OSCILLATION CRITERIA FOR TWO DIMENSIONAL LINEAR NEUTRAL DELAY DIFFERENCE SYSTEMS

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Abstract. In this work, necessary and sufficient conditions for the oscillation of solutions of 2-dimensional linear neutral delay difference systems of the form

$$\Delta \begin{bmatrix} x(n)+p(n)x(n-m) \\ y(n)+p(n)y(n-m) \end{bmatrix} = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix} \begin{bmatrix} x(n-\alpha) \\ y(n-\beta) \end{bmatrix}$$

are established, where $m>0$, $\alpha \geqslant 0$, $\beta \geqslant 0$ are integers and $a(n)$, $b(n)$, $c(n)$, $d(n)$, $p(n)$ are sequences of real numbers.

Keywords: oscillation; nonoscillation; system of neutral equations; Krasnoselskii's fixed point theorem

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1. Introduction

Consider the 2-dimensional difference system

$$(\mathrm{S}_1) \qquad \Delta \begin{bmatrix} x(n)+p(n)x(n-m) \\ y(n)+p(n)y(n-m) \end{bmatrix} = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix} \begin{bmatrix} x(n-\alpha) \\ y(n-\beta) \end{bmatrix},$$

where $m>0$, $\alpha \geqslant 0$, $\beta \geqslant 0$ are integers and $a(n)$, $b(n)$, $c(n)$, $d(n)$, $p(n)$ are sequences of real numbers for $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, n_0+2, \ldots\}$, $n_0 \geqslant 0$. If $\alpha = 0$, $\beta = 0$ and $p(n) \equiv 0$ for all n, then (S_1) reduces to

$$(\mathrm{S}_2) \qquad \begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} a_1(n) & b_1(n) \\ c_1(n) & d_1(n) \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}.$$

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In [17], Tripathy has studied the oscillatory behaviour of solutions of the system (S_2) along with the oscillatory behaviour of solutions of the system

$$(S_3) \qquad \begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} a_1(n) & b_1(n) \\ c_1(n) & d_1(n) \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} + \begin{bmatrix} f_1(n) \\ f_2(n) \end{bmatrix}.$$

Indeed, (S_1) and (S_2) are not viewed as the direct discrete analogue of their continuous counterparts, so the work [17] is challenging, being done with the help of the work [11]. In this work, the oscillation and nonoscillation criteria for (S_1) are established unlike to the work [17]. Of course, the study of (S_1) is not so much simple when $\alpha > 0$, $\beta > 0$ and $p(n) \neq 0$ for all n.

In [7], [8], [9], Graef and Thandapani, Jiang and Tang, and Li have studied the oscillatory and asymptotic behaviour of all vector solutions of the system of the form

$$(S_4) \qquad \begin{bmatrix} \Delta x(n) \\ \Delta y(n-1) \end{bmatrix} = \begin{bmatrix} 0 & b(n) \\ -c(n) & 0 \end{bmatrix} \begin{bmatrix} f(x(n)) \\ g(y(n)) \end{bmatrix},$$

where $f, g \in C(\mathbb{R}, \mathbb{R})$ such that $uf(u) > 0$ and $ug(u) > 0$ for $u \neq 0$. We may note that (S_4) is a special case of (S_1), if we let $f(u) = u$ and $g(u) = u$. It is known that a similar kind of results can be obtained for

$$(S_5) \qquad \Delta \begin{bmatrix} x(n) + p(n)x(n-m) \\ y(n) + p(n)y(n-m) \end{bmatrix} = \begin{bmatrix} 0 & b(n) \\ c(n) & 0 \end{bmatrix} \begin{bmatrix} x(n-\alpha) \\ y(n-\beta) \end{bmatrix}$$

as long as the works [7], [8] and [9] are concerned.

Consider a particular case of (S_1) as

$$(S_6) \qquad \Delta \begin{bmatrix} x(n) + p(n)x(n-m) \\ y(n) + p(n)y(n-m) \end{bmatrix} = \begin{bmatrix} a(n) & 0 \\ 0 & d(n) \end{bmatrix} \begin{bmatrix} x(n-\alpha) \\ y(n-\beta) \end{bmatrix}$$

from which we find two first-order neutral delay difference equations

$$(1.1) \qquad \Delta[x(n) + p(n)x(n-m)] - a(n)x(n-\alpha) = 0,$$

$$(1.2) \qquad \Delta[y(n) + p(n)y(n-m)] - d(n)y(n-\beta) = 0.$$

A close observation reveals that the oscillation properties of (1.1) and (1.2) are studied by Parhi and Tripathy in their works [12] and [13] and hence the fact that (1.1) and (1.2) are oscillatory implies that (S_6) is oscillatory when $a(n)d(n) \neq 0$ for all n. Hence, we do not discuss the oscillation properties of (S_1) when either $a(n) = 0 = d(n)$ (as in (S_5)) or $b(n) = 0 = c(n)$ (as in (S_6)) for all n. In this work, our objective is to present the oscillatory behaviour of all vector solutions of (S_1) when $a(n) \neq 0$, $b(n) \neq 0$, $c(n) \neq 0$, $d(n) \neq 0$ for all n. Up to our best understanding, the present work is a new finding in the literature. However, there are some works (see, e.g., [4], [5], [10], [14], [15], [16]) in which the authors have studied oscillation and nonoscillation properties of some kind of neutral and nonneutral systems of equations that are not in the closed forms like (S_1), (S_2) and (S_3). Concerning difference equations and systems of difference equations, we refer to the monographs by Agarwal et al. (see [3], [1]) and by Elyadi (see [6]).

Definition 1.1. By a solution of (S_1) we mean a vector $X(n) = [x(n), y(n)]^\top$ which satisfies (S_1) for $n \in \mathbb{N}(-\varrho) = \{-\varrho, -\varrho+1, \ldots, 0, 1, 2, \ldots\}$, where $\varrho = \max\{m, \alpha, \beta\}$. We say that the solution $X(n)$ oscillates componentwise or simply oscillates or strongly oscillates, if each component oscillates. Otherwise, the solution $X(n)$ is called nonoscillatory. Therefore, a solution of (S_1) is nonoscillatory if it has a component which is eventually positive or eventually negative, and strongly nonoscillatory if both components of $X(n)$ are nonoscillatory. A vector solution $X(n)$ of (S_1) has the property that it oscillates or converges to zero as $n \to \infty$, if each component of $X(n)$ has this property.

Lemma 1.1 ([13]). *Let $f(n)$, $g(n)$ and $p(n)$ be real valued functions of discrete arguments defined for $n \geqslant n_0$ such that $f(n) = g(n) + p(n)g(n-m)$, $n \geqslant n_0 + m$, where $m \geqslant 0$ is an integer. Suppose that there exist real numbers b_1, b_2, b_3, b_4 such that $p(n)$ is in one of the following ranges:*
(1) $-\infty < b_1 \leqslant p(n) \leqslant 0$,
(2) $0 \leqslant p(n) \leqslant b_2 < 1$,
(3) $1 < b_3 \leqslant p(n) \leqslant b_4 < \infty$.
If $g(n) > 0$ for $n \geqslant n_0$, $\liminf\limits_{n\to\infty} g(n) = 0$, and $\lim\limits_{n\to\infty} f(n) = L$ exists, then $L = 0$.

Theorem 1.1 ([2]). *Let X be a Banach space. Let Ω be a bounded closed convex subset of X and let T_1, T_2 be maps of Ω into X such that $T_1x + T_2y \in \Omega$ for every pair $x, y \in \Omega$. If T_1 is a contraction and T_2 is completely continuous, then the equation $T_1x + T_2x = x$ has a solution in Ω.*

2. Oscillation criteria

In this section, necessary and sufficient conditions are established for the oscillation of all vector solutions of the system (S_1).

Theorem 2.1. *Let $0 < p(n) \leqslant r < 1$ for large n. Assume that $a(n) < 0$, $b(n) > 0$, $c(n) > 0$, $d(n) < 0$ are for large n such that*

(A_1) $\sum\limits_{n=0}^{\infty} b(n) < \infty$, $\sum\limits_{n=0}^{\infty} c(n) < \infty$.

Then every bounded vector solution of (S_1) *either strongly oscillates or converges to zero if and only if*

(A_2) $\sum\limits_{n=0}^{\infty} a(n) = -\infty$, $\sum\limits_{n=0}^{\infty} d(n) = -\infty$.

Proof. On the contrary, let $X(n) = [x(n), y(n)]^{\top}$ be a strongly nonoscillatory bounded vector solution of (S_1) such that $x(n) > 0$, $x(n-m) > 0$, $x(n-\alpha) > 0$, $x(n-\beta) > 0$ and $y(n) > 0$, $y(n-m) > 0$, $y(n-\alpha) > 0$, $y(n-\beta) > 0$ for $n \geqslant n_0 > \varrho$. Setting

$$K(n) = \sum_{i=n}^{\infty} b(i)y(i-\beta), \quad T(n) = \sum_{i=n}^{\infty} c(i)x(i-\alpha);$$
$$u(n) = x(n) + p(n)x(n-m), \quad v(n) = y(n) + p(n)y(n-m)$$

for (S_1), we find that

$$\Delta[u(n) + K(n)] = a(n)x(n-\alpha) \leqslant 0, \tag{2.1}$$
$$\Delta[v(n) + T(n)] = d(n)y(n-\beta) \leqslant 0 \tag{2.2}$$

for $n \geqslant n_1 > n_0$. Hence, there exists $n_2 > n_1$ such that $[u(n)+K(n)]$ and $[v(n)+T(n)]$ are monotonic for $n \geqslant n_2$. Since $u(n) > 0$, $v(n) > 0$ and $\lim\limits_{n\to\infty} K(n) < \infty$, $\lim\limits_{n\to\infty} T(n) < \infty$, then $\lim\limits_{n\to\infty} u(n)$ exists and $\lim\limits_{n\to\infty} v(n)$ exists. We claim that $\liminf\limits_{n\to\infty} x(n) = 0 = \liminf\limits_{n\to\infty} y(n)$. If not, we can find $n_3 > n_2$ such that $x(n-\alpha) > \gamma$ and $y(n-\beta) > \eta$ for $n \geqslant n_3$. Therefore, summing (2.1) and (2.2) from n_3 to ∞, we obtain contradictions to the hypothesis (A_2). So, our claim holds. By Lemma 1.1, it follows that $\lim\limits_{n\to\infty} u(n) = 0 = \lim\limits_{n\to\infty} v(n)$. Ultimately, $u(n) \geqslant x(n)$ and $v(n) \geqslant y(n)$ implies that $\lim\limits_{n\to\infty} x(n) = 0 = \lim\limits_{n\to\infty} y(n)$. The above argument is analogous, if we assume that $x(n) < 0$, $x(n-m) < 0$, $x(n-\alpha) < 0$, $x(n-\beta) < 0$ and $y(n) < 0$, $y(n-m) < 0$, $y(n-\alpha) < 0$, $y(n-\beta) < 0$ for $n \geqslant n_0 > \varrho$.

Next, we consider the case when $x(n) > 0$, $x(n-m) > 0$, $x(n-\alpha) > 0$, $x(n-\beta) > 0$ and $y(n) < 0$, $y(n-m) < 0$, $y(n-\alpha) < 0$, $y(n-\beta) < 0$ for $n \geqslant n_0 > \varrho$. Then

$$\Delta[u(n) + K(n)] = a(n)x(n-\alpha) \leqslant 0, \tag{2.3}$$
$$\Delta[v(n) + T(n)] = d(n)y(n-\beta) \geqslant 0 \tag{2.4}$$

and hence $[u(n) + K(n)]$ and $[v(n) + T(n)]$ are monotonic as well as bounded also for $n \geqslant n_2$. Consequently, $\lim\limits_{n\to\infty} [u(n) + K(n)]$ and $\lim\limits_{n\to\infty} [v(n) + T(n)]$ exist. Using the above argument, it is easy to see that $\lim\limits_{n\to\infty} X(n) = [0,0]^{\top}$. The case $x(n) < 0$, $x(n-m) < 0$, $x(n-\alpha) < 0$, $x(n-\beta) < 0$ and $y(n) > 0$, $y(n-m) > 0$, $y(n-\alpha) > 0$, $y(n-\beta) > 0$ for $n \geqslant n_0 > \varrho$ is similar.

Conversely, let us assume that (A_2) fails to hold. Let $\mathbf{B}$ denote the Banach space of all bounded sequences in $\mathbb{R}^2$ with the supremum norm, i.e., $\mathbf{B} = \Big\{X\colon \mathbb{N} \to \mathbb{R}^2\colon \|X\| = \sup_{n\in\mathbb{N}} |X| < \infty\Big\}$. For a fixed real number $k > 0$, put

$$\Omega = \{X \in \mathbf{B}\colon\ x(n),\ y(n) \in I,\ n \in \mathbb{N}\},$$

where $I = [\frac{1}{3}k(1-r), k]$. Indeed, $\Omega \subset \mathbf{B}$ is closed, bounded and convex. Due to (A_1), we can find $n_1 > 0$ such that

$$\sum_{n=n_1}^{\infty} |a(n)| < \frac{(1-r)}{6}, \quad \sum_{n=n_1}^{\infty} |b(n)| < \frac{(1-r)}{6},$$
$$\sum_{n=n_1}^{\infty} |c(n)| < \frac{(1-r)}{6}, \quad \sum_{n=n_1}^{\infty} |d(n)| < \frac{(1-r)}{6}.$$

Let us define the maps $G, H\colon \Omega \to \mathbf{B}$ such that

$$(GX)(n) = \begin{bmatrix} \dfrac{(2+r)k}{3} - p(n)x(n-m) - \displaystyle\sum_{s=n}^{\infty} a(s)x(s-\alpha) \\ \dfrac{(2+r)k}{3} - p(n)y(n-m) - \displaystyle\sum_{s=n}^{\infty} d(s)y(s-\beta) \end{bmatrix} \quad \text{for } n \geqslant n_1,$$
$$(GX)(n) = (GX)(n_1) \quad \text{for } 0 < n < n_1$$

and

$$(HX)(n) = \begin{bmatrix} -\displaystyle\sum_{s=n}^{\infty} b(s)y(s-\beta) \\ -\displaystyle\sum_{s=n}^{\infty} c(s)x(s-\alpha) \end{bmatrix} \quad \text{for } n \geqslant n_1,$$
$$(HX)(n) = (HX)(n_1) \quad \text{for } 0 < n < n_1.$$

We rewrite G, H as

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

Let $X, Y \in \Omega$. Then for $n \geqslant n_1$,

$$\begin{aligned}(G_1X)(n) + (H_1Y)(n) &= \frac{(2+r)k}{3} - p(n)x(n-m) \\ &\quad - \sum_{s=n}^{\infty} a(s)x(s-\alpha) - \sum_{s=n}^{\infty} b(s)y(s-\beta) \\ &\leqslant \frac{(2+r)k}{3} + \sum_{s=n}^{\infty} |a(s)|x(s-\alpha) + \sum_{s=n}^{\infty} |b(s)|y(s-\beta) \\ &\leqslant \frac{(2+r)k}{3} + \frac{(1-r)k}{6} + \frac{(1-r)k}{6} = k\end{aligned}$$

and

$$
\begin{aligned}
&(G_1X)(n) + (H_1Y)(n) \\
&\quad = \frac{(2+r)k}{3} - p(n)x(n-m) - \sum_{s=n}^{\infty} a(s)x(s-\alpha) - \sum_{s=n}^{\infty} b(s)y(s-\beta) \\
&\quad \geqslant \frac{(2+r)k}{3} - p(n)x(n-m) - \sum_{s=n}^{\infty} |a(s)|x(s-\alpha) - \sum_{s=n}^{\infty} |b(s)|y(s-\beta) \\
&\quad \geqslant \frac{(2+r)k}{3} - rk - \frac{(1-r)k}{6} - \frac{(1-r)k}{6} = \frac{k(1-r)}{3}.
\end{aligned}
$$

A similar observation can be made for $(G_2X)(n) + (H_2Y)(n)$, $n \geqslant n_1$. Hence, $GX + HY \in \Omega$. For X_1, $X_2 \in \Omega$, it is easy to verify that

$$
\begin{aligned}
|(G_1X_1)(n) - (G_1X_2)(n)| &\leqslant r|x_1(n-m) - x_2(n-m)| \\
&\qquad + \sum_{s=n}^{\infty} |a(s)||x_1(s-\alpha) - x_2(s-\alpha)| \\
&\leqslant \left[r + \frac{(1-r)}{6}\right] \|x_1 - x_2\| = \frac{(5r+1)}{6}\|x_1 - x_2\|,
\end{aligned}
$$

and

$$
|(G_2X_1)(n) - (G_2X_2)(n)| \leqslant \frac{(5r+1)}{6}\|y_1 - y_2\|
$$

for $n \geqslant n_1$ implies that

$$
\|GX_1 - GX_2\| \leqslant \frac{(5r+1)}{6}\|X_1 - X_2\|,
$$

that is, G is a contraction mapping.

Next, we show that H is continuous. Let $X_j = [x_j, y_j]^\top \in \Omega$ for any $j \in \mathbb{N}$. Let $X_j(n)$ be such that $x_j(n) \to x(n)$ and $y_j(n) \to y(n)$ as $j \to \infty$. If we choose $X = [x, y]^\top$, then $X_j \in \Omega$ implies that $X \in \Omega$ and hence $x(n), y(n) \in I$ for $n \geqslant n_1$. Therefore,

$$
\begin{aligned}
|(H_1X_j)(n) - (H_1X)(n)| &\leqslant \sum_{s=n}^{\infty} |b(s)||y_j(s-\beta) - y(s-\beta)| \to 0 \quad \text{as } j \to \infty, \\
|(H_2X_j)(n) - (H_2X)(n)| &\leqslant \sum_{s=n}^{\infty} |c(s)||x_j(s-\alpha) - x(s-\alpha)| \to 0 \quad \text{as } j \to \infty
\end{aligned}
$$

imply that

$$
\|(HX_j) - (HX)\| \to 0 \quad \text{as } j \to \infty,
$$

that is, H is continuous. To complete the proof of the theorem, we need to show that $H\Omega$ is uniformly Cauchy. Indeed, for $\varepsilon > \frac{2}{3}k(1-r) > 0$, we can find $n_2 > n_1$

such that for $n \geqslant n_2$

$$\sum_{s=n}^{\infty} |b(s)||y(s-\beta)| < \frac{\varepsilon}{2}, \quad \sum_{s=n}^{\infty} |c(s)||x(s-\alpha)| < \frac{\varepsilon}{2}.$$

Hence for $n_4 > n_3 > n_2$, it follows that

$$\begin{aligned}|(H_1X)(n_4) - (H_1X)(n_3)| &= \left|\sum_{s=n_4}^{\infty} b(s)y(s-\beta) - \sum_{s=n_3}^{\infty} b(s)y(s-\beta)\right| \\ &\leqslant \sum_{s=n_4}^{\infty} |b(s)||y(s-\beta)| + \sum_{s=n_3}^{\infty} |b(s)||y(s-\beta)| < \varepsilon\end{aligned}$$

and

$$\begin{aligned}|(H_2X)(n_4) - (H_2X)(n_3)| &= \left|\sum_{s=n_4}^{\infty} c(s)x(s-\alpha) - \sum_{s=n_3}^{\infty} c(s)x(s-\alpha)\right| \\ &\leqslant \sum_{s=n_4}^{\infty} |c(s)||x(s-\alpha)| + \sum_{s=n_3}^{\infty} |c(s)||x(s-\alpha)| < \varepsilon,\end{aligned}$$

that is, $H\Omega$ is uniformly Cauchy.

Hence by Krasnoselskii's fixed point theorem, there exists a solution $X(n) = [x(n), y(n)]^\top$ of (S_1) in Ω such that $(GX)(n)+(HX)(n) = X(n)$ for $n \geqslant n_1$. Keeping in view that

$$(G_1X)(n) + (H_1X)(n) = x(n), \quad (G_2X)(n) + (H_2X)(n) = y(n) \quad \text{for } n \geqslant n_1,$$

it is easy to verify that $X(n) = [x(n), y(n)]^\top$ is the required vector solution of (S_1). This completes the proof of the theorem. $\square$

Theorem 2.2. *Let* $1 < t \leqslant p(n) \leqslant t_1 \leqslant \frac{1}{2}t^2 < \infty$ *for large* n. *If* (A_1) *holds, then the conclusion of Theorem* 2.1 *remains intact.*

P r o o f. The sufficient part of the proof is the same as in Theorem 2.1. For the necessary part, let $\mathbf{B}$ denote the Banach space of all bounded sequences in $\mathbb{R}^2$ with the sup norm, i.e.,

$$\mathbf{B} = \left\{X\colon \mathbb{N} \to \mathbb{R}^2\colon \|X\| = \sup_{n\in\mathbb{N}} |X| < \infty\right\}.$$

For a fixed real number $k > 0$, put

$$\Omega_1 = \{X \in \mathbf{B}\colon x(n), y(n) \in I_1, n \in \mathbb{N}\},$$

where $I_1 = [k(t-1)/(8tt_1), k]$. It is easy to see that $\Omega_1 \subset \mathbf{B}$ is closed, bounded and convex. Because of (A_1), we can find $n_1 > 0$ such that

$$\sum_{n=n_1}^{\infty} |a(n)| < \frac{(t-1)}{4t}, \quad \sum_{n=n_1}^{\infty} |b(n)| < \frac{(t-1)}{8t_1},$$
$$\sum_{n=n_1}^{\infty} |c(n)| < \frac{(t-1)}{8t_1}, \quad \sum_{n=n_1}^{\infty} |d(n)| < \frac{(t-1)}{4t}.$$

We define the maps $G, H\colon \Omega_1 \to \mathbf{B}$ as

$$(GX)(n) = \begin{bmatrix} \frac{(2t^2+t-1)k}{4tp(n+m)} - \frac{x(n+m)}{p(n+m)} - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} a(s)x(s-\alpha) \\ \frac{(2t^2+t-1)k}{4tp(n+m)} - \frac{y(n+m)}{p(n+m)} - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} d(s)y(s-\beta) \end{bmatrix}$$

for $n \geqslant n_1$,

$$(GX)(n) = (GX)(n_1) \quad \text{for } 0 < n < n_1$$

and

$$(HX)(n) = \begin{bmatrix} -\frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} b(s)y(s-\beta) \\ -\frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} c(s)x(s-\alpha) \end{bmatrix} \quad \text{for } n \geqslant n_1,$$
$$(HX)(n) = (HX)(n_1) \quad \text{for } 0 < n < n_1.$$

We rewrite G, H as

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

Let $X, Y \in \Omega_1$. Then for $n \geqslant n_1$,

$$\begin{aligned} (G_1X)(n) + (H_1Y)(n) &= \frac{(2t^2+t-1)k}{4tp(n+m)} - \frac{x(n+m)}{p(n+m)} - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} a(s)x(s-\alpha) \\ &\quad - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} b(s)y(s-\beta) \\ &\leqslant \frac{(2t^2+t-1)k}{4t^2} + \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} |a(s)|x(s-\alpha) \\ &\quad + \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} |b(s)|y(s-\beta) \\ &\leqslant \frac{(2t^2+t-1)k}{4t^2} + \frac{(t-1)k}{8tt_1} + \frac{(t-1)k}{4t^2} \\ &\leqslant \frac{(2t^2+t-1)k}{4t^2} + \frac{(t-1)k}{8t^2} + \frac{(t-1)k}{4t^2} = k\frac{4t^2+5t-5}{8t^2} < k \end{aligned}$$

and

$$(G_1X)(n)+(H_1Y)(n)=\frac{(2t^2+t-1)k}{4tp(n+m)}-\frac{x(n+m)}{p(n+m)}-\frac{1}{p(n+m)}\sum_{s=n+m}^{\infty}a(s)x(s-\alpha)$$

$$-\frac{1}{p(n+m)}\sum_{s=n+m}^{\infty}b(s)y(s-\beta)$$

$$\geqslant\frac{(2t^2+t-1)k}{4tt_1}-\frac{x(n+m)}{p(n+m)}-\frac{1}{p(n+m)}\sum_{s=n+m}^{\infty}|b(s)|y(s-\beta)$$

$$\geqslant\frac{(2t^2+t-1)k}{4tt_1}-\frac{k}{t}-\frac{(t-1)k}{8tt_1}$$

$$=k\frac{4t^2+t-8t_1-1}{8tt_1}>k\frac{t-1}{8tt_1}.$$

A similar observation can be obtained for $(G_2X)(n)+(H_2Y)(n)$, $n\geqslant n_1$. Hence, $GX+HY\in\Omega_1$. For $X_1,X_2\in\Omega_1$, it is easy to verify that

$$|(G_1X_1)(n)-(G_1X_2)(n)|\leqslant\frac{1}{t}|x_1(n+m)-x_2(n+m)|$$

$$+\frac{1}{p(n+m)}\sum_{s=n+m}^{\infty}|a(s)||x_1(s-\alpha)-x_2(s-\alpha)|$$

$$\leqslant\Big[\frac{1}{t}+\frac{(t-1)}{4t}\Big]\|x_1-x_2\|=\frac{(3+t)}{4t}\|x_1-x_2\|,$$

and

$$|(G_2X_1)(n)-(G_2X_2)(n)|\leqslant\frac{(3+t)}{4t}\|y_1-y_2\|$$

for $n\geqslant n_1$ implies that

$$\|GX_1-GX_2\|\leqslant\frac{(3+t)}{4t}\|X_1-X_2\|,$$

that is, G is a contraction mapping.

Proceeding as in the proof of Theorem 2.1, we can show that H is continuous and $H\Omega_1$ is uniformly Cauchy. Hence by Krasnoselskii's fixed point theorem, there exists a solution $X(n)=[x(n),y(n)]^\top$ of (S_1) in Ω_1 such that $(GX)(n)+(HX)(n)=X(n)$ for $n\geqslant n_1$. Therefore, the theorem is proved. □

Theorem 2.3. *Let* $-1<r_1\leqslant p(n)\leqslant 0$ *for large* n. *If* (A_1) *holds, then the conclusion of Theorem* 2.1 *remains intact.*

P r o o f. Proceeding as in the proof of Theorem 2.1, we can find an $n_2>n_1$ such that $[u(n)+K(n)]$ and $[v(n)+T(n)]$ are monotonic for $n\geqslant n_2$. Since $\lim\limits_{n\to\infty}K(n)<\infty$ and $\lim\limits_{n\to\infty}T(n)<\infty$, then $\lim\limits_{n\to\infty}u(n)$ exists and $\lim\limits_{n\to\infty}v(n)$ exists. Using the same argument as in the proof of Theorem 2.1, we can show that $\liminf_{n\to\infty} x(n) = 0 = \liminf_{n\to\infty} y(n)$. By Lemma 1.1, it follows that $\lim_{n\to\infty} u(n) = 0 = \lim_{n\to\infty} v(n)$. Therefore,

$$\begin{aligned}0 = \lim_{n\to\infty} u(n) &= \limsup_{n\to\infty}(x(n) + p(n)x(n-m)) \geqslant \limsup_{n\to\infty}(x(n) + r_1 x(n-m))\\ &\geqslant \limsup_{n\to\infty} x(n) + \liminf_{n\to\infty}(r_1 x(n-m)) = (1+r_1)\limsup_{n\to\infty} x(n)\end{aligned}$$

implies that $\lim_{n\to\infty} x(n) = 0$. Similarly, we can show that $\lim_{n\to\infty} y(n) = 0$. The rest of the sufficient part follows from Theorem 2.1.

Conversely, assume that (A_2) fails to hold. Let $\mathbf{B}$ denote the Banach space of all bounded sequences in $\mathbb{R}^2$ with the supremum norm defined by $\mathbf{B} = \Big\{X\colon \mathbb{N} \to \mathbb{R}^2\colon \|X\| = \sup_{n\in\mathbb{N}} |X| < \infty\Big\}$. For a fixed real number $k > 0$, put

$$\Omega_2 = \{X \in \mathbf{B}\colon\ x(n),\ y(n) \in I_2,\ n \in \mathbb{N}\},$$

where $I_2 = [\frac{1}{12}k(1+r_1), k]$. Indeed, $\Omega_2 \subset \mathbf{B}$ is closed, bounded and convex. Due to (A_1), we can find $n_1 > 0$ such that

$$\begin{aligned}\sum_{n=n_1}^{\infty} |a(n)| < \frac{(1+r_1)}{24}, \quad \sum_{n=n_1}^{\infty} |b(n)| < \frac{(1+r_1)}{24},\\ \sum_{n=n_1}^{\infty} |c(n)| < \frac{(1+r_1)}{24}, \quad \sum_{n=n_1}^{\infty} |d(n)| < \frac{(1+r_1)}{24}.\end{aligned}$$

Let us define the maps $G, H\colon\ \Omega_2 \to \mathbf{B}$ such that

$$\begin{aligned}(GX)(n) &= \begin{bmatrix} \dfrac{(1+r_1)k}{6} - p(n)x(n-m) - \displaystyle\sum_{s=n}^{\infty} a(s)x(s-\alpha) \\ \dfrac{(1+r_1)k}{6} - p(n)y(n-m) - \displaystyle\sum_{s=n}^{\infty} d(s)y(s-\beta) \end{bmatrix} \quad \text{for } n \geqslant n_1,\\ (GX)(n) &= (GX)(n_1) \quad \text{for } 0 < n < n_1\end{aligned}$$

and

$$\begin{aligned}(HX)(n) &= \begin{bmatrix} -\displaystyle\sum_{s=n}^{\infty} b(s)y(s-\beta) \\ -\displaystyle\sum_{s=n}^{\infty} c(s)x(s-\alpha) \end{bmatrix} \quad \text{for } n \geqslant n_1,\\ (HX)(n) &= (HX)(n_1) \quad \text{for } 0 < n < n_1.\end{aligned}$$

We note that

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

The rest of the proof follows from the proof of Theorem 2.1 and hence the details are omitted. □

Theorem 2.4. *Let $-\infty < r_2 \leqslant p(n) \leqslant r_3 < -1$ for large n. If (A_1) holds, then the conclusion of Theorem 2.1 remains intact.*

Proof. The sufficient part of the proof is similar to that of Theorem 2.3. By Lemma 1.1, it follows that $\lim_{n\to\infty} u(n) = 0 = \lim_{n\to\infty} v(n)$. Hence,

$$\begin{aligned}0 = \lim_{n\to\infty} u(n) &= \liminf_{n\to\infty}(x(n) + p(n)x(n-m)) \leqslant \liminf_{n\to\infty}(x(n) + r_3x(n-m))\\ &\leqslant \limsup_{n\to\infty} x(n) + \liminf_{n\to\infty}(r_3x(n-m)) = (1+r_3)\limsup_{n\to\infty} x(n)\end{aligned}$$

implies that $\lim_{n\to\infty} x(n) = 0$. Similarly, we can show that $\lim_{n\to\infty} y(n) = 0$.

For the necessary part of the proof, let $\mathbf{B}$ denote the Banach space of all bounded sequences in $\mathbb{R}^2$ with the sup norm, i.e., $\mathbf{B} = \Big\{X\colon \mathbb{N} \to \mathbb{R}^2\colon \|X\| = \sup_{n\in\mathbb{N}}|X| < \infty\Big\}$. For a fixed real number $k > 0$, put

$$\Omega_3 = \{X \in \mathbf{B}\colon\ x(n),\ y(n) \in I_3,\ n \in \mathbb{N}\},$$

where $I_3 = [-kr_3/(M - r_3), Lk]$, and

$$M > \max\Big\{-r_2, r_3 + \frac{r_3}{1+r_3}\Big\},\quad L = \frac{2M - (M+1)r_3}{(r_3 - M)(1+r_3)} > 0.$$

Indeed, $\Omega_3 \subset \mathbf{B}$ is closed, bounded and convex. Due to (A_1), we can find $n_1 > 0$ such that

$$\begin{aligned}&\sum_{n=n_1}^{\infty} |a(n)| < \frac{-r_3}{(M-r_3)},\quad \sum_{n=n_1}^{\infty} |b(n)| < \frac{-r_3}{(M-r_3)},\\ &\sum_{n=n_1}^{\infty} |c(n)| < \frac{-r_3}{(M-r_3)},\quad \sum_{n=n_1}^{\infty} |d(n)| < \frac{-r_3}{(M-r_3)}.\end{aligned}$$

Let us define the maps $G, H\colon \Omega_3 \to \mathbf{B}$ such that

$$(GX)(n) = \begin{bmatrix} \dfrac{-(2-r_3)Mk}{(M-r_3)p(n+m)} - \dfrac{x(n+m)}{p(n+m)} - \dfrac{1}{p(n+m)}\displaystyle\sum_{s=n+m}^{\infty} a(s)x(s-\alpha)\\ \dfrac{-(2-r_3)Mk}{(M-r_3)p(n+m)} - \dfrac{y(n+m)}{p(n+m)} - \dfrac{1}{p(n+m)}\displaystyle\sum_{s=n+m}^{\infty} d(s)y(s-\beta)\end{bmatrix}$$

for $n \geqslant n_1$,

$(GX)(n) = (GX)(n_1)\quad$ for $0 < n < n_1$

and

$$(HX)(n) = \begin{bmatrix} -\dfrac{1}{p(n+m)}\displaystyle\sum_{s=n+m}^{\infty} b(s)y(s-\beta)\\ -\dfrac{1}{p(n+m)}\displaystyle\sum_{s=n+m}^{\infty} c(s)x(s-\alpha)\end{bmatrix}\quad \text{for } n \geqslant n_1,$$

$(HX)(n) = (HX)(n_1)\quad$ for $0 < n < n_1$.

We note that

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

Let $X, Y \in \Omega_3$. Then for $n \geqslant n_1$,

$$\begin{aligned}
(G_1X)(n) + (H_1Y)(n) &= \frac{-(2-r_3)Mk}{(M-r_3)p(n+m)} - \frac{x(n+m)}{p(n+m)} \\
&\quad - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} a(s)x(s-\alpha) \\
&\quad - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} b(s)y(s-\beta) \\
&\leqslant \frac{-(2-r_3)Mk}{(M-r_3)r_3} - \frac{x(n+m)}{p(n+m)} - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} b(s)y(s-\beta) \\
&\leqslant \frac{-(2-r_3)Mk}{(M-r_3)r_3} - \frac{Lk}{r_3} + \frac{Lk}{(M-r_3)} \\
&= -k\Big[\frac{L(M-r_3) + 2M - (M+1)r_3}{(M-r_3)r_3}\Big] = kL
\end{aligned}$$

and

$$\begin{aligned}
(G_1X)(n) + (H_1Y)(n) &= \frac{-(2-r_3)Mk}{(M-r_3)p(n+m)} - \frac{x(n+m)}{p(n+m)} \\
&\quad - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} a(s)x(s-\alpha) \\
&\quad - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} b(s)y(s-\beta) \\
&\geqslant \frac{-(2-r_3)Mk}{(M-r_3)r_2} - \frac{kr_3}{(M-r_3)r_2} + \frac{1}{r_2} \sum_{s=n+m}^{\infty} |a(s)|x(s-\alpha) \\
&\geqslant \frac{-(2-r_3)Mk}{(M-r_3)r_2} - \frac{2kr_3}{(M-r_3)r_2} \geqslant -\frac{kr_3}{(M-r_3)}.
\end{aligned}$$

A similar observation can be obtained for $(G_2X)(n) + (H_2Y)(n)$, $n \geqslant n_1$. Hence, $GX + HY \in \Omega_3$. For $X_1, X_2 \in \Omega_3$, it is easy to verify that

$$\begin{aligned}
|(G_1X_1)(n) - (G_1X_2)(n)| &\leqslant -\frac{1}{r_3}|x_1(n+m) - x_2(n+m)| \\
&\quad - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} |a(s)||x_1(s-\alpha) - x_2(s-\alpha)| \\
&\leqslant \Big[-\frac{1}{r_3} + \frac{1}{M-r_3}\Big]\|x_1 - x_2\|
\end{aligned}$$

and

$$|(G_2X_1)(n)-(G_2X_2)(n)| \leqslant \Big[-\frac{1}{r_3}+\frac{1}{M-r_3}\Big]\|y_1-y_2\|$$

for $n \geqslant n_1$ implies that

$$\|GX_1-GX_2\| \leqslant \Big[-\frac{1}{r_3}+\frac{1}{M-r_3}\Big]\|X_1-X_2\|,$$

that is, G is a contraction mapping.

Proceeding as in the proof of Theorem 2.3, we can show that H is continuous and $H\Omega_3$ is uniformly Cauchy. Hence by Krasnoselskii's fixed point theorem, there exists a solution $X(n)=[x(n),y(n)]^\top$ of (S_1) in Ω_3 such that $(GX)(n)+(HX)(n)=X(n)$ for $n \geqslant n_1$. Therefore, the theorem is proved. □

R e m a r k 2.1. It would be interesting to keep this work up for any solution of the system (S_1) (i.e., not necessarily the bounded solution).

E x a m p l e 2.1. Consider a 2-dimensional linear neutral difference system of the form:

$$(\mathrm{S}_7)\quad \Delta\begin{bmatrix} x(n)+\mathrm{e}^{-n}x(n-2)\\ y(n)+\mathrm{e}^{-n}y(n-2)\end{bmatrix} = \begin{bmatrix} -(2+\mathrm{e}^{-n}+2\mathrm{e}^{-(n+1)}) & \mathrm{e}^{-(n+2)}\\ \mathrm{e}^{-n} & -(2+\mathrm{e}^{-n}+2\mathrm{e}^{-(n+1)})\end{bmatrix} \times\begin{bmatrix} x(n-4)\\ y(n-6)\end{bmatrix}\quad \text{for } n>6.$$

Clearly, (A_1) and (A_2) are satisfied for (S_7). By Theorem 2.1, every bounded vector solution $X(n)$ of (S_7) is strongly oscillatory. Indeed, $X(n)=[(-1)^n,\mathrm{e}(-1)^n]^\top$ is one of such solutions of (S_7).

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