INVESTIGATIONS ON UNIQUE RANGE SETS OF MEROMORPHIC FUNCTIONS IN AN ANGULAR DOMAIN

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Abstract. We study unique range sets of meromorphic functions over an angular domain in the light of weighted sharing. One of our main results generalizes and improves a result of Xu et al. (2014). Most importantly, we have pointed out a gap in the proofs of some main results of Rathod (2021) and subsequently rectifying the gap we have conveniently improved the results.

Keywords: angular domain; meromorphic function; unique range set

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1. Introduction

In 1929, Nevanlinna first investigated the uniqueness of meromorphic functions in the whole complex plane by obtaining his famous five values theorem. After this result, there was vast research work done on the uniqueness of meromorphic functions sharing values and sets in the whole complex plane, the unit disc and the angular domain. In this paper we focus on uniqueness problem of meromorphic functions sharing one set in an angular domain by using Tsuji's characteristic.

First we recall some basic value distribution theory on an angular domain (see [9], [17]). Let $f(z)$ be a meromorphic function on an angular domain $\Omega := \Omega(\alpha,\beta) = \{z\colon \alpha < \arg z < \beta\}$, where $0 \leqslant \alpha < \beta \leqslant 2\pi$ and consider $\omega = \pi/(\beta-\alpha)$.

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Define

$$\mathfrak{M}_{\alpha,\beta}(r,f) = \frac{1}{2\pi}\int_{\arcsin r^{-\omega}}^{\pi-\arcsin r^{-\omega}} \log^+ |f(re^{i(\alpha+\omega^{-1}\theta)}\sin^{\omega^{-1}}\theta)|\frac{1}{r^\omega \sin^2\theta}\,d\theta,$$

$$\mathfrak{N}_{\alpha,\beta}(r,f) = \sum_{1<|b_n|<r(\sin(\omega(\beta_n-\alpha)))^{\omega^{-1}}} \Big(\frac{\sin\omega(\beta_n-\alpha)}{|b_n|^\omega} - \frac{1}{r^\omega}\Big),$$

where $b_n = |b_n|e^{i\beta_n}$ are the poles of $f(z)$ in $\Xi(\alpha,\beta;r) = \{z = te^{i\theta}\colon \alpha < \theta < \beta, 1 < t \leqslant r(\sin(\omega(\beta_n-\alpha)))^{\omega^{-1}}\}$ appearing often according to their multiplicities and then Tsuji's characteristic of f is

$$\mathfrak{T}_{\alpha,\beta}(r,f) = \mathfrak{M}_{\alpha,\beta}(r,f) + \mathfrak{N}_{\alpha,\beta}(r,f).$$

We denote by $\mathfrak{n}_{\alpha,\beta}(r,f)$ the number of poles of $f(z)$ in $\Xi(\alpha,\beta;r)$. Then

$$\mathfrak{N}_{\alpha,\beta}(r,f) = \int_1^r \Big(\frac{1}{t^\omega} - \frac{1}{r^\omega}\Big)\,d\mathfrak{n}_{\alpha,\beta}(t,f) = \omega\int_1^r \frac{\mathfrak{n}_{\alpha,\beta}(t,f)}{t^{\omega+1}}\,dt,$$

where pole b_n is counted in the sum $\sum\limits_{1<|b_n|<r(\sin(\omega(\beta_n-\alpha)))^{\omega^{-1}}}$ only once and we denote it by $\overline{\mathfrak{N}}_{\alpha,\beta}(r,f)$. For meromorphic function f in Ω, if

$$\limsup_{r\to\infty}\frac{\mathfrak{T}_{\alpha,\beta}(r,f)}{\log r} = \infty,$$

then f is called transcendental in Tsuji's sense. For simplicity throughout the paper we write $\mathfrak{M}(r,f)$, $\mathfrak{N}(r,f)$, $\mathfrak{T}(r,f)$, $\overline{\mathfrak{N}}(r,f)$ instead of $\mathfrak{M}_{\alpha,\beta}(r,f)$, $\mathfrak{N}_{\alpha,\beta}(r,f)$, $\mathfrak{T}_{\alpha,\beta}(r,f)$, $\overline{\mathfrak{N}}_{\alpha,\beta}(r,f)$, respectively. Sometimes we write $\mathfrak{N}(r,1/(f-a))$ as $\mathfrak{N}(r,a;f)$. For any complex number a, we have

$$\mathfrak{T}\Big(r,\frac{1}{f-a}\Big) = \mathfrak{T}(r,f) + O(1).$$

Let S be a set of distinct elements in $\overline{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$, define

$$E(S,\Omega,f) = \bigcup_{a\in S}\{z\in\Omega\colon\ f(z)-a=0,\ \text{counting multiplicities}\},$$

$$\overline{E}(S,\Omega,f) = \bigcup_{a\in S}\{z\in\Omega\colon\ f(z)-a=0,\ \text{ignoring multiplicities}\}.$$

If $E(S,\Omega,f) = E(S,\Omega,g)$, then we say f and g share the set S CM (counting multiplicities) in Ω. If $\overline{E}(S,\Omega,f) = \overline{E}(S,\Omega,g)$, then we say f and g share the set S IM (ignoring multiplicities) in Ω. In 2001, Lahiri (see [8]) introduced the notion of weighted sharing over $\overline{\mathbb{C}}$. Similarly, we can define it for angular domain.

Definition 1.1. For $k \in \mathbb{Z}^+ \cup \{\infty\}$, the set of all a-points of f with multiplicity m counted m times if $m \leqslant k$ and counted $k+1$ times if $m > k$, is denoted by $E_k(a, \Omega, f)$. For two functions f, g in Ω, if $E_k(a, \Omega, f) = E_k(a, \Omega, g)$, then we say f, g share the value a with weight k.

Inspired from the definition of weighted sharing of sets as introduced in [7], we demonstrate the analogous definition over Ω as follows:

Definition 1.2. We say f, g share the set S with weight k if $E_k(S, \Omega, f) = E_k(S, \Omega, g)$, where

$$E_k(S, \Omega, f) = \bigcup_{a \in S} E_k(a, \Omega, f) \quad \text{and} \quad E_k(S, \Omega, g) = \bigcup_{a \in S} E_k(a, \Omega, g).$$

We write f, g share (S, k) to mean that f, g share the set S with weight k. In particular, if $S = \{a\}$, then we write f, g share (a, k).

Remark 1.3. In [15] and [12], the authors denoted

$$E_1(S, \Omega, f) = \bigcup_{a \in S} \{z \in \Omega\colon \text{ all the simple zeros of } f(z) - a\},$$

and called f, g share the set S with weight 1 if $E_1(S, \Omega, f) = E_1(S, \Omega, g)$. However, this definition does not conform with the definition of actual weighted sharing (see Definition 1.2) as the features of multiple a points are not considered here. In fact, the basic criterion of finite weighted sharing is that two functions have to share the value IM first. So it will be appropriate to say the definition of [15], [12] as truncated sharing up to multiplicity 1, rather to say weighted 1 sharing and to justify the definition it will be reasonable to use the notation $E_{1)}(S, \Omega, f)$ instead of $E_1(S, \Omega, f)$. From onward, for weighted sharing we follow Definition 1.2.

Definition 1.4. For any two meromorphic functions f, g in Ω if $E_k(S, \Omega, f) = E_k(S, \Omega, g)$ implies $f \equiv g$, then we say the set S is a unique range set for meromorphic functions with weight k in Ω, in brief we write URSMk. In particular, for $k = 0$ and ∞ we write URSM-IM and URSM, respectively.

Fujimoto in [6] introduced the following definition and called it “Property H”, which was later well known as “Critical Injection Property”.

Definition 1.5 ([2])**.** Let $P(z)$ be a polynomial such that $P'(z)$ has l distinct zeros, namely $z_1, z_2, \ldots, z_l$. If $P(z_i) \neq P(z_j)$ for $i \neq j$, $i, j \in \{1, 2, \ldots, l\}$, then $P(z)$ is said to satisfy the critical injection property.

In 2004, Zheng in [16] studied the uniqueness problem under the condition that five values are shared in some angular domain in $\mathbb{C}$. During the last few years, many authors obtained results on uniqueness of meromorphic function sharing values and sets over angular domain (see [11], [13], [14]). In 2006 and 2011, Lin et al. (see [10]) and Chen-Lin (see [4]), respectively, dealt with the uniqueness problem on meromorphic functions sharing three finite sets in an angular domain. In 2010, by using Tsuji's characteristic, Zheng in [17] proved the following theorem to extend the five IM values theorem of Nevanlinnas to an angular domain.

Theorem A ([17]). *Let $f(z)$ and $g(z)$ be both meromorphic functions in an angular domain $\Omega = \{z\colon \alpha < \arg z < \beta\}$ with $0 \leqslant \alpha < \beta \leqslant 2\pi$ and let $f(z)$ be transcendental in Tsuji's sense. Assume that a_j $(j = 1, 2, \ldots, 5)$ are five distinct complex numbers. If $\overline{E}(a_j, \Omega, f) = \overline{E}(a_j, \Omega, g)$ for $j = 1, 2, \ldots, 5$, then $f(z) \equiv g(z)$.*

Recently in 2014, Xu et al. in [15] considered the set which contains all zeros of famous Frank-Reinders (see [5]) polynomial $P_{\mathrm{FR}}(z)$, where

$$(1.1)\qquad P_{\mathrm{FR}}(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c,$$

and c is a complex number such that $c \neq 0, 1$. They obtained the following results:

Theorem B (Theorem 5 of [15]). *Let $f(z)$ and $g(z)$ be two meromorphic functions in an angular domain $\Omega = \{z\colon \alpha < \arg z < \beta\}$ with $0 \leqslant \alpha < \beta \leqslant 2\pi$ and $f(z)$ be transcendental in Tsuji's sense. Consider $S = \{z \in \Omega\colon P_{\mathrm{FR}}(z) = 0\}$, where $P_{\mathrm{FR}}(z)$ is defined in* (1.1). *If n is an integer $\geqslant 11$ and $E(S, \Omega, f) = E(S, \Omega, g)$, then $f(z) \equiv g(z)$.*

Theorem C (Theorem 9 of [15]). *Under the same situation as in Theorem* B, *if n is an integer $\geqslant 15$ and $E_{1)}(S, \Omega, f) = E_{1)}(S, \Omega, g)$, then $f(z) \equiv g(z)$.*

Now the following question comes if one tries to improve Theorem B.

Question 1.6. In Theorem B, can it be possible to relax the CM sharing up to weight 2 sharing?

Recently, the present authors [3] introduced a new polynomial of degree $m+n+1$ over a non-Archimedean field, which is the generalization of Frank-Reinders (see [5]) polynomial $P_{\mathrm{FR}}(z)$. Here we consider the same polynomial over complex field as:

$$(1.2)\qquad \begin{aligned} P(z) &= \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{m+n+1-j} z^{m+n+1-j} a^j \\ &\quad + \sum_{i=1}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \frac{(-1)^{i+j}}{m+n+1-i-j} z^{m+n+1-i-j} a^j b^i + c \\ &= Q(z) + c, \end{aligned}$$

where a and b are distinct such that $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$, $c \in \mathbb{C} \setminus \{0, -Q(a), -Q(b)\}$. It is easy to verify that

$$P'(z) = (z-a)^n (z-b)^m.$$

N o t e 1.7. From Remark 1.10 of [3], we see that $P(z)$ is a critically injective polynomial.

The following theorem is one of our main results which answers Question 1.6 affirmatively for more generalized polynomial than Frank-Reinders polynomial.

Theorem 1.8. *Let f, g be two non-constant meromorphic functions in an angular domain $\Omega = \{z\colon \alpha < \arg z < \beta\}$ with $0 \leqslant \alpha < \beta \leqslant 2\pi$. Let m, n be two positive integers such that* $\min\{m, n\} \geqslant 3$, $m + n \geqslant 10$. *Consider the polynomial* (1.2) *such that $P(a) \neq -1$ and $S = \{z \in \Omega\colon P(z) = 0\}$. Now*

(i) *$P(b) \neq 1$, $n \geqslant m + 3$, or*
(ii) *$P(b) = 1$.*

Then for both cases, S is URSM2.

Notice that Theorem 1.8 is an improvement as well as a generalization of Theorem B. Recently, to study the unique range set problem in an angular domain, Rathod (see [12]) considered the following polynomial:

$$\widetilde{P}(z) = z^n + bz^{n-m} + c, \tag{1.3}$$

where b and c are two nonzero constants such that $\widetilde{P}(z)$ has only simple zeros. Using similar methods as in the proofs of Theorem B and Theorem C, Rathod in [12] obtained analogous two results for the polynomial $\widetilde{P}(z)$.

Theorem D (Theorem 4.1 of [12])**.** *Let $f(z)$ and $g(z)$ be two meromorphic functions in an angular domain $\Omega = \{z\colon \alpha < \arg z < \beta\}$ with $0 \leqslant \alpha < \beta \leqslant 2\pi$ and $f(z)$ be transcendental in Tsuji's sense. Consider $\widetilde{S} = \{z \in \Omega\colon \widetilde{P}(z) = 0\}$, where $\widetilde{P}(z)$ is defined in* (1.3)*. Let n, m be two positive integers such that* $\gcd(n, m) = 1$, *$m \geqslant 2$ and $n > 2m + 8$. If $E(\widetilde{S}, \Omega, f) = E(\widetilde{S}, \Omega, g)$, then $f(z) \equiv g(z)$.*

Theorem E (Theorem 4.3 of [12])**.** *Under the same situation as in Theorem D, if n, m be two positive integers such that* $\gcd(n, m) = 1$, *$m \geqslant 2$, $n > 2m + 12$ and $E_{1)}(\widetilde{S}, \Omega, f) = E_{1)}(\widetilde{S}, \Omega, g)$, then $f(z) \equiv g(z)$.*

R e m a r k 1.9. From Theorem D and Theorem E it can be noticed that the cardinalities of unique range set $\widetilde{S}$ are $\geqslant 13$ and $\geqslant 17$ for CM sharing and for truncated 1 sharing, respectively. However, there is a gap in the proof of Theorem D and Theorem E. Let us consider equation number (4.4) (see [12], page 101). The author writes

$$\widetilde{P}(z)-\widetilde{P}(1)=(z-1)Q_2(z),$$

where $Q_2(z)$ is a polynomial of degree $n-1$ and $Q_2(1)\neq 0$. But we claim that this is not at all true. Let $\Phi(z):=\widetilde{P}(z)-\widetilde{P}(1)=z^n+bz^{n-m}-(1+b)$. It is clear that $\Phi(1)=0$ and $\Phi'(z)=nz^{n-1}+b(n-m)z^{n-m-1}$. Note that if we choose $b=-n/(n-m)$, then $\Phi'(1)=0$. Thus, for $b=-n/(n-m)$, $\widetilde{P}(z)-\widetilde{P}(1)$ have a zero at 1 with multiplicity 2. Therefore equation number (4.4) (see [12], page 101) is not correct and the same equation used in many places such as equations number (4.15) and (4.28) (see [12], page 104 and 106) etc. So the wrong analysis has been carried forwarded several places throughout the proofs of Theorem D and Theorem E.

In our next result we deal with the unique range set corresponding to the zeros of the polynomial $\widetilde{P}(z)$ under weighted sharing hypothesis.

Theorem 1.10. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions in an angular domain $\Omega=\{z\colon \alpha<\arg z<\beta\}$ with $0\leqslant\alpha<\beta\leqslant 2\pi$. Consider $\widetilde{S}=\{z\in\Omega\colon \widetilde{P}(z)=0\}$, where $\widetilde{P}(z)$ is defined in* (1.3). *Let n, m be two positive integers such that* $\gcd(n,m)=1$, *$m\geqslant 2$ and $n\geqslant 2m+9$. Then $\widetilde{S}$ is URSM2.*

Note that Theorem 1.10 is a significant improvement of Theorem D since CM sharing is relaxed to weight 2 sharing with the cardinality of the set $\widetilde{S}$ remaining $\geqslant 13$. On the other hand, as in our proof of Theorem 1.10, we use some technique different from the proof of Theorem D, the gap as mentioned in Remark 1.9 has automatically been rectified.

2. Lemmas

Lemma 2.1 (Remark 14 of [15]). *Let $f(z)$ be a meromorphic function in an angular domain Ω. Then for any l $(\geqslant 2)$ distinct points $a_1,a_2,\dots,a_l\in\overline{\mathbb{C}}$,*

$$(l-2)\mathfrak{T}(r,f)\leqslant\sum_{j=1}^{l}\overline{\mathfrak{N}}(r,a_j;f)-\mathfrak{N}^0(r,0;f')+Q(r,f),$$

where $Q(r,f)=O(\log^+\mathfrak{T}(r,f)+\log r)$, $r\notin E$, E denotes a set of r with finite linear measure and $\mathfrak{N}^0(r,0;f')$ denotes the counting function of those zeros of f' which are not zeros of $f-a_j$ for all $j\in\{1,2,\dots,l\}$.

Lemma 2.2 ([17]). *Let $f(z)$ be a meromorphic function in an angular domain Ω. Then for $0 < r < R$ one has*

$$\mathfrak{M}\Big(r, \frac{f^{(p)}}{f}\Big) \leqslant K\Big(\log^+ \mathfrak{T}(R, f) + \log \frac{R}{R-r} + 1\Big),$$

where K is a constant independent of r and R.

Consider the polynomial $\widetilde{P}(z)$ as defined in (1.3). Let

$$F = \frac{f^{n-m}(f^m + b)}{-c} \quad \text{and} \quad G = \frac{g^{n-m}(g^m + b)}{-c}. \tag{2.1}$$

Then

$$F' = \frac{f^{n-m-1}(nf^m + b(n-m))f'}{-c} \quad \text{and} \quad G' = \frac{g^{n-m-1}(ng^m + b(n-m))g'}{-c}. \tag{2.2}$$

Define

$$H \equiv \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big). \tag{2.3}$$

Then $\mathfrak{M}(r, H) = Q(r)$, where $Q(r) = o(\mathfrak{T}(r))$ and $\mathfrak{T}(r) = \max\{\mathfrak{T}(r, F), \mathfrak{T}(r, G)\}$.

Lemma 2.3. *Let $H \not\equiv 0$ and F, G share $(1,1)$. Then*

$$\mathfrak{N}(r, 1; F \mid= 1) = \mathfrak{N}(r, 1; G \mid= 1) \leqslant \mathfrak{N}(r, H) + Q(r),$$

where $\mathfrak{N}(r, 1; F \mid= 1)$ denotes the counting function of 1-points of F with multiplicity 1 and similarly for $\mathfrak{N}(r, 1; G \mid= 1)$.

Proof. As F and G share $(1,1)$, each simple 1-point of F is also a simple 1-point of G and vice versa. Now each simple 1-point of F (i.e., simple 1-point of G) is a zero of H. Note that $\mathfrak{M}(r, H) = Q(r)$. Hence

$$\mathfrak{N}(r, 1; F \mid= 1) = \mathfrak{N}(r, 1; G \mid= 1) \leqslant \mathfrak{N}(r, 0; H) \leqslant \mathfrak{T}(r, \mathcal{H}) \leqslant \mathfrak{N}(r, H) + Q(r).$$

□

Lemma 2.4. *Let $\widetilde{S} = \{z \in \Omega \colon \widetilde{P}(z) = 0\}$, where $\widetilde{P}(z)$ is defined as in (1.3). Let $H \not\equiv 0$ and f, g be two meromorphic functions on Ω such that $E_2(\widetilde{S}, \Omega, f) = E_2(\widetilde{S}, \Omega, g)$. Then*

$$\begin{aligned}\mathfrak{N}(r, H) \leqslant{}& \overline{\mathfrak{N}}(r, 0; f) + \overline{\mathfrak{N}}(r, 0; nf^m + b(n-m)) + \overline{\mathfrak{N}}(r, f) + \overline{\mathfrak{N}}^0(r, 0; f') + \overline{\mathfrak{N}}(r, 0; g)\\ &+ \overline{\mathfrak{N}}(r, 0; ng^m + b(n-m)) + \overline{\mathfrak{N}}(r, g) + \overline{\mathfrak{N}}^0(r, 0; g') + \overline{\mathfrak{N}}^*(r, 1; F, G),\end{aligned}$$

where $\overline{\mathfrak{N}}^0(r, 0; f')$ denotes reduced counting function of those zeros of f' which are not zeros of $(F-1)f(nf^m + b(n-m))$ and $\overline{\mathfrak{N}}^0(r, 0; g')$ denotes similar counting function. $\overline{\mathfrak{N}}^*(r, 1; F, G)$ denotes the reduced counting function of those 1-points of F whose multiplicities differ from the multiplicities of the corresponding 1-points of G.

P r o o f. Recall the values of F' and G' from (2.2). As $E_2(\widetilde{S},\Omega,f)=E_2(\widetilde{S},\Omega,g)$, the lemma directly follows by calculating all the possible poles of H. □

Lemma 2.5. *Let F, G share $(1,k)$, where $1\leqslant k<\infty$. Then*

$$\overline{\mathfrak{N}}(r,1;F)+\overline{\mathfrak{N}}(r,1;G)-\mathfrak{N}(r,1;F\mid=1)+\Big(k-\frac{1}{2}\Big)\overline{\mathfrak{N}}^{*}(r,1;F,G)$$
$$\leqslant\frac{1}{2}(\mathfrak{N}(r,1;F)+\mathfrak{N}(r,1;G)).$$

Lemma 2.5 can be considered as angular domain analogue of Lemma 2.10 from [1]. Proof of the lemma is omitted as it can be done proceeding similarly as in [1], Lemma 2.10.

R e m a r k 2.6. In particular, for $k=2$, from Lemma 2.5 we get

$$\overline{\mathfrak{N}}(r,1;F)+\overline{\mathfrak{N}}(r,1;G)-\mathfrak{N}(r,1;F\mid=1)+\frac{3}{2}\overline{\mathfrak{N}}^{*}(r,1;F,G)$$
$$\leqslant\frac{1}{2}(\mathfrak{N}(r,1;F)+\mathfrak{N}(r,1;G)).$$

Lemma 2.7. *Let $P(z)$ be any polynomial of degree $n\geqslant 5$ without multiple zeros, whose derivative is of the form $(z-d_1)^{n_1}(z-d_2)^{n_2}\dots(z-d_l)^{n_l}$, where $d_1,d_2,\dots,d_l$ are distinct and $n_1+n_2+\dots+n_l=n-1$. Also assume $P(z)$ is a critically injective polynomial and there are two distinct meromorphic functions f and g in Ω such that*

$$\frac{1}{P(f)}=\frac{c_0}{P(g)}+c_1$$

for some constants $c_0\neq 0$ and c_1. If $l\geqslant 3$ or if $l=2$ and $\min\{n_1,n_2\}\geqslant 2$, then $c_1=0$.

We omit the proof of Lemma 2.7 as the same can be done in a similar fashion as adopted in Proposition 7.1 of [6].

3. Proofs of the theorems

P r o o f of Theorem 1.8. We omit the proof as this can be carried out in the line of proof of Theorem 1.13 of [3]. Note that the bounds $\min\{m,n\}\geqslant 2$, $m+n\geqslant 9$ and $n\geqslant m+2$ in Theorem 1.13 of [3] will be replaced in this theorem by $\min\{m,n\}\geqslant 3$, $m+n\geqslant 10$ and $n\geqslant m+3$, respectively. In our present theorem the bounds have been increased by 1 just because of the absent of the term $-\log r$ in the second fundamental theorem but which is present for the case of non-Archimedean field. □

P r o o f of Theorem 1.10. We have

$$\widetilde{P}(z) = z^n + bz^{n-m} + c,$$

where b and c are two nonzero constants such that $\widetilde{P}(z)$ has only simple zeros. Now

$$\widetilde{P}'(z) = z^{n-m-1}(nz^m + b(n-m)).$$

Denote $\xi(z) := nz^m + b(n-m)$. Now $\xi'(z) = 0$ implies $z = 0$ is a zero of $\xi'(z)$ of multiplicity $m-1$. But $\xi(0) \neq 0$. Thus, $\xi(z)$ has m distinct simple zeros, say δ_j, $j = 1, 2, \ldots, m$. So we can write

$$\widetilde{P}'(z) = z^{n-m-1}(z-\delta_1)(z-\delta_2)\ldots(z-\delta_m). \tag{3.1}$$

Next, we claim that $\widetilde{P}(z)$ is critically injective polynomial. Choose any two distinct zeros δ_i, δ_j of $\xi(z)$. Hence, $\delta_i^m = -b(n-m)/n$ and $\delta_j^m = -b(n-m)/n$, which implies $(\delta_i/\delta_j)^m = 1$. We will show that $\widetilde{P}(\delta_i) \neq \widetilde{P}(\delta_j)$. On the contrary, let us assume $\widetilde{P}(\delta_i) = \widetilde{P}(\delta_j)$. This implies

$$\delta_i^{n-m}(\delta_i^m + b) = \delta_j^{n-m}(\delta_j^m + b) \Rightarrow \delta_i^{n-m}\Big(-\frac{b(n-m)}{n} + b\Big) = \delta_j^{n-m}\Big(-\frac{b(n-m)}{n} + b\Big)$$
$$\Rightarrow \Big(\frac{\delta_i}{\delta_j}\Big)^n = \Big(\frac{\delta_i}{\delta_j}\Big)^m = 1,$$

since $(\delta_i/\delta_j)^m = 1$. As $\gcd(n,m) = 1$, $\delta_i/\delta_j = 1$. This implies $\delta_i = \delta_j$, a contradiction. Therefore $\widetilde{P}(\delta_i) \neq \widetilde{P}(\delta_j)$.

On the other hand, we know that $\delta_i \neq 0$ for all $i = 1, 2, \ldots, m$. Now we claim $\widetilde{P}(\delta_i) \neq \widetilde{P}(0)$. On the contrary, let us assume $\widetilde{P}(\delta_i) = \widetilde{P}(0)$, which implies

$$\delta_i^{n-m}(\delta_i^m + b) = 0 \Rightarrow -\frac{b(n-m)}{n} + b = 0 \Rightarrow \frac{bm}{n} = 0,$$

a contradiction. Thus $\widetilde{P}(\delta_i) \neq \widetilde{P}(0)$. Therefore $\widetilde{P}(z)$ is a critically injective polynomial. Note that if f, g share $(\widetilde{S}, 2)$, then F, G share $(1, 2)$.

Now we discuss the following two cases:

Case 1: First assume $H \not\equiv 0$, where H is defined in (2.3). In view of Lemma 2.1 we get

$$(n+m)\mathfrak{T}(r,f) \leqslant \overline{\mathfrak{N}}(r,0;f) + \overline{\mathfrak{N}}(r,f) + \sum_{j=1}^{m}\overline{\mathfrak{N}}(r,\delta_j;f) + \overline{\mathfrak{N}}(r,1;F) - \mathfrak{N}^0(r,0;f') + Q(r,f). \tag{3.2}$$

Similarly for g:

$$(n+m)\mathfrak{T}(r,g) \leqslant \overline{\mathfrak{N}}(r,0;g) + \overline{\mathfrak{N}}(r,g) + \sum_{j=1}^{m}\overline{\mathfrak{N}}(r,\delta_j;g) + \overline{\mathfrak{N}}(r,1;G) - \mathfrak{N}^0(r,0;g') + Q(r,g). \tag{3.3}$$

Adding (3.2) and (3.3),

$$\begin{aligned}(n+m)(\mathfrak{T}(r,f)+\mathfrak{T}(r,g)) \leqslant{}& \overline{\mathfrak{N}}(r,0;f)+\overline{\mathfrak{N}}(r,f)+\sum_{j=1}^{m}\overline{\mathfrak{N}}(r,\delta_j;f)+\overline{\mathfrak{N}}(r,1;F)\\ &+\overline{\mathfrak{N}}(r,0;g)+\overline{\mathfrak{N}}(r,g)+\sum_{j=1}^{m}\overline{\mathfrak{N}}(r,\delta_j;g)+\overline{\mathfrak{N}}(r,1;G)\\ &-\mathfrak{N}^0(r,0;f')-\mathfrak{N}^0(r,0;g')+Q(r),\end{aligned}$$

which implies

(3.4)

$$(n-2)(\mathfrak{T}(r,f)+\mathfrak{T}(r,g)) \leqslant \overline{\mathfrak{N}}(r,1;F)+\overline{\mathfrak{N}}(r,1;G)-\mathfrak{N}^0(r,0;f')-\mathfrak{N}^0(r,0;g')+Q(r).$$

Now using Remark 2.6 and Lemma 2.3, 2.4 from (3.4) we get

$$\begin{aligned}(3.5)\quad &(n-2)(\mathfrak{T}(r,f)+\mathfrak{T}(r,g))\\ &\leqslant \mathfrak{N}(r,1;F\mid=1)-\frac{3}{2}\overline{\mathfrak{N}}^{*}(r,1;F,G)+\frac{1}{2}\mathfrak{N}(r,1;F)+\frac{1}{2}\mathfrak{N}(r,1;G)\\ &\quad-\mathfrak{N}^0(r,0;f')-\mathfrak{N}^0(r,0;g')+Q(r)\\ &\leqslant \mathfrak{N}(r,H)-\frac{3}{2}\overline{\mathfrak{N}}^{*}(r,1;F,G)+\frac{1}{2}\mathfrak{N}(r,1;F)+\frac{1}{2}\mathfrak{N}(r,1;G)\\ &\quad-\mathfrak{N}^0(r,0;f')-\mathfrak{N}^0(r,0;g')+Q(r)\\ &\leqslant \overline{\mathfrak{N}}(r,0;f)+\sum_{j=1}^{m}\overline{\mathfrak{N}}(r,\delta_j;f)+\overline{\mathfrak{N}}(r,f)+\overline{\mathfrak{N}}(r,0;g)+\sum_{j=1}^{m}\overline{\mathfrak{N}}(r,\delta_j;g)\\ &\quad+\overline{\mathfrak{N}}(r,g)-\frac{1}{2}\overline{\mathfrak{N}}^{*}(r,1;F,G)+\frac{n}{2}\mathfrak{T}(r,f)+\frac{n}{2}\mathfrak{T}(r,g)+Q(r)\\ &\leqslant \Big(\frac{n}{2}+m+2\Big)(\mathfrak{T}(r,f)+\mathfrak{T}(r,g))+Q(r).\end{aligned}$$

Hence, $(\frac{1}{2}n-m-4)(\mathfrak{T}(r,f)+\mathfrak{T}(r,g)) \leqslant Q(r)$ is a contradiction as we assume $n \geqslant 2m+9$, $m \geqslant 2$.

Case 2: Next assume $H \equiv 0$. Integrating (2.3) we get

$$(3.6)\qquad \frac{1}{F-1} \equiv \frac{A}{G-1}+B \Rightarrow \frac{1}{\widetilde{P}(f)} \equiv \frac{A}{\widetilde{P}(g)}-\frac{B}{c},$$

where A, B are integrating constants with $A \neq 0$. Recall that as $n \geqslant 2m+9$, $m \geqslant 2$, degree of $\widetilde{P}(z) \geqslant 13$ and we have already proved $\widetilde{P}(z)$ is critically injective. Also from (3.1) and $m \geqslant 2$ we can say the number of distinct zeros of $\widetilde{P}'(z)$ is $\geqslant 3$. Now from (3.6) and Lemma 2.7 we get $B/c=0$. Let $1/A=A_1$, then from (3.6),

$$(3.7)\qquad \widetilde{P}(f) \equiv A_1\widetilde{P}(g).$$

Sub-case 2.1: Assume $A_1 \neq 1$. Notice that

$$\widetilde{P}(z) - \widetilde{P}(0) = z^{n-m} R_1(z),$$

where $R_1(z)$ is an m degree polynomial, $R_1(0) \neq 0$ and all zeros of $R_1(z)$ are simple, namely α_j $(j = 1, 2, \ldots, m)$. From (3.7) we can write

$$\widetilde{P}(f) - A_1\widetilde{P}(0) \equiv A_1(\widetilde{P}(g) - \widetilde{P}(0)) \Rightarrow \widetilde{P}(f) - A_1\widetilde{P}(0) \equiv A_1 g^{n-m} \prod_{j=1}^{m} (g - \alpha_j). \tag{3.8}$$

Denote $\Psi(z) = \widetilde{P}(z) - A_1\widetilde{P}(0)$. From (3.1) we get $\Psi'(z) = \widetilde{P}'(z) = z^{n-m-1} \times \prod_{j=1}^{m} (z - \delta_j)$. Now similarly as $\widetilde{P}(z)$ it can be shown that $\Psi(z)$ is critically injective. $\Psi(z)$ has at most one multiple zero as any critically injective polynomial has at most one multiple zero. Next we discuss the following two cases:

Sub-case 2.1.1: First assume $\Psi(z)$ has exactly one multiple zero. As $\Psi(0) \neq 0$, the only possible multiple zero is one of the δ_j $(j = 1, 2, \ldots, m)$. Let us denote the multiple zero by δ_* and its multiplicity is equal to 2. Hence, we get

$$\Psi(z) = (z - \delta_*)^2 \prod_{j=1}^{n-2} (z - \beta_j),$$

where β_j $(j = 1, 2, \ldots, n-2)$ are distinct. Thus, (3.8) can be written as

$$(f - \delta_*)^2 \prod_{j=1}^{n-2} (f - \beta_j) \equiv A_1 g^{n-m} \prod_{j=1}^{m} (g - \alpha_j). \tag{3.9}$$

From (3.7) it is clear that $\mathfrak{T}(r, f) = \mathfrak{T}(r, g) + Q(r)$. Using Lemma 2.1 and (3.9) we obtain

$$\begin{aligned}(n-3)\mathfrak{T}(r, f) &\leqslant \overline{\mathfrak{N}}(r, \delta_*; f) + \sum_{j=1}^{n-2} \overline{\mathfrak{N}}(r, \beta_j; f) + Q(r, f) \\ &\leqslant \overline{\mathfrak{N}}(r, 0; g) + \sum_{j=1}^{m} \overline{\mathfrak{N}}(r, \alpha_j; g) + Q(r, f) \\ &\leqslant (m+1)\mathfrak{T}(r, f) + Q(r).\end{aligned}$$

Thus, we get $(n - m - 4)\mathfrak{T}(r, f) \leqslant Q(r)$, this is a contradiction as $n \geqslant 2m + 9$.

Sub-case 2.1.2: Next assume $\Psi(z)$ has no multiple zero. Let us denote all simple zeros of $\Psi(z)$ as γ_j $(j = 1, 2, \ldots, n)$. From (3.8)

$$\prod_{j=1}^{n} (f - \gamma_j) \equiv A_1 g^{n-m} \prod_{j=1}^{m} (g - \alpha_j). \tag{3.10}$$

By Lemma 2.1 and (3.10) we deduce

$$
\begin{aligned}
(n-2)\mathfrak{T}(r,f) &\leqslant \sum_{j=1}^{n}\overline{\mathfrak{N}}(r,\gamma_j;f)+Q(r,f)\leqslant \overline{\mathfrak{N}}(r,0;g)+\sum_{j=1}^{m}\overline{\mathfrak{N}}(r,\alpha_j;g)+Q(r,f)\\
&\leqslant (m+1)\mathfrak{T}(r,f)+Q(r).
\end{aligned}
$$

Thus, we get $(n-m-3)\mathfrak{T}(r,f)\leqslant Q(r)$. This is a contradiction as $n\geqslant 2m+9$.

Sub-case 2.2: Next assume $A_1=1$. Thus, from (3.7)

$$
\widetilde{P}(f)\equiv \widetilde{P}(g)\Rightarrow f^n+bf^{n-m}\equiv g^n+bg^{n-m}. \tag{3.11}
$$

Let $h\equiv f/g$. From (3.11) we get

$$
g^m(h^n-1)\equiv -b(h^{n-m}-1). \tag{3.12}
$$

First we assume that h is a non-constant function. Then we can write (3.12) as

$$
g^m\equiv -b\frac{(h-v)(h-v^2)\dots(h-v^{n-m-1})}{(h-u)(h-u^2)\dots(h-u^{n-1})}, \tag{3.13}
$$

where $v=\exp(2\pi\mathrm{i}/(n-m))$, $u=\exp(2\pi\mathrm{i}/n)$. Since $\gcd(n,m)=1$, $v^j\neq u^l$ for $j=1,2,\dots,n-m-1$, $l=1,2,\dots,n-1$. Suppose z_l be zero of $h-u^l$ for $l=1,2,\dots,n-1$. Then from (3.13) it is easy to see that the multiplicity of z_l is $\geqslant m$. Thus,

$$
\overline{\mathfrak{N}}\Big(r,\frac{1}{h-u^l}\Big)\leqslant \frac{1}{m}\mathfrak{N}\Big(r,\frac{1}{h-u^l}\Big)\leqslant \frac{1}{2}\mathfrak{T}(r,h)+Q(r). \tag{3.14}
$$

By Lemma 2.1 and (3.14) we obtain

$$
(n-3)\mathfrak{T}(r,h)\leqslant \sum_{l=1}^{n-1}\overline{\mathfrak{N}}\Big(r,\frac{1}{h-u^l}\Big)+Q(r)\leqslant \frac{n-1}{2}\mathfrak{T}(r,h)+Q(r).
$$

This implies $\frac{1}{2}(n-5)\mathfrak{T}(r,h)\leqslant Q(r)$, a contradiction arises as $n\geqslant 2m+9$ and $m\geqslant 2$. Thus, h is a constant function. But as g is a non-constant meromorphic function, from (3.12) we get

$$
h^n-1=0 \quad\text{and}\quad h^{n-m}-1=0.
$$

Since $\gcd(n,m)=1$, $h=1$. Therefore $f\equiv g$. This completes the proof. □

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