COVERING ENERGY OF POSETS AND ITS BOUNDS

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Cordially dedicated to Professor N. K. Thakare on his 84th birthday

Abstract. The concept of covering energy of a poset is known and its McClelland type bounds are available in the literature. In this paper, we establish formulas for the covering energy of a crown with $2n$ elements and a fence with n elements. A lower bound for the largest eigenvalue of a poset is established. Using this lower bound, we improve the McClelland type bounds for the covering energy for some special classes of posets.

Keywords: covering energy of poset; eigenvalue; spectrum; upper bound; lower bound

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1. Introduction and preliminaries

The concept of the energy of graph was introduced by Gutman in 1978 (see [8]). This concept has its roots in the Hückel molecular orbital (HMO) theory, see Hückel [12]. The *energy of a graph* G is defined as the sum of the absolute values of all eigenvalues of the adjacency matrix $A(G)$ of G, denoted by $\mathcal{E}(G)$. We label the eigenvalues of $A(G)$ in the non-increasing order as $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \ldots \geqslant \lambda_n$. This set of eigenvalues is called the *spectrum* of G and denoted by $\mathrm{Spec}(G) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. For details of the theory of graph energy, see Li, Shi and Gutman [15] and the survey article by Gutman and Ramane [11].

McClelland in [16] gave simple bounds for the energy of a graph. A number of researchers have improved these bounds, e.g. Altindağ and Bozkurt [2], Das et al. [5]. During the last four decades, many researchers have worked in this area. In the context of graphs, more than 200 different types of "energies" were proposed, see Gutman and Furtula [9] and [10]. Recently, Pawar and Bhangale [19] proposed one more graph energy.

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Pawar and Bhamre (see [17] and [18]) extended the concept of the "energy" to posets by defining the covering energy. They obtained the McClelland type bounds for the covering energy of a poset. In the present paper, we give a lower bound for the largest eigenvalue of a poset. We improve the McClelland type bounds for the covering energy for some special classes of posets.

We recall some definitions. A nonempty set P, together with a binary relation $\leqslant$ which is reflexive, antisymmetric, and transitive, is called a *partially ordered set*, in short, a *poset*. The *Hasse diagram of a poset* is a representation of a poset in the plane. For a poset $(P, \leqslant)$ one represents each element of P as a vertex in the plane and draws a line segment that goes upward from x to y whenever y covers x (i.e., whenever $x < y$ and there is no z such that $x < z < y$, denoted by $x \prec y$). We call such a line as an *edge* and denote the set of all edges in P by $e(P)$. The Hasse diagram of a poset P considered as a graph is called the *covering graph*. We denote it by $G(P)$. Hasse diagrams of some posets are depicted in Figure 1.

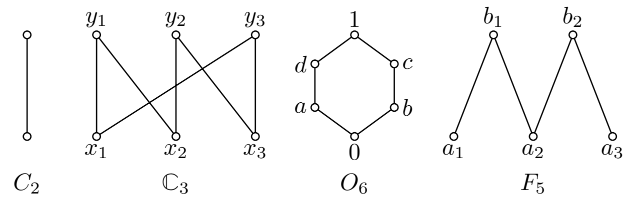

Figure 1. Hasse diagrams of some posets.

Two elements $a, b \in P$ are said to be *comparable* if either $a \leqslant b$ or $b \leqslant a$; otherwise they are said to be *incomparable*. A poset in which every pair of elements is comparable is called a *chain*, and if every pair of elements is incomparable, it is called an *antichain*.

A *lattice* is a poset in which every pair of elements has the supremum (called their *join*) and the infimum (called their *meet*). If a and b are elements in a lattice L, then their join and meet are denoted by $a \vee b$ and $a \wedge b$, respectively. An element x in a poset P is called *doubly-irreducible* if it covers and is covered by at most one element. The set of all doubly irreducible elements in P is denoted by $\mathrm{Irr}(P)$. A *reducible element* is an element in P, which is not doubly irreducible. The set of all reducible elements in P is denoted by $R(P)$. If there is only one poset under discussion, then the notation $R(P)$ will be replaced by R.

A partially ordered set $F_n = \{x_1, x_2, \ldots, x_n\}$ is called *fence,* if either $x_1 < x_2$, $x_2 > x_3, \ldots, x_{2m-1} < x_{2m}$, $x_{2m} > x_{2m+1}, \ldots, x_{n-1} < x_n$ if n is even ($x_{n-1} > x_n$ if n is odd) or $x_1 > x_2$, $x_2 < x_3, \ldots, x_{2m-1} > x_{2m}$, $x_{2m} < x_{2m+1}, \ldots, x_{n-1} > x_n$ if n is even ($x_{n-1} < x_n$, if n is odd) are the only comparability relations. A fence F_n is called a *lower fence* if $x_1 < x_2$ and an *upper fence* if $x_1 > x_2$, e.g. the fence F_5 as depicted in Figure 1 is a lower fence and its dual F_n^* is an upper fence.

For an integer $n \geqslant 3$, a *crown of order* n is a poset $P = \{x_1, x_2, x_3, \ldots, x_n, y_1, y_2, y_3, \ldots, y_n\}$ whose elements satisfy precisely the comparabilities $x_1 < y_1$, $y_1 > x_2$, $x_2 < y_2$, $y_2 > x_3$, $x_3 < y_3$, $y_3 > x_4, \ldots, x_{n-1} < y_{n-1}$, $y_{n-1} > x_n$, $x_n < y_n$, $y_n > x_1$. We denote a crown of order n by $\mathbb{C}_n$. The crown $\mathbb{C}_3$ is as depicted in Figure 1. The concepts of a doubly irreducible element, crown, and fence are useful to study dismantlable lattices, see, e.g. Kelly and Rival [14], Rival [20], and Thakare et al. [21].

Throughout this paper, P denotes a finite poset. For undefined terms and notations from lattice theory, refer to Grätzer [7] or Davey and Priestley [6], and for graph theoretic terms, see Cvetković et al. [4] or Li, Shi and Gutman [15]. The following two results are used in the next sections.

Lemma 1.1 ([15]). *For a path $P_n, n \geqslant 2$,*

$$\operatorname{Spec}(P_n) = \left\{2\cos\frac{\pi r}{n+1} :\ r = 1, 2, \ldots, n\right\}$$

and

$$\mathcal{E}(P_n) = \begin{cases} \dfrac{2}{\sin(\pi/2(n+1))} - 2 & \text{if } n \equiv 0 \pmod 2, \\ \dfrac{2\cos(\pi/2(n+1))}{\sin(\pi/2(n+1))} - 2 & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

Lemma 1.2 ([13]). *For $p \geqslant 3$, let $\mathbf{C}_p$ and $\overline{\mathbf{C}_p}$ denote a p-cycle and the compliment of $\mathbf{C}_p$, respectively, then*

$$\operatorname{Spec}(\mathbf{C}_p) = \left\{2, 2\cos\frac{2\pi}{p}, 2\cos\frac{4\pi}{p}, 2\cos\frac{6\pi}{p}, \ldots, 2\cos\frac{2(p-1)\pi}{p}\right\}$$

and

$$\operatorname{Spec}(\overline{\mathbf{C}_p}) = \left\{p-3, -1-2\cos\frac{2\pi}{p}, -1-2\cos\frac{4\pi}{p}, \ldots, -1-2\cos\frac{2(p-1)\pi}{p}\right\}.$$

2. The covering energy of a poset

When the energy of a graph G was defined by Gutman [8], no loop edges were allowed. However, it is natural to ask what happens if some loops are present. It does not look promising to allow that vertices with loops form an *arbitrary subset* X of the vertex set of G. Much after Gutman [8], Adiga et al. [1] allowed X to be a minimal covering set of the graph. Now if G is the covering graph of a poset P, then we can use the language of posets to uniquely define a vertex set X in a natural way, and we can allow loops exactly at the vertices belonging to X. This is how Adiga et al. [1]

motivated Pawar and Bhamre [17] to introduce the concept of covering energy of a poset P, in which $R(P)$ plays the role of X. We recall the definition and some of the results proved in [17] and [18].

Definition 2.1 ([17]). Let $P = \{v_1, v_2, v_3, \dots, v_n\}$ be a poset. The *covering matrix* of P denoted by $C(P)$ is an $n \times n$ matrix $C(P) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j \text{ and } v_i \prec v_j \text{ or } v_j \prec v_i, \\ 1 & \text{if } i = j \text{ and } v_i \notin \mathrm{Irr}(P), \\ 0 & \text{otherwise.} \end{cases}$$

The *characteristic polynomial of* P denoted by $\psi(P, \lambda)$ is the determinant $\det(\lambda I_n - C(P))$. The eigenvalues of $C(P)$ are called the *eigenvalues of the poset* P. The sum of absolute values of all these eigenvalues is called the *covering energy* of P and denoted by $E(P)$. If $E(P)$ is an integer, then P is called an *integral poset.*

Note that $C(P)$ depends on how (in which order) we list the elements of P, but this causes no problem since a different listing of elements gives a matrix similar to $C(P)$ and similar matrices have the same characteristic polynomial and eigenvalues. Hence, we can always fix a list $v_1, v_2, \dots, v_n$ and work with $C(P)$ defined by this list. As $C(P)$ is a real and symmetric matrix, its all eigenvalues are real numbers. We label them in the non-increasing order as $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \dots \geqslant \lambda_n$. This set of eigenvalues is called the *spectrum* of P and denoted by $\mathrm{Spec}(P)$. It is clear that

$$E(P) = \sum_{i=1}^{n} |\lambda_i|.$$

We compute covering energies of some simple posets depicted in Figure 1.

Example 2.2. The covering matrix of the chain C_2 is $C(C_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This leads to $\psi(C_2, \lambda) = \lambda^2 - 1$, $\mathrm{Spec}(C_2) = \{1, -1\}$ and $E(C_2) = 2$.

Example 2.3. For the lattice $O_6 = \{0, a, b, c, d, 1\}$ in Figure 1, the covering matrix, characteristic polynomial, spectrum and covering energy of O_6 are given by

$$C(O_6) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

$$\psi(O_6, \lambda) = -3 - 6\lambda + 7\lambda^2 + 8\lambda^3 - 5\lambda^4 - 2\lambda^5 + \lambda^6,$$

$$\mathrm{Spec}(O_6) = \{1 + \sqrt{2}, \sqrt{3}, 1, 1 - \sqrt{2}, -1, -\sqrt{3}\}, \quad E(O_6) = 2(1 + \sqrt{2} + \sqrt{3}).$$

E x a m p l e 2.4. The crown $\mathbb{C}_3 = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ in Figure 1 is an integral poset. The covering matrix, characteristic polynomial, spectrum and covering energy of $\mathbb{C}_3$ are

$$C(\mathbb{C}_3) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$\psi(\mathbb{C}_3, \lambda) = -12\lambda^2 + 4\lambda^3 + 9\lambda^4 - 6\lambda^5 + \lambda^6 = \lambda^2(\lambda+1)(\lambda-2)^2(\lambda-3),$$

$$\operatorname{Spec}(\mathbb{C}_3) = \{3, 2, 2, 0, 0, -1\}, \quad E(\mathbb{C}_3) = 8.$$

E x a m p l e 2.5. Let $F_5 = \{a_1, b_1, a_2, b_2, a_3\}$ be the poset shown in Figure 1. The covering matrix, characteristic polynomial, spectrum and covering energy of F_5 are

$$C(F_5) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\psi(F_5, \lambda) = 1 - \lambda - 5\lambda^2 + \lambda^3 + 3\lambda^4 - \lambda^5 = -(\lambda+1)(\lambda^2 - 3\lambda + 1)(\lambda^2 - \lambda - 1),$$

$$\operatorname{Spec}(F_5) = \Big\{\frac{1}{2}(3+\sqrt{5}), \frac{1}{2}(1+\sqrt{5}), \frac{1}{2}(3-\sqrt{5}), \frac{1}{2}(1-\sqrt{5}), -1\Big\}, \quad E(F_5) = 4 + \sqrt{5}.$$

Definition 2.6. Let P and Q be two disjoint posets. Then $P \cup Q$ is a poset under the partial order defined by $x \leqslant y$ in $P \cup Q$ if and only if either $x, y \in P$ and $x \leqslant y$ in P or $x, y \in Q$ and $x \leqslant y$ in Q.

Note that, if $C(P)$, $C(Q)$ and $C(P \cup Q)$ are the covering matrices of P, Q and $P \cup Q$, respectively, then $C(P \cup Q) = \begin{bmatrix} C(P) & 0 \\ 0 & C(Q) \end{bmatrix}$. Hence, $\psi(P \cup Q, \lambda) = \psi(P, \lambda)\psi(Q, \lambda)$. This leads to next claim.

Proposition 2.7. *Let P and Q be two disjoint posets. Then*

$$E(P \cup Q) = E(P) + E(Q).$$

Corollary 2.8. *Let $P_1, P_2, \ldots, P_k$ be k disjoint posets. Then*

$$E\Big(\bigcup_{i=1}^{k} P_i\Big) = \sum_{i=1}^{k} E(P_i).$$

We note that if $P_1 \cong P_2$, then $E(P_1) = E(P_2)$. But, two non-isomorphic posets may have the same covering energy, e.g. a poset and its dual have the same covering matrix. This fact motivates us to define the known concepts of equienergetic and co-spectral graphs for posets as follows.

Definition 2.9. Two posets P and Q are said to be *co-spectral* if they have the same spectra and are said to be *equienergetic* if $E(P) = E(Q)$.

Naturally, dual posets are co-spectral. Moreover, two co-spectral posets are equienergetic but not conversely. Here is an example.

Example 2.10. Consider the poset P consisting of four disjoint copies of C_2. From Corollary 2.8 and Example 2.2, we have

$$\mathrm{Spec}(P) = \{1, 1, 1, 1, -1, -1, -1, -1\} \quad \text{and} \quad E(P) = 8.$$

Thus from Example 2.4, it is observed that $\mathbb{C}_3$ and P are equienergetic posets but not co-spectral. Moreover they are of different orders.

For an integer $n \geqslant 2$, the covering matrix of the chain C_n is the same as the adjacency matrix of the path P_n. This observation and Lemma 1.1 lead to the following results.

Observation 2.11. For a chain C_n, $n \geqslant 2$,

$$\mathrm{Spec}(C_n) = \Big\{2\cos\frac{\pi r}{n+1}\colon\ r = 1, 2, 3, \ldots, n\Big\}, \tag{1}$$

$$E(C_n) = \begin{cases} \dfrac{2}{\sin(\pi/2(n+1))} - 2 & \text{if } n \text{ is even,} \\[2ex] \dfrac{2\cos(\pi/2(n+1))}{\sin(\pi/2(n+1))} - 2 & \text{if } n \text{ is odd.} \end{cases} \tag{2}$$

The crown $\mathbb{C}_3$ and lattices O_6 as depicted in Figure 1 have the same covering graph. Interestingly, these posets are not equienergetic. For an integer $n \geqslant 3$, let us write $\mathcal{P}_n = \{P\colon\ |P| = n,\ G(P) \cong P_n\}$. Then in the class of posets $\mathcal{P}_n$, the chain C_n and the fence F_n (maybe an upper fence or a lower fence) are posets with maximum and minimum number of doubly irreducible elements, respectively. Similarly, the crown $\mathbb{C}_n$ is the poset with no doubly irreducible element and the covering graph $G(\mathbb{C}_n)$ is a cycle with $2n$ edges. These observations motivate us to compute the covering energy of the crown $\mathbb{C}_n$ and the fence F_n in terms of n. One can easily derive the following result by using de Moivre's theorem.

Lemma 2.12. *For an integer $m \geqslant 1$ and $\alpha \in \mathbb{R}$, we have*

$$\sum_{r=1}^{m} \cos(\alpha r) = \frac{1}{2\sin(\frac{1}{2}\alpha)}\Big(\sin\frac{(2m+1)\alpha}{2} - \sin\frac{\alpha}{2}\Big)$$

and

$$\sum_{r=1}^{m} \sin(\alpha r) = \frac{1}{2\sin(\frac{1}{2}\alpha)}\Big(\cos\frac{\alpha}{2} - \cos\frac{(2m+1)\alpha}{2}\Big).$$

Using this result, we compute the covering energy of $\mathbb{C}_n$ and F_n.

Theorem 2.13. *For an integer $n \geqslant 3$, the spectrum and covering energy of the crown $\mathbb{C}_n$ are*

$$\operatorname{Spec}(\mathbb{C}_n) = \Big\{3, 1 + 2\cos\frac{\pi}{n}, 1 + 2\cos\frac{2\pi}{n}, \ldots, 1 + 2\cos\frac{(2n-1)\pi}{n}\Big\},$$

$$E(\mathbb{C}_n) = \begin{cases} \dfrac{2n}{3} + 2\sqrt{3}\cot\dfrac{\pi}{2n} & \text{if } n \equiv 0 \pmod 3, \\ \dfrac{2}{3}(n-1) + 4\operatorname{cosec}\dfrac{\pi}{2n}\cos\Big(\dfrac{\pi}{6} - \dfrac{\pi}{6n}\Big) & \text{if } n \equiv 1 \pmod 3, \\ \dfrac{2}{3}(n+1) + 4\operatorname{cosec}\dfrac{\pi}{2n}\cos\Big(\dfrac{\pi}{6} + \dfrac{\pi}{6n}\Big) & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. In the crown $\mathbb{C}_n$, $n \geqslant 3$, every element is a reducible element and hence in the covering matrix $C(\mathbb{C}_n)$ all diagonal entries are 1. Also, if we denote the adjacency matrix of the graph $G(\mathbb{C}_n)$ by $A(\mathbb{C}_n)$, then $C(\mathbb{C}_n) = I + A(\mathbb{C}_n)$. Hence, the characteristic polynomial of $\mathbb{C}_n$ can be expressed as

$$\psi(\mathbb{C}_n, \lambda) = |\lambda I - C(\mathbb{C}_n)| = |(\lambda - 1)I - A(\mathbb{C}_n)|.$$

Thus, if $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$ and $\beta_1, \beta_2, \ldots, \beta_{2n}$ are the eigenvalues of $A(\mathbb{C}_n)$ and $C(\mathbb{C}_n)$, respectively, then $\beta_i = \lambda_i + 1$ for all $i = 1, 2, \ldots, 2n$. Therefore Lemma 1.2 leads to

$$\begin{aligned} \operatorname{Spec}(\mathbb{C}_n) &= \Big\{3, 1 + 2\cos\frac{\pi}{n}, 1 + 2\cos\frac{2\pi}{n}, \ldots, 1 + 2\cos\frac{(2n-1)\pi}{n}\Big\}, \\ E(\mathbb{C}_n) &= 3 + \sum_{r=1}^{2n-1}\Big|1 + 2\cos\frac{\pi r}{n}\Big| \\ &= 3 + \sum_{\substack{r=1 \\ 1+2\cos(\pi r/n)\geqslant 0}}^{2n-1}\Big(1 + 2\cos\frac{\pi r}{n}\Big) - \sum_{\substack{r=1 \\ 1+2\cos(\pi r/n)<0}}^{2n-1}\Big(1 + 2\cos\frac{\pi r}{n}\Big). \end{aligned}$$

In the interval $(0, 2\pi)$, it is $1 + 2\cos(\pi r/n) < 0$ if and only if $\frac{2}{3}\pi < \pi r/n < \frac{4}{3}\pi$, that is, $\frac{2}{3} < r/n < \frac{4}{3}$.

Case (1): If $n \equiv 0 \pmod 3$, that is, $n = 3k$, then $1 + 2\cos(\pi r/n) < 0$ if and only if $r = 2k+1, 2k+2, \ldots, 4k-1$. Hence, by symmetry of the curve $1 + 2\cos\alpha$, we have

$$\begin{aligned} E(\mathbb{C}_n) &= 3 + 2\sum_{r=1}^{2k}\Big(1 + 2\cos\frac{\pi r}{n}\Big) - \sum_{r=2k+1}^{4k-1}\Big(1 + 2\cos\frac{\pi r}{n}\Big) \\ &= 3 + 4k - (2k-1) + 4\sum_{r=1}^{2k}\cos\frac{\pi r}{n} - 2\sum_{r=2k+1}^{4k-1}\cos\frac{\pi r}{n} \\ &= 2k + 4 + 6\sum_{r=1}^{2k}\cos\frac{\pi r}{n} - 2\sum_{r=1}^{4k-1}\cos\frac{\pi r}{n}. \end{aligned}$$

Using Lemma 2.12, after simplification, we obtain

$$E(\mathbb{C}_n) = 2k + 2\sqrt{3}\cot\frac{\pi}{2n} = \frac{2n}{3} + 2\sqrt{3}\cot\frac{\pi}{2n}.$$

Case (2): If $n \equiv 1 \pmod 3$, that is, $n = 3k+1$, then $1 + 2\cos(\pi r/n) < 0$ if and only if $\frac{2}{3} < r/n < \frac{4}{3}$ if and only if $2k + \frac{1}{3} < r < 4k + 1 + \frac{1}{3}$. That is, $r = 2k+1, 2k+2, \ldots, 4k+1$. Thus

$$\begin{aligned} E(\mathbb{C}_n) &= 3 + 2\sum_{r=1}^{2k}\Big(1 + 2\cos\frac{\pi r}{n}\Big) - \sum_{r=2k+1}^{4k+1}\Big(1 + 2\cos\frac{\pi r}{n}\Big) \\ &= 3 + 4k - 2k - 1 + 6\sum_{r=1}^{2k}\cos\frac{\pi r}{n} - 2\sum_{r=1}^{4k+1}\cos\frac{\pi r}{n}. \end{aligned}$$

Using Lemma 2.12, after simplification, we obtain

$$E(\mathbb{C}_n) = \frac{2}{3}(n-1) + 4\cos\Big(\frac{\pi}{6} - \frac{\pi}{6n}\Big)\operatorname{cosec}\frac{\pi}{2n}.$$

Case (3): If $n \equiv 2 \pmod 3$, that is, $n = 3k-1$ then $1 + 2\cos(\pi r/n) < 0$ if and only if $\frac{2}{3} < r/n < \frac{4}{3}$ if and only if $\frac{2}{3}n < r < \frac{4}{3}n$. That is, $r = 2k, 2k+1, \ldots, 4k-2$. Therefore

$$E(\mathbb{C}_n) = 3 + 2\sum_{r=1}^{2k-1}\Big(1 + 2\cos\frac{\pi r}{n}\Big) - \sum_{r=2k}^{4k-2}\Big(1 + 2\cos\frac{\pi r}{n}\Big).$$

Using Lemma 2.12, after simplification, we obtain

$$E(\mathbb{C}_n) = \frac{2}{3}(n+1) + 4\operatorname{cosec}\frac{\pi}{2n}\cos\Big(\frac{\pi}{6} + \frac{\pi}{6n}\Big).$$

□

Corollary 2.14. *Among all crowns, $\mathbb{C}_3$ is the only integral crown.*

Corollary 2.15. *If λ is an eigenvalue of the crown $\mathbb{C}_n$, then $-1 \leqslant \lambda \leqslant 3$ and $\lambda^* = 2 - \lambda$ is another eigenvalue.*

Proof. The first part is trivial. If $\lambda = 1 + 2\cos(\pi r/n)$ is an eigenvalue of $\mathbb{C}_n$, then $\lambda^* = 2 - \lambda = 1 + 2\cos(\pi(n + r)/n)$ is also an eigenvalue of $\mathbb{C}_n$. $\square$

Theorem 2.16. *Let $F_n = \{v_1, v_2, \ldots, v_n\}$ and $(F_n, \leqslant)$ be a fence. Then*

$$\text{(1)} \qquad \operatorname{Spec}(F_n) = \Big\{ -1, 1 + 2\cos\frac{\pi}{n}, 1 + 2\cos\frac{2\pi}{n}, \ldots, 1 + 2\cos\frac{(n-1)\pi}{n} \Big\}.$$

(2) *The covering energy of the fence F_n is given by*

$$E(F_n) = \begin{cases} \dfrac{n-3}{3} + \sqrt{3}\cot\dfrac{\pi}{2n} & \text{if } n \equiv 0 \pmod 3, \\ \dfrac{n-4}{3} + \Big(\sqrt{3}\cos\dfrac{\pi}{6n} + \sin\dfrac{\pi}{6n}\Big)\operatorname{cosec}\dfrac{\pi}{2n} & \text{if } n \equiv 1 \pmod 3, \\ \dfrac{n-2}{3} + \Big(\sqrt{3}\cos\dfrac{\pi}{6n} - \sin\dfrac{\pi}{6n}\Big)\operatorname{cosec}\dfrac{\pi}{2n} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. For an integer $n \geqslant 3$, the covering graph $G(F_n)$ is the path P_n. Let $\psi(F_n, \lambda)$ and $\varphi(P_n, \lambda)$ denote the characteristic polynomials of F_n and P_n, respectively. Then the covering matrix $C(F_n)$, the adjacency matrix $A(P_n)$ and the characteristic polynomial of F_n can be expressed as

$$C(F_n) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 1 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \end{bmatrix}, \quad A(P_n) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \end{bmatrix}$$

and

$$\psi(F_n, \lambda) = \begin{vmatrix} \lambda & -1 & 0 & \ldots & 0 & 0 & 0 \\ -1 & \lambda - 1 & -1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & \lambda - 1 & -1 \\ 0 & 0 & 0 & \ldots & 0 & -1 & \lambda \end{vmatrix}.$$

If we write $\lambda' = \lambda - 1$ and split the determinant using the first row, then

$$\psi(F_n,\lambda) = \begin{vmatrix} \lambda' & -1 & 0 & \dots & 0 & 0 \\ -1 & \lambda' & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda' & -1 \\ 0 & 0 & 0 & \dots & -1 & \lambda'+1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & \lambda' & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda' & -1 \\ 0 & 0 & 0 & \dots & -1 & \lambda'+1 \end{vmatrix}$$
$$= \begin{vmatrix} \lambda' & -1 & 0 & \dots & 0 & 0 \\ -1 & \lambda' & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda' & -1 \\ 0 & 0 & 0 & \dots & -1 & \lambda'+1 \end{vmatrix} + \begin{vmatrix} \lambda' & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda' & -1 \\ 0 & 0 & \dots & -1 & \lambda'+1 \end{vmatrix}.$$

Applying the same technique to split these two determinants with respect to the last row, we obtain

$$\psi(F_n,\lambda) = \varphi(P_n,\lambda') + 2\varphi(P_{n-1},\lambda') + \varphi(P_{n-2},\lambda').$$

It is well known that, for $n \geqslant 3$, $\varphi(P_n,\lambda) = \lambda\varphi(P_{n-1},\lambda) - \varphi(P_{n-2},\lambda)$. Using this recurrence relation, we have

$$\begin{aligned} \psi(F_n,\lambda) &= \lambda'\varphi(P_{n-1},\lambda') - \varphi(P_{n-2},\lambda') + 2\varphi(P_{n-1},\lambda') + \varphi(P_{n-2},\lambda'), \\ \psi(F_n,\lambda) &= (\lambda+1)\varphi(P_{n-1},\lambda') = (\lambda+1)\prod(\lambda' - \alpha_i), \end{aligned}$$

where α_i, $i = 1, 2, \dots, n-1$, are eigenvalues of the matrix $A(P_{n-1})$. Hence,

$$\psi(F_n,\lambda) = (\lambda+1)\prod_{r=1}^{n-1}\Big(\lambda - 1 - 2\cos\frac{\pi r}{n}\Big).$$

Thus,

$$\operatorname{Spec}(F_n) = \Big\{-1, 1 + 2\cos\frac{\pi}{n}, 1 + 2\cos\frac{2\pi}{n}, \dots, 1 + 2\cos\frac{(n-1)\pi}{n}\Big\},$$

and

$$\begin{aligned} E(F_n) &= 1 + \sum_{r=1}^{n-1}\Big|1 + 2\cos\frac{\pi r}{n}\Big| \\ &= 1 + \sum_{\substack{r=1 \\ \frac12<\cos(\pi r/n)<1}}^{n-1}\Big(1 + 2\cos\frac{\pi r}{n}\Big) + \sum_{\substack{r=1 \\ -\frac12\leqslant\cos(\pi r/n)\leqslant\frac12}}^{n-1}\Big(1 + 2\cos\frac{\pi r}{n}\Big) \\ &\quad - \sum_{\substack{r=1 \\ -1<\cos(\pi r/n)<-\frac12}}^{n-1}\Big(1 + 2\cos\frac{\pi r}{n}\Big). \end{aligned}$$

Using the symmetry of the curve $1 + 2\cos\theta$, $0 < \theta < \pi$, we have

$$\begin{aligned}
E(F_n) &= 1 + 4 \sum_{r=1}^{k=\max\{r\colon 3r<n\}} \Big(\cos\frac{\pi r}{n}\Big) + (n - 1 - 2k)\\
&= n - 2k + \frac{2}{\sin(\pi/2n)}\Big(\sin\frac{(2k+1)\pi}{2n} - \sin\frac{\pi}{2n}\Big) \quad \text{(using Lemma 2.12)}\\
&= n - 2k - 2 + 2\operatorname{cosec}\frac{\pi}{2n}\sin\frac{(2k+1)\pi}{2n}.
\end{aligned}$$

Case (1): If $n \equiv 0 \pmod 3$, then $k = \frac{1}{3}n - 1$ and after routine calculations we have

$$E(F_n) = n - \frac{2n}{3} + 2\sin\Big(\frac{\pi}{3} - \frac{\pi}{2n}\Big)\operatorname{cosec}\frac{\pi}{2n} = \frac{n-3}{3} + \sqrt{3}\cot\frac{\pi}{2n}.$$

Case (2) : If $n \equiv 1 \pmod 3$, then $k = \frac{1}{3}(n-1)$ and after simple calculations we obtain

$$E(F_n) = \frac{n-4}{3} + \Big(\sqrt{3}\cos\frac{\pi}{6n} + \sin\frac{\pi}{6n}\Big)\operatorname{cosec}\frac{\pi}{2n}.$$

Case (3) : If $n \equiv 2 \pmod 3$, then $k = \frac{1}{3}(n-2)$, which leads to

$$E(F_n) = \frac{n-2}{3} + \Big(\sqrt{3}\cos\frac{\pi}{6n} - \sin\frac{\pi}{6n}\Big)\operatorname{cosec}\frac{\pi}{2n}.$$

$\square$

To compute the covering energy of other poset classes whose covering graphs are isomorphic to P_n is a separate subject. We do not go into those details. Now, we turn our attention to the bounds of the covering energy of posets.

Pawar and Bhamre (see [17] and [18]) have studied the covering energy of some special classes of posets. They have obtained formulas for coefficients of λ^n, λ^{n-1}, λ^{n-2} and λ^{n-3} in the characteristic polynomial $\psi(P, \lambda)$ of a poset P in terms of its number of vertices, edges, reducible and doubly irreducible elements.

The following results are from Pawar and Bhamre (see [17]).

Theorem 2.17 ([17]). *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of a poset P, then*

$$\sum_{i=1}^{n} \lambda_i = n - |\operatorname{Irr}(P)| = |R|, \tag{1}$$

$$\sum_{i=1}^{n} \lambda_i^2 = n + 2|e(P)| - |\operatorname{Irr}(P)| = 2|e(P)| + |R|. \tag{2}$$

Pawar and Bhamre (see [17]) have also obtained McClelland type bounds for the covering energy of a poset in terms of its number of vertices, number of edges and the determinant of the covering matrix $C(P)$.

Theorem 2.18 ([17]). *If a poset P has n elements, m edges and $D = \det(C(P))$, then*

$$\sqrt{2m + n - |\operatorname{Irr}(P)| + n(n-1)D^{2/n}} \leqslant E(P) \leqslant \sqrt{n(2m + n - |\operatorname{Irr}(P)|)}. \tag{2.1}$$

We improve these bounds for a particular class of posets in the next section.

3. Improvement of bounds for the covering energy of a poset

In the literature, we can find definitions of singular and non-singular graphs. We extend these concepts to posets as follows.

Definition 3.1. Let P be a poset with $|P| = n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of P. Then P is called a singular poset if $\lambda_i = 0$ for some $1 \leqslant i \leqslant n$. Otherwise, if $\lambda_i \neq 0$ for each i, then P is called a non-singular poset.

We recall that a *Hermitian matrix* (or a *self adjoint matrix*) is a complex square matrix which equals its conjugate transpose. Every real symmetric matrix is a Hermitian matrix. In the next result, we obtain a lower bound for the largest eigenvalue of a poset.

Theorem 3.2. *Let P be a poset with n elements and m edges. Let r be the greatest eigenvalue of P and $R = \{v \in P\colon\ v \notin \operatorname{Irr}(P)\}$. Then*

$$\frac{2m + |R|}{n} \leqslant r. \tag{3.1}$$

Moreover, the equality holds if and only if P is C_2 or more copies of C_2 than one or $\operatorname{Irr}(P) = \emptyset$ and the covering graph $G(P)$ is regular.

Proof. Since the covering matrix $C(P) = (a_{ij})_{n\times n}$ is symmetric and real, it is a Hermitian matrix. It is well known that the problem of finding the maximal value of Rayleigh's quotient

$$Rq = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j}{\sum_{i=1}^{n} x_i^2} \tag{3.2}$$

(the x_i being arbitrary real numbers not all equal to zero) has the solution $Rq = r$. The maximum is attained if and only if the x_i, $i = 1, 2, \ldots, n$, are the components of the eigenvector of $C(P)$ belonging to r. If we put $x_i = 1$, $i = 1, 2, \ldots, n$, in equation (3.2), we have $Rq = \bar{d} = (1/n)\sum\limits_{i=1}^{n} d_i$, where

$$d_i = \sum_{j=1}^{n} a_{ij} = \begin{cases} d(v_i) & \text{if } v_i \in \operatorname{Irr}(P), \\ d(v_i) + 1 & \text{if } v_i \notin \operatorname{Irr}(P) \end{cases} \tag{3.3}$$

(here $d(v_i)$ denotes degree of the vertex v_i in the covering graph $G(P)$). So, $\bar{d}$ is a particular value of Rayleigh's quotient. This leads to

$$\bar{d} = \frac{2m + |R|}{n} \leqslant r. \tag{3.4}$$

Thus the first part of the theorem is proved.

In Case-I, we prove the second part of the theorem for a connected poset and in Case-II, as a consequence of Case-I, we can prove it for disconnected posets.

Case-I: For the chain C_2, it is trivial to show that the equality holds in (3.1). If P is a connected poset, for which $\mathrm{Irr}(P) = \varphi$, and $G(P)$ is regular of order k, then

$$\frac{2m + |R|}{n} = k + 1 = r,$$

i.e., equality holds in the inequality (3.1).

Conversely, if the equality holds in (3.1), then the values $x_i = 1$, $i = 1, 2, \ldots, n$, constitute an eigenvector for $C(P)$ belonging to r and $\sum\limits_{j=1}^{n} a_{ij} x_j = r x_i$, $i = 1, 2, \ldots, n$, implies $d_i = \sum\limits_{j=1}^{n} a_{ij} = r$, $i = 1, 2, \ldots, n$. As d_i is an integer for each i, hence $|R| = 0$ or $|R| = n$. In both the cases $G(P)$ is regular. As $G(P)$ is connected, $|R| = 0$ if and only if P is a chain C_2, and $|R| = n$ if and only if $\mathrm{Irr}(P) = \emptyset$. This proves the result for connected posets.

Case-II: Let us consider a disconnected poset P with s connected components $P_1, P_2, \ldots, P_s$ with $n_1, n_2, \ldots, n_s$ elements and $m_1, m_2, \ldots, m_s$ edges, respectively.

Case (a): P is regular of order one if and only if each component P_i is regular of order one, i.e., $P_i = C_2$, $i = 1, 2, \ldots, s$. Hence $m = s$, $n = 2s$, $|R| = 0$ and $r = 1 = (2m + |R|)/n$.

Case (b): The poset P is regular of order $k > 1$ and no element of P is doubly irreducible if and only if for each $i = 1, 2, \ldots, s$, P_i is regular of order k and no element of P_i is doubly irreducible. Using Case-I for each P_i, $1 \leqslant i \leqslant s$, the largest eigenvalues of P_i,

$$r_i = \frac{2m_i + |R(P_i)|}{n_i} = \frac{kn_i + n_i}{n_i} = k + 1.$$

Hence

$$r = \max\{r_1, r_2, \ldots, r_s\} = k + 1 = \frac{kn + n}{n} = \frac{2m + |R|}{n}.$$

□

The Theorem 3.2 is a generalization of a result which is originally due to Collatz and Sinogowitz [3] (for the English version of this result, see [4]). The following two results are applications of Theorem 3.2.

Theorem 3.3. *Let P be a non-singular poset with n elements, m edges and $R = \{v \in P\colon\ v \notin \operatorname{Irr}(P)\}$. Then*

$$\frac{2m+|R|}{n} + n - 1 + \ln \frac{n|\det(C(P))|}{2m+|R|} \leqslant E(P). \tag{3.5}$$

Proof. As P is a non-singular poset, $|\lambda_i| > 0$ for each $1 \leqslant i \leqslant n$. Consider the function

$$f(x) = x - 1 - \ln(x) \quad \text{for } x > 0.$$

It is easy to show that $f(x)$ is decreasing for $x \in (0,1]$ and increasing for $x \geqslant 1$. Hence, $f(x) \geqslant f(1) = 0$ implies that

$$x \geqslant 1 + \ln(x) \quad \text{for } x > 0. \tag{3.6}$$

Moreover, equality holds if and only if $x = 1$. Also, using the inequality (3.6), we have

$$E(P) = \lambda_1 + \sum_{i=2}^{n} |\lambda_i| \geqslant \lambda_1 + n - 1 + \sum_{i=2}^{n} \ln|\lambda_i| = \lambda_1 + n - 1 + \ln \prod_{i=2}^{n} |\lambda_i|. \tag{3.7}$$

Thus,

$$E(P) \geqslant \lambda_1 + n - 1 + \ln|\det(C(P))| - \ln(\lambda_1). \tag{3.8}$$

By Theorem 3.2, we have $\lambda_1 \geqslant (2m+|R|)/n$. The function $g(x) = x + n - 1 + \ln|\det(C(P))| - \ln x$ is increasing for $x \in [1,n]$ and we conclude that

$$g(\lambda_1) \geqslant \frac{2m+|R|}{n} + n - 1 + \ln|\det(C(P))| - \ln\frac{2m+|R|}{n} \tag{3.9}$$

for $x \geqslant 2m/n$. Combining (3.9) with (3.8), we arrive at (3.5). □

Suppose that equality holds in (3.5). Then all the inequalities considered in the Theorem 3.3 must be equalities. From the equality (3.7), we obtain $|\lambda_2| = |\lambda_3| = \ldots = |\lambda_n| = 1$. Since P is connected, the condition $|\lambda_i| = 1$, $i = 2,3,\ldots,n$, is satisfied if and only if $P = C_2$. This discussion leads to the following claim.

Remark 3.4. Equality holds in (3.5) if and only if P is the chain C_2.

For crowns $\mathbb{C}_k$, $k = 4,5,7,8$, one can numerically verify that (3.5) is superior to (2.1). But for $k > 9$, the situation is the opposite. (If $3|k$, then $\det(C(\mathbb{C}_k)) = 0$ and the inequality (3.5) is not applicable). That is, in general, (3.5) may not be superior to (2.1).

The next result gives a sufficient condition for (3.5) to be superior to (2.1).

Observation 3.5. Let Ω be the class of connected posets with n elements and m edges, and for which the following conditions are satisfied:

$$16 \leqslant \frac{n}{2} \leqslant \frac{2m+|R|}{n} \leqslant |\det(C(P))| \leqslant n-1. \tag{3.10}$$

Then for each poset $P \in \Omega$, the inequality (3.5) is better than the inequality (2.1).

Proof. As $P \in \Omega$, by the inequality (3.10), we have

$$\ln|\det(C(P))| - \ln\frac{2m+|R|}{n} = \ln\frac{n|\det(C(P))|}{2m+|R|} \geqslant 0 \quad \text{and} \quad n \geqslant 32,$$

which leads to $(n-1)^{2/n} \leqslant \frac{5}{4}$. Thus, using Theorem 3.3, we obtain

$$E(P) \geqslant \frac{2m+|R|}{n} + n - 1 + \ln\frac{n|\det(C(P))|}{2m+|R|} \geqslant \frac{2m+|R|}{n} + n - 1. \tag{3.11}$$

Also,

$$2m+|R|+n(n-1)|\det(C(P))|^{2/n} \leqslant 2m+|R|+n(n-1)(n-1)^{2/n} \leqslant 2m+|R|+n(n-1)\frac{5}{4}. \tag{3.12}$$

In the light of inequalities (3.11) and (3.12), to prove the result it is sufficient to show that

$$2m+|R|+n(n-1)\frac{5}{4} \leqslant \Big(\frac{2m+|R|}{n} + n - 1\Big)^2.$$

That is, to show that,

$$2m+|R|+n(n-1)\frac{5}{4} \leqslant \Big(\frac{2m+|R|}{n} - 1\Big)^2 + n^2 + 4m + 2|R| - 2n,$$

i.e.,

$$\frac{n^2+3n}{4} \leqslant \Big(\frac{2m+|R|}{n} - 1\Big)^2 + 2m + |R|. \tag{3.13}$$

As $(2m+|R|)/n \geqslant \frac{1}{2}n$, we have

$$\Big(\frac{2m+|R|}{n} - 1\Big)^2 + 2m + |R| \geqslant \Big(\frac{n}{2} - 1\Big)^2 + \frac{n^2}{2} = \frac{3n^2-4n-4}{4}. \tag{3.14}$$

But for $n \geqslant 16$, the inequality

$$\frac{3n^2-4n-4}{4} \geqslant \frac{n^2+3n}{4}$$

is always true. Thus, using (3.13) and (3.14), the proof is complete. $\square$

Now, we improve the McClelland type upper bound for the covering energy of a poset whose covering graph $G(P)$ contains no isolated vertex.

Theorem 3.6. *Let P be a poset with n elements and m edges. If $G(P)$ contains no isolated vertex and $R = \{v \in P\colon\ v \notin \mathrm{Irr}(P)\}$, then*

$$E(P) \leqslant \frac{2m+|R|}{n} + \sqrt{(n-1)\Big(2m+|R| - \Big(\frac{2m+|R|}{n}\Big)^2\Big)}. \tag{3.15}$$

Proof. As $G(P)$ does not contain any isolated vertex, $2m \geqslant n$. Suppose $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ are the eigenvalues of P, then by the inequality (3.1), we have

$$1 \leqslant \frac{2m+|R|}{n} \leqslant \lambda_1. \tag{3.16}$$

Moreover, by Theorem 2.17 (2), we have

$$\sum_{i=2}^{n} \lambda_i^2 = 2m + |R| - \lambda_1^2. \tag{3.17}$$

Using (3.17) together with the Cauchy-Schwarz inequality applied to the vectors $(|\lambda_2|, |\lambda_3|, \ldots, |\lambda_n|)$ and $(1, 1, \ldots, 1)$ with $n-1$ entries, we obtain the inequality

$$\sum_{i=2}^{n} |\lambda_i| \leqslant \sqrt{(n-1)(2m+|R|-\lambda_1^2)}. \tag{3.18}$$

Thus, we have

$$E(P) \leqslant \lambda_1 + \sqrt{(n-1)(2m+|R|-\lambda_1^2)}. \tag{3.19}$$

Now, as the function $F(x) = x + \sqrt{(n-1)(2m+|R|-x^2)}$ is decreasing on the interval $\big(\sqrt{(2m+|R|)/n}, \sqrt{2m+|R|}\,\big]$, in view of the fact $2m+|R| \geqslant n$, we observe that $\sqrt{(2m+|R|)/n} \leqslant (2m+|R|)/n \leqslant \lambda_1$ holds. Hence,

$$F(\lambda_1) \leqslant F\Big(\frac{2m+|R|}{n}\Big) \leqslant F\Big(\sqrt{\frac{2m+|R|}{n}}\Big). \tag{3.20}$$

The inequalities (3.19) and (3.20) lead to (3.15). $\square$

It is easy to verify that $F\big(\sqrt{(2m+|R|)/n}\big) = \sqrt{n(2m+|R|)}$. Hence in the light of the inequality (3.20), we observe that the bound given in Theorem 3.6 is an improved bound for the given class of posets. Also, if P is $\frac{1}{2}n$ copies of C_2, then the eigenvalues for P are ± 1 (both with multiplicity $\frac{1}{2}n$) and $|R| = 0$. It is easy to check that for the poset P, equality holds in the relation (3.15).

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