumption (H1) reduces to $0 < \sum\limits_{i=1}^{l} \alpha_i < 1$.

(ii) If $\alpha[u] = (\eta_2 - \eta_1)^{-1} \int_{\eta_1}^{\eta_2} \alpha t u(t) \, \mathrm{d}t$ with $0 < \eta_1 < \eta_2 < 1$, and α is a positive constant, then assumption (H1) reduces to $0 < \alpha(\eta_1 + \eta_2) < 2$.

(iii) If $\alpha[u] = \alpha \int_0^1 t^m u(t) \, \mathrm{d}t$, $m > -1$, then (H1) reduces to $0 < \alpha < m + 1$.

In view of the above observations, our assumption (H1) is more general than the assumptions considered for (1.1) and (1.2). This motivates us to study the positive solutions of BVP (1.3).

This work is divided into four sections. Section 1 is the introduction. All basic results and two fixed point theorems are presented in Section 2. Section 3 contains the main results in this paper. Two examples are given in Section 4 to enhance our results.

2. Preliminaries

Consider the BVP

$$\begin{cases} u''''(t) + h(t) = 0, & t \in [0,1], \\ u'(0) = u'(1) = u''(0) = 0, & u(0) = \alpha[u], \end{cases} \tag{2.1}$$

where $h \geqslant 0$ and $\alpha[u] = \int_0^1 u(t) \, \mathrm{d}A(t)$ is a Riemann-Stieltjes integral with $A \geqslant 0$ being a function of bounded variation. It will be convenient to define

$$\varrho(t) = \min\{t^3, t^2(1-t)\} = \begin{cases} t^3, & t \leqslant \frac{1}{2}, \\ t^2(1-t), & t \geqslant \frac{1}{2}, \end{cases} \tag{2.2}$$

and

$$\delta := \theta^3 \left(1 - \alpha[1] + \int_\theta^{1-\theta} \mathrm{d}A(t)\right) < 1. \tag{2.3}$$

BVP (2.1) can be expressed as an equivalent integral equation, which is given in the following lemma.

Lemma 2.1. *For any $h \in C[0,1]$, BVP (2.1) has the unique solution $u(t)$ given by*

$$u(t) = \int_0^1 G(t,s) h(s) \, \mathrm{d}s + \frac{1}{(1 - \alpha[1])} \int_0^1 \int_0^1 G(t,s) h(s) \, \mathrm{d}s \, \mathrm{d}A(t), \tag{2.4}$$

where $G(t,s)$ is Green's kernel

$$G(t,s) = \frac{1}{6} \begin{cases} t^3(1-s)^2 - (t-s)^3, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^3(1-s)^2, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{2.5}$$

P r o o f. Repeated integration of the equation $u''''(t)+h(t)=0$ from 0 to t gives

$$u''(t) = -\int_0^t (t-s)h(s)\,\mathrm{d}s + c_1 t + c_2, \tag{2.6}$$

$$u'(t) = -\frac{1}{2}\int_0^t (t-s)^2 h(s)\,\mathrm{d}s + \frac{c_1}{2}t^2 + c_2 t + c_3 \tag{2.7}$$

and

$$u(t) = -\frac{1}{6}\int_0^t (t-s)^3 h(s)\,\mathrm{d}s + \frac{c_1}{6}t^3 + \frac{c_2}{2}t^2 + c_3 t + c_4, \tag{2.8}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. The conditions $u'(0)=0$ and $u''(0)=0$ imply $c_2=0$ and $c_3=0$. Hence, from (2.7) we have

$$u'(t) = -\frac{1}{2}\int_0^t (t-s)^2 h(s)\,\mathrm{d}s + \frac{c_1}{2}t^2,$$

which, together with the condition $u'(1)=0$, gives

$$c_1 = \int_0^1 (1-s)^2 h(s)\,\mathrm{d}s.$$

Thus, (2.8) becomes

$$u(t) = \frac{t^3}{6}\int_0^1 (1-s)^2 h(s)\,\mathrm{d}s - \frac{1}{6}\int_0^t (t-s)^3 h(s)\,\mathrm{d}s + c_4. \tag{2.9}$$

Multiplying both sides of (2.9) by $\mathrm{d}A(t)$ and integrating the resulting identity from 0 to 1, we obtain

$$\alpha[u] = \frac{1}{6}\alpha[t^3]\int_0^1 (1-s)^2 h(s)\,\mathrm{d}s - \frac{1}{6}\int_0^1\int_0^t (t-s)^3 h(s)\,\mathrm{d}s\,\mathrm{d}A(t) + c_4\alpha[1], \tag{2.10}$$

where $\alpha[t^3] = \int_0^1 t^3\,\mathrm{d}A(t)$. Thus, from (2.9), (2.10), and the boundary condition $u(0)=\alpha[u]$, we obtain

$$c_4 = \frac{1}{6(1-\alpha[1])}\Big(\alpha[t^3]\int_0^1 (1-s)^2 h(s)\,\mathrm{d}s - \int_0^1\int_0^t (t-s)^3 h(s)\,\mathrm{d}s\,\mathrm{d}A(t)\Big).$$

Hence, from (2.9) we have

$$
\begin{aligned}
u(t) &= \frac{t^3}{6}\int_0^1 (1-s)^2 h(s)\,\mathrm{d}s - \frac{1}{6}\int_0^t (t-s)^3 h(s)\,\mathrm{d}s \\
&\quad + \frac{1}{6(1-\alpha[1])}\biggl(\alpha[t^3]\int_0^1 (1-s)^2 h(s)\,\mathrm{d}s - \int_0^1\int_0^t (t-s)^3 h(s)\,\mathrm{d}s\,\mathrm{d}A(t)\biggr) \\
&= \frac{1}{6}\int_0^t [t^3(1-s)^2 - (t-s)^3] h(s)\,\mathrm{d}s + \frac{1}{6}\int_t^1 t^3(1-s)^2 h(s)\,\mathrm{d}s + \frac{1}{6(1-\alpha[1])} \\
&\quad \times \biggl(\int_0^1\int_0^t t^3(1-s)^2 h(s)\,\mathrm{d}s\,\mathrm{d}A(t) + \int_0^1\int_t^1 t^3(1-s)^2 h(s)\,\mathrm{d}s\,\mathrm{d}A(t)\biggr) \\
&\quad - \frac{1}{6(1-\alpha[1])}\int_0^1\int_0^t (t-s)^3 h(s)\,\mathrm{d}s\,\mathrm{d}A(t) \\
&= \frac{1}{6}\int_0^t [t^3(1-s)^2 - (t-s)^3] h(s)\,\mathrm{d}s + \frac{1}{6}\int_t^1 t^3(1-s)^2 h(s)\,\mathrm{d}s + \frac{1}{6(1-\alpha[1])} \\
&\quad \times \int_0^1 \biggl(\int_0^t [t^3(1-s)^2 - (t-s)^3] h(s)\,\mathrm{d}s + \int_t^1 t^3(1-s)^2 h(s)\,\mathrm{d}s\biggr)\,\mathrm{d}A(t) \\
&= \int_0^1 G(t,s)h(s)\,\mathrm{d}s + \frac{1}{(1-\alpha[1])}\int_0^1\int_0^1 G(t,s)h(s)\,\mathrm{d}s\,\mathrm{d}A(t).
\end{aligned}
$$

This proves the lemma. $\square$

Green's function $G(t,s)$ defined in (2.5) is the same as the one obtained in [3]. We apply the following lemma, which can be found in [3], Lemma 2.3 or [5], Lemma 2.6.

Lemma 2.2. *Let $\theta \in (0,\frac{1}{2})$ be fixed. Then Green's function $G(t,s)$ has the following properties:*

(i) *$G(t,s) \geqslant 0$ for all $t,s \in [0,1]$;*

(ii) *for all $t,s \in [0,1]$ we have*

$$\frac{1}{6}\varrho(t)s(1-s)^2 \leqslant G(t,s) \leqslant \frac{1}{6}s(1-s)^2. \tag{2.11}$$

Since (2.11) is true for any $t,s \in [0,1]$, for any fixed $\theta \in (0,\frac{1}{2})$ we have

$$\frac{1}{6}\theta^3 s(1-s)^2 \leqslant G(t,s) \leqslant \frac{1}{6}s(1-s)^2 \text{ for all } s \in [0,1] \text{ and } t \in [\theta, 1-\theta]. \tag{2.12}$$

In this paper, we take $X = \mathcal{C}[0,1]$ to be the Banach space with the standard norm

$$\|u\| = \max_{0\leqslant t\leqslant 1} |u(t)|. \tag{2.13}$$

Then we have the following lemma.

Lemma 2.3. *Let $\theta \in (0, \frac{1}{2})$. Any solution $u(t)$ of (2.1) in X satisfies*

$$\min_{t\in[\theta,1-\theta]} u(t) \geqslant \delta\|u\|. \tag{2.14}$$

Proof. From (2.4) and (2.11) we have

$$\begin{aligned}
\|u\| &= \max_{0\leqslant t\leqslant 1}\left|\int_0^1 G(t,s)h(s)\,\mathrm{d}s + \frac{1}{(1-\alpha[1])}\int_0^1\int_0^1 G(t,s)h(s)\,\mathrm{d}s\,\mathrm{d}A(t)\right| \\
&\leqslant \frac{1}{6}\int_0^1 s(1-s)^2h(s)\,\mathrm{d}s + \frac{1}{6(1-\alpha[1])}\int_0^1\int_0^1 s(1-s)^2h(s)\,\mathrm{d}s\,\mathrm{d}A(t) \\
&= \frac{1}{6}\Big(\int_0^1 s(1-s)^2h(s)\,\mathrm{d}s + \frac{\alpha[1]}{(1-\alpha[1])}\int_0^1 s(1-s)^2h(s)\,\mathrm{d}s\Big) \\
&= \frac{1}{6(1-\alpha[1])}\int_0^1 s(1-s)^2h(s)\,\mathrm{d}s.
\end{aligned} \tag{2.15}$$

On the other hand, from (2.12),

$$\begin{aligned}
\min_{t\in[\theta,1-\theta]} u(t) &= \int_0^1 \min_{t\in[\theta,1-\theta]} G(t,s)h(s)\,\mathrm{d}s \\
&\quad + \frac{1}{(1-\alpha[1])}\int_0^1\int_0^1 \min_{t\in[\theta,1-\theta]} G(t,s)h(s)\,\mathrm{d}s\,\mathrm{d}A(t) \\
&\geqslant \frac{\theta^3}{6}\int_0^1 s(1-s)^2h(s)\,\mathrm{d}s + \frac{\theta^3}{6(1-\alpha[1])}\int_0^1 s(1-s)^2h(s)\,\mathrm{d}s\int_\theta^{1-\theta}\,\mathrm{d}A(t) \\
&= \frac{\theta^3}{6}\frac{(1-\alpha[1]+\int_\theta^{1-\theta}\,\mathrm{d}A(t))}{(1-\alpha[1])}\int_0^1 s(1-s)^2h(s)\,\mathrm{d}s,
\end{aligned}$$

which, in view of (2.15), implies (2.14). The lemma is now proved. □

We define two cones K and P on X by

$$K = \{u \in X\colon\ u(t) \geqslant 0,\ u(t) \geqslant \varrho(t)\|u\|,\ 0 \leqslant t \leqslant 1\} \tag{2.16}$$

and

$$P = \Big\{u \in X\colon\ u(t) \geqslant 0,\ \min_{t\in[\theta,1-\theta]} u(t) \geqslant \delta\|u\|\Big\}, \tag{2.17}$$

and an operator $T\colon\ K \to X$ by

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))\,\mathrm{d}s + \frac{1}{(1-\alpha[1])}\int_0^1\int_0^1 G(t,s)f(s,u(s))\,\mathrm{d}s\,\mathrm{d}A(t), \tag{2.18}$$

where $G(t,s)$ is Green's function given in (2.5). Then in view of Lemmas 2.1–2.3, we have the following lemma.

Lemma 2.4. *A continuous function $u(t)$ is a positive solution of BVP* (1.3) *if and only if $u(t)$ is a fixed point of the operator T on the cone K. Also, $T(K)\subset K$.*

Let us denote, for any $r>0$,

$$K_r=\{u\in K\colon\ \|u\|<r\}\quad\text{and}\quad\partial K_r=\{u\in K\colon\ \|u\|=r\}.$$

Lemma 2.5. *The operator $T\colon\ \overline{K}_{r_2}\setminus K_{r_1}\to K$ is completely continuous, where r_1 and r_2 are positive real numbers with $r_1<r_2$.*

Proof. Since f is continuous on $[0,1]$, there exists a continuous function $P_f\colon (0,1)\to[0,\infty)$ and positive numbers r_1 and r_2 with $r_1<r_2$, such that $f(t,u)\leqslant P_f(t)$ for $0\leqslant t\leqslant 1$ and $\delta r_1\leqslant u\leqslant r_2$, and $\int_0^1 s(1-s)^2P_f(s)\,\mathrm{d}s<\infty$. Clearly, T is continuous on $\overline{K}_{r_2}\setminus K_{r_1}$. For any $u\in\overline{K}_{r_2}\setminus K_{r_1}$ we have

$$\begin{aligned}|Tu|&=\int_0^1 G(t,s)f(s,u(s))\,\mathrm{d}s+\frac{1}{(1-\alpha[1])}\int_0^1\int_0^1 G(t,s)f(s,u(s))\,\mathrm{d}s\,\mathrm{d}A(t)\\&\leqslant\frac{1}{6}\int_0^1 s(1-s)^2f(s,u(s))\,\mathrm{d}s+\frac{1}{6(1-\alpha[1])}\int_0^1\int_0^1 s(1-s)^2f(s,u(s))\,\mathrm{d}s\,\mathrm{d}A(t)\\&=\frac{1}{6}\Big(\int_0^1 s(1-s)^2f(s,u(s))\,\mathrm{d}s+\frac{\alpha[1]}{(1-\alpha[1])}\int_0^1 s(1-s)^2f(s,u(s))\,\mathrm{d}s\Big)\\&=\frac{1}{6(1-\alpha[1])}\int_0^1 s(1-s)^2P_f(s)\,\mathrm{d}s,\end{aligned}$$

which implies that T is uniformly bounded. Since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous there, so for every $\varepsilon>0$ there exists $\delta_1>0$ such that for all $(t_1,s),(t_2,s)\in[0,1]\times[0,1]$ with $|t_1-t_2|<\delta_1$, we have $|G(t_1,s)-G(t_2,s)|<\varepsilon$. So, for any $u\in\overline{K}_{r_2}\setminus K_{r_1}$ and $t_1,t_2\in[0,1]$ with $|t_1-t_2|<\delta_1$, we have

$$|(Tu)(t_2)-(Tu)(t_1)|\leqslant\varepsilon\frac{1}{(1-\alpha[1])}\int_0^1 P_f(s)\,\mathrm{d}s.$$

Hence, T is equicontinuous. Consequently, T is relatively compact on $\overline{K}_{r_2}\setminus K_{r_1}$, and hence compact on $\overline{K}_{r_2}\setminus K_{r_1}$. Thus, the operator $T\colon\ \overline{K}_{r_2}\setminus K_{r_1}\to K$ is completely continuous, and this proves the lemma. □

In this paper, we shall use two very valuable fixed point theorems, Krasnosel'skii's fixed point theorem and the Avery-Peterson fixed point theorem, to prove our results.

Theorem 2.1 ([8], Krasnosel'skii's fixed point theorem)**.** *Let X be a real Banach space and $K\subset X$ be a cone in X. Assume that K_1 and K_2 are bounded open subsets of X with $0\in K_1$, $\overline{K_1}\subset K_2$, and $T\colon\ K\cap(\overline{K_2}\setminus K_1)\to K$ is a completely continuous operator such that either*

(i) $\|Tu\| \leqslant \|u\|$, $u \in K \cap \partial K_1$ *and* $\|Tu\| \geqslant \|u\|$, $u \in K \cap \partial K_2$*; or*
(ii) $\|Tu\| \geqslant \|u\|$, $u \in K \cap \partial K_1$ *and* $\|Tu\| \leqslant \|u\|$, $u \in K \cap \partial K_2$.

Then T *has a fixed point in* $K \cap (\overline{K_2} \setminus K_1)$.

In order to state the Avery-Peterson fixed point theorem, we need to define some functions in a cone. A map Φ is said to be a nonnegative continuous concave functional on a cone K in a real Banach space X if $\Phi\colon K \to [0, \infty)$ is continuous and

$$\Phi(tx + (1-t)y) \geqslant t\Phi(x) + (1-t)\Phi(y)$$

for all x, $y \in K$ and $t \in [0, 1]$. Similarly, the map φ is a nonnegative continuous convex functional on K if $\varphi\colon K \to [0, \infty)$ is continuous and

$$\varphi(tx + (1-t)y) \leqslant t\varphi(x) + (1-t)\varphi(y)$$

for all $x, y \in K$ and $t \in [0, 1]$. We use the following notations as introduced by Avery and Peterson (see [2]). Let φ and Θ be nonnegative convex functionals on K and let Φ be a nonnegative continuous concave functional on K. Also, let ψ be a nonnegative continuous functional on K. Then, for positive numbers r_1, r_2, r_3 and r_4, we define the sets:

$$\begin{aligned} K(\varphi, r_4) &= \{u \in K\colon\ \varphi(x) < r_4\}, \quad \overline{K(\varphi, r_4)} = \{u \in K\colon\ \varphi(x) \leqslant r_4\}, \\ K(\varphi, \Phi, r_2, r_4) &= \{u \in K\colon\ r_2 \leqslant \Phi(u), \varphi(u) \leqslant r_4\}, \\ K(\varphi, \Theta, \Phi, r_2, r_3, r_4) &= \{u \in K\colon\ r_2 \leqslant \Phi(u), \Theta(u) \leqslant r_3, \varphi(u) \leqslant r_4\}, \\ K(\varphi, \psi, r_1, r_4) &= \{u \in K\colon\ r_1 \leqslant \psi(u), \varphi(u) \leqslant r_4\}. \end{aligned} \tag{2.19}$$

The following theorem will be used to establish the existence of multiple positive solutions to our BVP.

Theorem 2.2 ([2], Avery-Peterson fixed point theorem). *Let K be a cone in a real Banach space X, let φ and Θ be nonnegative continuous convex functionals on K, let Φ be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K satisfying $\psi(ku) \leqslant k\psi(u)$ for $0 \leqslant k \leqslant 1$, and such that for some positive numbers M and r_4 we have $\Phi(u) \leqslant \psi(u)$ and $\|u\| \leqslant M\varphi(u)$ for all $u \in K(\varphi, r_4)$. Assume that $T\colon \overline{K(\varphi, r_4)} \to \overline{K(\varphi, r_4)}$ is a completely continuous operator and there exist constants r_1, r_2 and r_3 with $r_1 < r_2$ such that:*

(S1) *$\{u \in K(\varphi, \Theta, \Phi, r_2, r_3, r_4)\colon\ \varphi(u) > r_2\}$ is nonempty and $\Phi(Tu) > r_2$ for $u \in K(\varphi, \Theta, \Phi, r_2, r_3, r_4)$;*
(S2) *$\Phi(Tu) > r_2$ for $u \in K(\varphi, \Phi, r_2, r_4)$ with $\Theta(Tu) > r_3$;*
(S3) *$0 \notin K(\varphi, \psi, r_1, r_4)$ and $\psi(Tu) < r_1$ for $u \in K(\varphi, \psi, r_1, r_4)$ with $\psi(u) = r_1$.*

Then T has at least three fixed points $u_1, u_2, u_3 \in \overline{K(\varphi, r_4)}$ such that $\varphi(u_i) \leqslant r_4$, $i = 1, 2, 3$, $r_2 < \Phi(u_1)$, $r_1 < \Phi(u_2)$, $\Phi(u_2) < r_2$ and $\psi(u_3) < r_1$.

3. Main results

In this section, we shall apply Theorems 2.1 and 2.2 to obtain the existence of positive solutions to BVP (1.3). We introduce the following height function to control the growth of the nonlinear term $f(t,u)$. For any $r>0$ and $0\leqslant t\leqslant 1$, let

$$f_1(t,r)=\min\{f(t,u)\colon\ \varrho(t)r\leqslant u\leqslant r\}$$

and

$$f_2(t,r)=\max\{f(t,u)\colon\ \varrho(t)r\leqslant u\leqslant r\}.$$

Proceeding as in the lines of the proof of Theorem 4.5 in [12], we can prove the following theorem.

Theorem 3.1. *Assume there exist constants r_1 and r_2 with $0<r_1<r_2$ such that either*

(H2) $r_1\leqslant \frac{1}{48}(1-\alpha[1])^{-1}\int_0^1 s(1-s)^2 f_1(s,r_1)\,\mathrm{d}s<\infty$ *and* $\frac{1}{6}(1-\alpha[1])^{-1}\int_0^1 s(1-s)^2\times f_2(s,r_2)\,\mathrm{d}s\leqslant r_2$, *or*

(H3) $\frac{1}{6}(1-\alpha[1])^{-1}\int_0^1 s(1-s)^2 f_2(s,r_1)\,\mathrm{d}s\leqslant r_1$ *and* $r_2\leqslant \frac{1}{48}(1-\alpha[1])^{-1}\int_0^1 s(1-s)^2\times f_1(s,r_2)\,\mathrm{d}s<\infty$.

Then BVP (1.3) has at least one positive solution $u^*(t)$ in K with $r_1\leqslant u^*(t)\leqslant r_2$ for $0\leqslant t\leqslant 1$.

Remark 3.1. Although the conditions in Theorem 3.1 look simple and examples can be easily constructed, the following result for the existence of positive solutions to (1.3) covers a wide range of functions and is easily verifiable.

For convenience, we introduce the following notations:

$$f_0=\lim_{u\to 0+}\Big\{\min_{t\in[0,1]}\frac{f(t,u)}{u}\Big\},\quad f^0=\lim_{u\to 0+}\Big\{\max_{t\in[0,1]}\frac{f(t,u)}{u}\Big\},$$
$$f_\infty=\lim_{u\to \infty}\Big\{\min_{t\in[0,1]}\frac{f(t,u)}{u}\Big\},\quad f^\infty=\lim_{u\to \infty}\Big\{\max_{t\in[0,1]}\frac{f(t,u)}{u}\Big\}.$$

Set

$$(3.1)\quad \Lambda_\theta=\frac{72(1-\alpha[1])}{\theta^3(1-2\theta)(1-2\theta^2+2\theta)\big(1-\alpha[1]+\int_\theta^{1-\theta}\mathrm{d}A(t)\big)}\quad \text{for any } \theta\in(0,\tfrac{1}{2}).$$

Theorem 3.2. *If either*

(H4) $f^0=0$ *and* $f_\infty=\infty$, *or*

(H5) $f_0=\infty$ *and* $f^\infty=0$,

then problem (1.3) has at least one positive solution.

P r o o f. We consider the cone P defined in (2.17) to prove this theorem. Fix $\theta \in (0, \frac{1}{2})$. First, suppose that (H4) holds. Since $f^0 = 0$, there exists $r_1 > 0$ such that $f(t,u) \leqslant \varepsilon u$ for all $0 < u \leqslant r_1$ and $t \in [0,1]$, where $\varepsilon > 0$ is chosen so that $\frac{1}{72}\varepsilon/(1-\alpha[1]) \leqslant 1$. Set $\Omega_{r_1} = \{u \in P\colon\ \|u\| < r_1\}$. Then for $u \in P \cap \partial\Omega_{r_1}$ we have $\delta r_1 = \delta\|u\| \leqslant \min\limits_{t\in[\theta,1-\theta]} u(t) \leqslant \|u\| = r_1$, and so using (2.12),

$$\begin{aligned}
\|Tu\| &= \max_{0\leqslant t\leqslant 1} \left| \int_0^1 G(t,s) f(s,u(s))\,\mathrm{d}s + \frac{1}{(1-\alpha[1])} \int_0^1 \int_0^1 G(t,s) f(s,u(s))\,\mathrm{d}s\,\mathrm{d}A(t) \right| \\
&\leqslant \frac{1}{6} \int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s + \frac{1}{6(1-\alpha[1])} \int_0^1 \int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s\,\mathrm{d}A(t) \\
&= \frac{1}{6(1-\alpha[1])} \int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s \leqslant \frac{1}{6(1-\alpha[1])} \int_0^1 s(1-s)^2 \varepsilon u(s)\,\mathrm{d} \\
&\leqslant \|u\|\varepsilon \frac{1}{72(1-\alpha[1])} \leqslant \|u\|.
\end{aligned}$$

Thus, $\|Tu\| \leqslant \|u\|$ for $u \in P \cap \partial\Omega_{r_1}$. Since $f_\infty = \infty$, there exists $\overline{r}_2 > 0$ such that $f(t,u) \geqslant \mu u$ for all $u > \overline{r}_2$ and $t \in [\theta, 1-\theta]$, where $\mu > 0$ is chosen so that $\mu\delta\Lambda_\theta \geqslant 1$. Now set $r_2 = \max\{r_1/\delta, \overline{r}_2/\delta\}$ and $\Omega_{r_2} = \{u \in P\colon\ \|u\| < r_2\}$. Then for all $u \in P \cap \partial\Omega_{r_2}$ we have $\delta r_2 = \delta\|u\| \leqslant \min\limits_{t\in[\theta,1-\theta]} u(t) \leqslant \|u\| = r_2$ and so

$$\begin{aligned}
\|Tu\| \geqslant \min_{t\in[\theta,1-\theta]} Tu(t) = {} & \int_0^1 \Big(\min_{t\in[\theta,1-\theta]} G(t,s) \Big) f(s,u(s))\,\mathrm{d}s \\
& + \frac{1}{(1-\alpha[1])} \int_0^1 \int_0^1 \Big(\min_{t\in[\theta,1-\theta]} G(t,s) \Big) f(s,u(s))\,\mathrm{d}s\,\mathrm{d}A(t) \\
\geqslant {} & \frac{\theta^3}{6} \int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s \\
& + \frac{\theta^3}{6(1-\alpha[1])} \int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s \int_\theta^{1-\theta} \mathrm{d}A(t) \\
= {} & \frac{\theta^3}{6} \frac{(1-\alpha[1] + \int_\theta^{1-\theta} \mathrm{d}A(t))}{(1-\alpha[1])} \int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s \\
\geqslant {} & \mu \frac{\theta^3}{6} \frac{(1-\alpha[1] + \int_\theta^{1-\theta} \mathrm{d}A(t))}{(1-\alpha[1])} \int_0^1 s(1-s)^2 u(s)\,\mathrm{d}s \\
\geqslant {} & \mu \frac{\theta^3}{6} \frac{(1-\alpha[1] + \int_\theta^{1-\theta} \mathrm{d}A(t))}{(1-\alpha[1])} \int_\theta^{1-\theta} s(1-s)^2 u(s)\,\mathrm{d}s \\
\geqslant {} & \mu\delta\|u\| \frac{\theta^3}{6} \frac{(1-\alpha[1] + \int_\theta^{1-\theta} \mathrm{d}A(t))}{(1-\alpha[1])} \int_\theta^{1-\theta} s(1-s)^2\,\mathrm{d}s \geqslant \|u\|
\end{aligned}$$

since $\int_\theta^{1-\theta} s(1-s)^2\,\mathrm{d}s = \frac{1}{12}(1 - 6\theta^2 + 4\theta^3)$. This implies that $\|Tu\| \geqslant \|u\|$ for $u \in P \cap \partial\Omega_{r_2}$. Hence, by Theorem 2.1 (i), BVP (1.3) has a positive solution.

Next, suppose that (H5) holds. Since $f_0 = \infty$, there exists $r_1 > 0$ such that $f(t,u) \geqslant \mu u$ for all $0 < u \leqslant r_1$, where $\mu > 0$ is chosen so that $\mu\delta\Lambda_\theta \geqslant 1$. Then for all $u \in P \cap \partial\Omega_{r_1}$, a similar calculation shows that

$$\|Tu\| \geqslant \min_{t\in[\theta,1-\theta]} Tu(t) \geqslant \|u\|.$$

Hence, $\|Tu\| \geqslant \|u\|$ for $u \in P \cap \partial\Omega_{r_1}$.

Now, $f^\infty = 0$, so there exists $\tilde{r}_2 > r_1$ such that $f(t,u) \leqslant \varepsilon u$ for all $t \in [0,1]$ and $u \geqslant \tilde{r}_2$, where $\varepsilon > 0$ is chosen so that $0 < \frac{1}{72}\varepsilon/(1-\alpha[1]) \leqslant 1$. We consider two possibilities.

If f is bounded, then there exists $L > 0$ such that $f(t,u) \leqslant L$. Let $\Omega_{r_2} = \{u \in P\colon \|u\| < r_2\}$, where $r_2 = \max\{r_1/\delta, L\Lambda_\theta\}$. Then, proceeding as in the first part of the proof of this theorem, where (H4) holds, we obtain $\|Tu\| \leqslant r_2 = \|u\|$ for $u \in P \cap \Omega_{r_2}$.

If f is unbounded, then there exists $\varrho > 0$ such that $f(t,u) \leqslant \varepsilon\varrho$ with $0 < u \leqslant \tilde{r}_2$ and $t \in [0,1]$. Let $\Omega_{r_2} = \{u \in P\colon \|u\| < r_2\}$, where $r_2 = \max\{\varrho, \tilde{r}_2\}$. So, for $u \in P \cap \partial\Omega_{r_2}$ we have $f(t,u) \leqslant \varepsilon r_2$, and proceeding as above, we obtain $\|Tu\| \leqslant r_2 = \|u\|$.

Hence, by Theorem 2.1 (ii), BVP (1.3) has at least one positive solution. This completes the proof of the theorem. $\square$

Now, we shall apply Theorem 2.2 to find sufficient conditions for the existence of three positive solutions to BVP (1.3). We let K be a cone in a Banach space X and define sets as in (2.19).

Theorem 3.3. *Let $\theta \in (0, \frac{1}{2})$ be fixed and assume that there exists a continuous function $f_0\colon [0,1] \to [0,\infty)$ such that*

(H6) $\int_0^1 s(1-s)^2 f_0(s)\,\mathrm{d}s \leqslant 6(1-\alpha[1])$.

In addition, assume that there exist constants r_1, r_2, r_3 and r_4 with

$$0 < r_1 < r_2 < \frac{r_2}{\delta} = r_3 \leqslant r_4$$

such that:

(H7) $f(t,u) < f_0(t) r_1$ *for* $0 \leqslant u(t) \leqslant r_1$ *and* $0 \leqslant t \leqslant 1$;

(H8) $f(t,u) > \Lambda_\theta r_2$ *for* $r_2 \leqslant u(t) \leqslant r_2/\delta$ *and* $\theta \leqslant t \leqslant 1-\theta$;

(H9) $f(t,u) \leqslant f_0(t) r_4$ *for* $0 \leqslant u(t) \leqslant r_4$ *and* $0 \leqslant t \leqslant 1$.

Then BVP (1.3) has at least three positive solutions u_i with $\|u_i\| \leqslant r_4$, $i = 1,2,3$, $r_2 < \min_{t\in[\theta,1-\theta]} |u_1(t)|$, $r_1 < \min_{t\in[\theta,1-\theta]} |u_2(t)| < r_2$ and $\|u_3\| < r_1$.

P r o o f. We define a nonnegative continuous concave functional Φ on the cone K by

$$\Phi(u)=\min_{t\in[\theta,1-\theta]}|u(t)|$$

so that $\Phi(u)\leqslant\|u\|$. We also consider two nonnegative continuous convex functionals φ and Θ on K given by $\Theta(u)=\varphi(u)=\|u\|$, and a nonnegative continuous functional ψ on K defined by $\psi(u)=\|u\|$. Then

$$\begin{aligned}\psi(ku)&=\|ku\|\leqslant|k|\|u\|\leqslant|k|\psi(u)=k\psi(u)\quad\text{for } 0\leqslant k\leqslant 1,\\ \Phi(u)&=\min_{t\in[\theta,1-\theta]}|u(t)|\leqslant\|u\|=\psi(u),\end{aligned}$$

and we can find a constant $M\geqslant 1$ such that

$$\|u\|=\varphi(u)\leqslant M\varphi(u)\quad\text{for every } u\in\overline{K(\varphi,r_4)}.$$

We consider the operator $T\colon K\to X$ defined in the same way as in (2.18). Clearly, $u(t)$ is a solution of BVP (1.3) if and only if it is a fixed point of T on K. It can also be shown that $T(K)\subseteq K$.

Let $u\in\overline{K(\varphi,r_4)}$. Then $\varphi(u)=\|u\|\leqslant r_4$ for $0\leqslant u\leqslant r_4$ and $0\leqslant t\leqslant 1$. Hence, by (2.12), (H6) and (H9), we have

$$\begin{aligned}\varphi(Tu)&=\|Tu\|\\ &=\max_{0\leqslant t\leqslant 1}\Big|\int_0^1 G(t,s)f(s,u(s))\,\mathrm{d}s+\frac{1}{(1-\alpha[1])}\int_0^1\int_0^1 G(t,s)f(s,u(s))\,\mathrm{d}s\,\mathrm{d}A(t)\Big|\\ &\leqslant\frac{1}{6}\int_0^1 s(1-s)^2f(s,u(s))\,\mathrm{d}s+\frac{1}{6(1-\alpha[1])}\int_0^1\int_0^1 s(1-s)^2f(s,u(s))\,\mathrm{d}s\,\mathrm{d}A(t)\\ &=\frac{1}{6(1-\alpha[1])}\int_0^1 s(1-s)^2f(s,u(s))\,\mathrm{d}s\\ &\leqslant r_4\frac{1}{6(1-\alpha[1])}\int_0^1 s(1-s)^2f_0(s)\,\mathrm{d}s\leqslant r_4.\end{aligned}$$

This shows that $T\colon\overline{K(\varphi,r_4)}\to\overline{K(\varphi,r_4)}$.

Next, we prove that $T\colon\overline{K(\varphi,r_4)}\to\overline{K(\varphi,r_4)}$ is completely continuous. From the continuity of $G(t,s)$ and $f(t,u)$ for $t,s\in[0,1]$, it follows that T is continuous on K. Setting

$$M_1=\max_{0\leqslant t\leqslant 1,\,u\in[0,r_4]}f(t,u),$$

we have

$$|(Tu)(t)|\leqslant\frac{M_1}{72(1-\alpha[1])},$$

which implies that T is uniformly bounded on $\overline{K(\varphi,r_4)}$. Since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous there. Hence, for every $\varepsilon>0$ there exists

$\delta_1 > 0$ such that $|G(t_1,s) - G(t_2,s)| < \varepsilon$ for (t_1,s) and $(t_2,s) \in [0,1] \times [0,1]$ with $|t_1 - t_2| < \delta_1$. Consequently, for any $u \in \overline{K(\varphi, r_4)}$ and $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta_1$, we have

$$|(Tu)(t_2) - (Tu)(t_1)| \leqslant \varepsilon \frac{1}{(1-\alpha[1])} M_1.$$

That is, $T(\overline{K(\varphi, r_4)})$ is equicontinuous and so the set $T(\overline{K(\varphi, r_4)})$ is relatively compact. Thus, by the Arzelà-Ascoli theorem, for the convex functional $\varphi(u) = \|u\|$, the mapping $T\colon \overline{K(\varphi, r_4)} \to \overline{K(\varphi, r_4)}$ is completely continuous. Set

$$\begin{aligned} u_0(t) &= \frac{r_2 + r_3}{2} = \frac{1}{2}\left(r_2 + \frac{r_2}{\delta}\right) = \frac{r_2}{2\delta}(1+\delta) < \frac{r_2}{\delta} = r_3, \\ \Phi(u_0) &= \min_{t \in [\theta, 1-\theta]} |u_0| = \frac{r_2 + r_3}{2} > \frac{2r_2}{2} = r_2 \end{aligned}$$

and

$$\varphi(u_0) = \frac{r_2 + r_3}{2} = \frac{r_2}{2\delta} < \frac{r_2}{\delta} = r_3 \leqslant r_4.$$

This implies that the set $\{u \in K(\varphi, \Theta, \Phi, r_2, r_3, r_4)\colon \Phi(x) > r_2\}$ is nonempty.

Now, let $r_2 \leqslant u(t) \leqslant r_3 = r_2/\delta$ with $t \in [\theta, 1-\theta]$. Then, by (2.12), (H6), (H8) and (3.1), we have

$$\begin{aligned} \Phi(Tu) = \min_{t \in [\theta, 1-\theta]} Tu(t) &= \int_0^1 \min_{t \in [\theta, 1-\theta]} G(t,s) f(s, u(s))\, \mathrm{d}s \\ &\quad + \frac{1}{(1-\alpha[1])} \int_0^1 \int_0^1 \min_{t \in [\theta, 1-\theta]} G(t,s) f(s, u(s))\, \mathrm{d}s\, \mathrm{d}A(t) \\ &\geqslant \frac{\theta^3}{6} \int_0^1 s(1-s)^2 f(s, u(s))\, \mathrm{d}s \\ &\quad + \frac{\theta^3}{6(1-\alpha[1])} \int_0^1 s(1-s)^2 f(s, u(s))\, \mathrm{d}s \int_\theta^{1-\theta} \mathrm{d}A(t) \\ &= \frac{\theta^3}{6} \frac{(1 - \alpha[1] + \int_\theta^{1-\theta} \mathrm{d}A(t))}{(1-\alpha[1])} \int_0^1 s(1-s)^2 f(s, u(s))\, \mathrm{d}s > r_2 \end{aligned}$$

for $u \in K(\varphi, \Theta, \Phi, r_2, r_3, r_4)$. Hence, condition (S1) of Theorem 2.2 is satisfied.

Next assume that $u \in K(\varphi, \Phi, r_2, r_4)$ with $\Theta(Tu) > r_3$. Since $T(P) \subset P$,

$$\Phi(Tu) = \min_{t \in [\theta, 1-\theta]} (Tu)(t) \geqslant \delta \|Tu\| = \delta \Theta(Tu) > \delta r_3 = r_2,$$

which shows that (S2) of Theorem 2.2 is satisfied.

Since $\varphi(0) = 0 < r_1$ implies that $\varphi \in R(\varphi, \psi, r_1, r_4)$, for $u \in R(\varphi, \psi, r_1, r_4)$ with $\psi(u) = \|u\| = r_1$ we have, using (2.12), (H6) and (H7),

$$\begin{aligned}
\psi(Tu) &= \|Tu\| \\
&= \max_{0 \leqslant t \leqslant 1} \left| \int_0^1 G(t,s) f(s, u(s))\, \mathrm{d}s + \frac{1}{(1-\alpha[1])} \int_0^1 \int_0^1 G(t,s) f(s,u(s))\, \mathrm{d}s\, \mathrm{d}A(t) \right| \\
&\leqslant \frac{1}{6} \int_0^1 s(1-s)^2 f(s,u(s))\, \mathrm{d}s + \frac{1}{6(1-\alpha[1])} \int_0^1 \int_0^1 s(1-s)^2 f(s,u(s))\, \mathrm{d}s\, \mathrm{d}A(t) \\
&= \frac{1}{6(1-\alpha[1])} \int_0^1 s(1-s)^2 f(s,u(s))\, \mathrm{d}s \\
&\leqslant r_1 \frac{1}{6(1-\alpha[1])} \int_0^1 s(1-s)^2 f_0(s)\, \mathrm{d}s < r_1.
\end{aligned}$$

Hence, (S3) of Theorem 2.2 is satisfied. Therefore, by Theorem 2.2, BVP (1.3) has at least three positive solutions u_i with $\|u_i\| \leqslant r_4$, $i = 1,2,3$, $r_2 < \min\limits_{t \in [\theta, 1-\theta]} |u_1(t)|$, $r_1 < \min\limits_{t \in [\theta, 1-\theta]} |u_2(t)| < r_2$, and $\|u_3\| < r_1$. The proof of the theorem is now complete. $\square$

The use of conditions (H7) and (H9) in Theorem 3.3 forces us to assume that condition (H6) holds. These conditions can also be replaced by some easily verifiable conditions as given in the following theorem.

Theorem 3.4. *Let $\theta \in (0, \frac{1}{2})$ be fixed and assume that there exists a continuous function $f_0\colon [0,1] \to [0,\infty]$ such that* (H6) *holds,*

(H10) $\limsup\limits_{u \to \infty} \max\limits_{0 \leqslant t \leqslant 1} f(t,u)/f_0(t)u = 0$,

(H11) $\limsup\limits_{u \to 0} \max\limits_{0 \leqslant t \leqslant 1} f(t,u)/f_0(t)u = 0$, *and*

(H12) *there is a constant $c_2 > 0$ such that $f(t,u) > \Lambda_\theta c_2$ for $c_2 \leqslant u(t) \leqslant c_2/\delta$ and $\theta \leqslant t \leqslant 1-\theta$.*

Then BVP (1.3) *has at least three positive solutions.*

Proof. By (H10), there exists

$$0 < \varepsilon < \frac{6(1-\alpha[1])}{\int_0^1 s(1-s)^2 f_0(s)\, \mathrm{d}s} \tag{3.2}$$

and $\eta > 0$ such that $f(t,u) \leqslant \varepsilon f_0(t) u(t)$ for $x(t) \geqslant \eta$ and $0 \leqslant t \leqslant 1$. Set

$$M_f = \max_{0 \leqslant u \leqslant \eta,\, t \in [0,1]} f(t,u).$$

Then we have $f(t,u) \leqslant \varepsilon f_0(t) u(t) + M_f$ for $u(t) \geqslant 0$ and $0 \leqslant t \leqslant 1$. Choose a constant $c_4 > 0$ such that

$$c_4 > \max\left\{ \frac{c_2}{\delta}, \frac{M_f}{12\bigl(6(1-\alpha[1]) - \varepsilon \int_0^1 s(1-s)^2 f_0(s)\, \mathrm{d}s\bigr)} \right\}$$

and consider a nonnegative continuous concave functional φ on the cone K defined by $\varphi(u) = \|u\|$. Then for $u \in \overline{K(\varphi, c_4)}$ we have

$$\begin{aligned}
\varphi(Tu) &= \|Tu\| \\
&= \max_{0\leqslant t\leqslant 1}\left|\int_0^1 G(t,s)f(s,u(s))\,\mathrm{d}s + \frac{1}{(1-\alpha[1])}\int_0^1\int_0^1 G(t,s)f(s,u(s))\,\mathrm{d}s\,\mathrm{d}A(t)\right| \\
&\leqslant \frac{1}{6}\int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s + \frac{1}{6(1-\alpha[1])}\int_0^1\int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s\,\mathrm{d}A(t) \\
&= \frac{1}{6}\left(\int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s + \frac{\alpha[1]}{(1-\alpha[1])}\int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s\right) \\
&= \frac{1}{6(1-\alpha[1])}\int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s \\
&\leqslant \frac{1}{6(1-\alpha[1])}\int_0^1 s(1-s)^2(\varepsilon f_0(s)u(s) + M_f)\,\mathrm{d}s \\
&\leqslant \frac{1}{6(1-\alpha[1])}\int_0^1 s(1-s)^2(\varepsilon f_0(s)c_4 + M_f)\,\mathrm{d}s \leqslant c_4.
\end{aligned}$$

Hence, $T\colon \overline{K(\varphi, c_4)} \to \overline{K(\varphi, c_4)}$. Proceeding along the lines of the proof of Theorem 3.3, we can show that the mapping T is completely continuous. Also, we can choose functionals Θ, ψ and Φ to show that conditions (S1) and (S2) of Theorem 2.2 are satisfied.

Thus, to complete the proof of the theorem, it remains to show the existence of a constant c_1, with $0 < c_1 < c_2$, such that condition (S3) of Theorem 2.2 is satisfied. The existence of such c_1 can be obtained from (H11). Indeed, by (H11), there exist constants ε and c_1 satisfying

$$0 < \varepsilon < \frac{6(1-\alpha[1])}{\int_0^1 s(1-s)^2 f_0(s)\,\mathrm{d}s} \quad \text{and} \quad 0 < c_1 < c_2,$$

such that $f(t,u) \leqslant \varepsilon f_0(t)u(t)$ for $0 \leqslant u(t) \leqslant c_1$ and $0 \leqslant t \leqslant 1$. Hence, for the continuous functional $\psi(u) = \|u\|$ on the cone K and $0 \leqslant t \leqslant 1$, we have

$$\begin{aligned}
\psi(Tu) &= \|Tu\| \\
&= \max_{0\leqslant t\leqslant 1}\left|\int_0^1 G(t,s)f(s,u(s))\,\mathrm{d}s + \frac{1}{(1-\alpha[1])}\int_0^1\int_0^1 G(t,s)f(s,u(s))\,\mathrm{d}s\,\mathrm{d}A(t)\right| \\
&\leqslant \frac{1}{6}\int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s + \frac{1}{6(1-\alpha[1])}\int_0^1\int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s\,\mathrm{d}A(t) \\
&\leqslant \varepsilon c_1 \frac{1}{6(1-\alpha[1])}\int_0^1 s(1-s)^2 f_0(s)\,\mathrm{d}s < c_1.
\end{aligned}$$

This proves the theorem. □

As an application of Theorem 3.4, consider the BVP

$$\begin{cases} u''''(t) + \dfrac{u^l}{1+u^m} = 0, & t \in [0,1], \\ u'(0) = u'(1) = u''(0) = 0, & u(0) = \alpha[u]. \end{cases} \tag{3.3}$$

The nonlinear term in this equation sometimes appears in models of red blood cell production. The following result provides a sufficient condition for the existence of three positive solutions to (3.3).

Corollary 3.1. *Let $\theta \in (0, \frac{1}{2})$ be fixed, let $l > 1$, $m > l - 1 > 0$, and*

$$\frac{m-l+1}{m}\Big(\frac{l-1}{m-l+1}\Big)^{(l-1)/m} > \Lambda_\theta, \tag{3.4}$$

where Λ_θ is given in (3.1). *Then BVP* (3.3) *has at least three positive solutions.*

Proof. Set $f_0(t) \equiv 1$. Then (H6) is equivalent to $\alpha[1] < \frac{71}{72}$. With $f(t,u) = u^l/(1+u^m)$ and $f_0(t) \equiv 1$, conditions (H10) and (H11) are satisfied. In order to apply Theorem 3.4, we need to find a positive constant c_2 such that condition (H12) is satisfied. Clearly, for $c_2 \leqslant u \leqslant c_2/\delta$ we have $f(t,u) = u^l/(1+u^m) \geqslant \delta^m c_2^l/(\delta^m + c_2^m)$. Hence, (H12) is satisfied if

$$\frac{\delta^m c_2^{l-1}}{\delta^m + c_2^m} > \Lambda_\theta. \tag{3.5}$$

Consider the function $g(r) = \delta^m r^{l-1}/(\delta^m + r^m)$. Then setting

$$c_2 = \delta\Big(\frac{l-1}{m-l+1}\Big)^{1/m},$$

we observe that $g'(r_2) = 0$, $g'(r) > 0$ for $r < c_2$, and $g'(r) < 0$ for $r > c_2$. Thus, the minimum value of $g(r)$ in the interval $[c_2, c_2/\delta]$ is attained at c_2/δ and is given by

$$\frac{m-l+1}{m}\Big(\frac{l-1}{m-l+1}\Big)^{(l-1)/m}.$$

Thus, (3.5) is satisfied if (3.4) is satisfied, and the theorem is proved. □

Remark 3.2. From the conditions of Theorem 2.2, it is clear that the theorem cannot be applied to the equations where the nonlinear function $f(t,u)$ satisfies

(H13) $\limsup\limits_{u\to 0} \max\limits_{0\leqslant t\leqslant 1} f(t,u)/f_0(t)u = \infty$ and $\limsup\limits_{u\to\infty} \max\limits_{0\leqslant t\leqslant 1} f(t,u)/f_0(t)u = \infty$,

which includes functions of the type $(\lambda/t^\beta)(c/u^\sigma + u^\gamma)$, where σ, β, λ, c and γ are constants with $c > 0$, $\lambda > 0$, $0 < \beta < 1$, $\gamma > 1$ and $\sigma + 1 > 0$. In this case, the following theorem provides a sufficient condition to obtain positive solutions of BVP (1.3). Its proof is based on Theorem 2.1.

Theorem 3.5. *Let $\theta \in (0, \frac{1}{2})$ be fixed. Assume that there exists a continuous function $f_0\colon [0,1] \to [0,\infty]$ such that condition* (H13) *is satisfied and*

(H14) *there exists a constant $r_2 > 0$ such that $\int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s \leqslant 6r_2(1-\alpha[1])$ for $\delta r_2 \leqslant u \leqslant r_2$ and $0 \leqslant t \leqslant 1$.*

Then problem (1.3) *has at least two positive solutions.*

P r o o f. To prove the theorem, we consider the cone P given in (2.17). From the first part of (H13) with

$$\mu_\theta = \left(\delta \frac{\theta^3}{6} \frac{(1-\alpha[1] + \int_\theta^{1-\theta} \mathrm{d}A(t))}{(1-\alpha[1])} \int_\theta^{1-\theta} s(1-s)^2 f_0(s)\,\mathrm{d}s\right)^{-1} \tag{3.6}$$

for any $\mu \geqslant \mu_\theta$, there exists $r_1 \in (0, \delta r_2]$ such that $f(t,u) \geqslant \mu f_0(t) u$ for all $t \in [0,1]$ and $0 < u \leqslant r_1$.

Set $\Omega_{r_1} = \{u \in P\colon \|u\| < r_1\}$. Then for $u \in P \cap \partial\Omega_{r_1}$ we have

$$\begin{aligned}
\|Tu\| &\geqslant \min_{t\in[\theta,1-\theta]} Tu(t)\\
&= \int_0^1 \Big(\min_{t\in[\theta,1-\theta]} G(t,s)\Big) f(s,u(s))\,\mathrm{d}s\\
&\quad + \frac{1}{(1-\alpha[1])} \int_0^1 \int_0^1 \Big(\min_{t\in[\theta,1-\theta]} G(t,s)\Big) f(s,u(s))\,\mathrm{d}s\,\mathrm{d}A(t)\\
&\geqslant \frac{\theta^3}{6} \int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s\\
&\quad + \frac{\theta^3}{6(1-\alpha[1])} \int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s \int_\theta^{1-\theta} \mathrm{d}A(t)\\
&= \frac{\theta^3}{6} \frac{(1-\alpha[1] + \int_\theta^{1-\theta} \mathrm{d}A(t))}{(1-\alpha[1])} \int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s\\
&\geqslant \mu \frac{\theta^3}{6} \frac{(1-\alpha[1] + \int_\theta^{1-\theta} \mathrm{d}A(t))}{(1-\alpha[1])} \int_0^1 s(1-s)^2 f_0(s) u(s)\,\mathrm{d}s\\
&\geqslant \mu \frac{\theta^3}{6} \frac{(1-\alpha[1] + \int_\theta^{1-\theta} \mathrm{d}A(t))}{(1-\alpha[1])} \int_\theta^{1-\theta} s(1-s)^2 f_0(s) u(s)\,\mathrm{d}s\\
&\geqslant \mu\delta\|u\| \frac{\theta^3}{6} \frac{(1-\alpha[1] + \int_\theta^{1-\theta} \mathrm{d}A(t))}{(1-\alpha[1])} \int_\theta^{1-\theta} s(1-s)^2 f_0(s)\,\mathrm{d}s \geqslant \|u\|,
\end{aligned}$$

which means that $\|Tu\| \geqslant \|u\|$ for $u \in P \cap \partial\Omega_{r_1}$. On the other hand, from the second part of (H13), there exists $\overline{r_3} > r_2$ such that $f(t,u) \geqslant \mu f_0(t) u$ for any $\mu \geqslant \mu_\theta$ and $u \geqslant \overline{r_3}$, where μ_θ is given in (3.6). Let $r_3 \geqslant \overline{r_3} > r_2/\delta$ and set $\Omega_{r_3} = \{u \in P\colon \|u\| < r_3\}$. Then proceeding as above, we can prove that $\|Tu\| \geqslant \|u\|$ for $u \in P \cap \partial\Omega_{r_3}$.

Finally, set $\Omega_{r_2} = \{u \in X\colon \|u\| < r_2\}$. Then for any $u \in P \cap \partial\Omega_{r_2}$, by (H14) we have

$$\begin{aligned}
\|Tu\| &= \max_{0\leqslant t\leqslant 1} \left| \int_0^1 G(t,s)f(s,u(s))\,\mathrm{d}s + \frac{1}{(1-\alpha[1])}\int_0^1\int_0^1 G(t,s)f(s,u(s))\,\mathrm{d}s\,\mathrm{d}A(t)\right| \\
&\leqslant \frac{1}{6}\int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s + \frac{1}{6(1-\alpha[1])}\int_0^1\int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s\,\mathrm{d}A(t) \\
&= \frac{1}{6}\biggl(\int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s + \frac{\alpha[1]}{(1-\alpha[1])}\int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s\biggr) \\
&= \frac{1}{6(1-\alpha[1])}\int_0^1 s(1-s)^2 f(s,u(s))\,\mathrm{d}s \leqslant r_2 = \|u\|,
\end{aligned}$$

which implies $\|Tu\| \leqslant \|u\|$ for $u \in P \cap \partial\Omega_{r_2}$. Since $r_1 < r_2 < r_3$, from the above estimates it follows from Theorem 2.1 that T has a fixed point $u_1 \in P \cap (\overline{\Omega}_{r_2} \setminus \Omega_{r_1})$ and another fixed point $u_2 \in P \cap (\overline{\Omega}_{r_3} \setminus \Omega_{r_2})$, which are positive solutions of (1.3). The proof of the theorem is now complete. □

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