ON THE DOMINATION OF TRIANGULATED DISCS

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Abstract. Let G be a 3-connected triangulated disc of order n with the boundary cycle C of the outer face of G. Tokunaga (2013) conjectured that G has a dominating set of cardinality at most $\frac{1}{4}(n+2)$. This conjecture is proved in Tokunaga (2020) for $G-C$ being a tree. In this paper we prove the above conjecture for $G-C$ being a unicyclic graph. We also deduce some bounds for the double domination number, total domination number and double total domination number in triangulated discs.

Keywords: domination; double domination; total domination; double total domination; planar graph; triangulated disc

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1. Introduction

For a simple graph G with vertex set $V = V(G)$, the number of vertices of G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of a vertex v, denoted by $\deg(v)$ (or $\deg_G(v)$ to refer to G), is the cardinality of its open neighborhood. For an integer $k \geqslant 2$, a graph G is called k-connected if it has more than k vertices and remains connected whenever fewer than k vertices are removed. The *contraction* of an edge is an operation that removes the edge while simultaneously merging the two vertices that it previously joined. A plane graph G is said to be a *triangulated disc* if it is 2-connected and all its faces are triangles except for the outer (infinite) face. The boundary cycle of the outer face of G is called the *outer cycle* of G and is denoted $C(G)$. The graph $G-C(G)$ is denoted by $\mathrm{In}(G)$.

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A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S. Note that by this definition any vertex of G dominates itself and its neighbors. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A subset $S \subseteq V$ in a graph with no isolated vertex is a *total dominating set* of G if every vertex in V has a neighbor in S. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. For a comprehensive survey on the subject of domination parameters in graphs the reader can refer to [9].

Harary and Haynes in [8] defined a generalization of domination, namely k-tuple domination, which is called double domination if $k = 2$. A subset S of vertices of a graph G is a *double dominating set* of G if for every vertex $v \in V(G)$, $|N[v] \cap S| \geqslant 2$. The *double domination number* $\gamma_{\times 2}(G)$ is the minimum cardinality of a double dominating set of G. The concept of double domination in graph was studied in, for example, [2], [3], [4], [7]. A subset S of V in a graph with no isolated vertex is a *double total dominating set* of G if for every vertex $v \in V$, $|N(v) \cap S| \geqslant 2$. The *double total domination number* $\gamma_{\times 2,t}(G)$ is the minimum cardinality of a double total dominating set of G. The concept of double total domination was introduced in [11] and further studied in, for example, [10], [12], [16]. This concept was also studied under the name *total* 2*-domination*, see for example, [1], [6].

Matheson and Tarjan in [15] proved that any triangulated disc G of order n has a dominating set of cardinality at most $\frac{1}{3}n$, and conjectured that $\gamma(G) \leqslant \frac{1}{4}n$ for every n-vertex triangulation G with sufficiently large n. King and Pelsmajer (see [13]) proved this conjecture for graphs of maximum degree 6. Campos and Wakabayashi in [5] and Tokunaga in [17] independently proved that $\gamma(G) \leqslant \frac{1}{4}(n + t)$ for each n-vertex outerplanar graph G of order n having t vertices of degree two. An improvement of the $\frac{1}{4}(n + t)$-bound is given by Li et al. in [14]. Tokunaga (see [17]) also posed the following conjecture.

Conjecture 1 (Tokunaga, [17]). If G is a 3-connected n-vertex triangulated disc, then $\gamma(G) \leqslant \frac{1}{4}(n + 2)$.

Recently, Tokunaga in [18] proved Conjecture 1 for triangulated discs G such that $\mathrm{In}(G)$ is a tree.

Theorem 2 (Tokunaga, [18]). *If G is an n-vertex triangulated disc such that $\mathrm{In}(G)$ is a tree and $C(G)$ is an induced cycle of G, then $\gamma(G) \leqslant \frac{1}{4}(n + 2)$.*

In this paper we proved a stronger version of Conjecture 1 for triangulated discs G such that $\mathrm{In}(G)$ is a unicyclic graph. We show that if G is an n-vertex triangulated disc such that $\mathrm{In}(G)$ is a unicyclic graph and $C(G)$ is an induced cycle of G, then $\gamma(G) \leqslant \frac{1}{4}(n+1)$. We also apply the methods for double domination, total domination and double total domination numbers in triangulated discs.

We use the same method given in [18] using proper colorings. For a given integer $k \geqslant 1$, a function $f\colon V(G) \to \{1,\dots,k\}$ is called a *proper k-coloring* if $f(u) \neq f(v)$ for every edge uv of G. If f is proper k-coloring and a vertex v is dominated by a vertex of color i for some $i \in \{1,\dots,k\}$, then we say that v is *dominated by color i*. We use the following key lemma of [18].

Lemma 3 (Tokunaga, [18])**.** *Let G be an n-vertex triangulated disc such that $\mathrm{In}(G)$ is a tree and $C(G)$ is an induced cycle of G, and let v be a vertex of $C(G)$ with $\deg_G(v) = 3$. Then $G - v$ has a proper 4-coloring such that each vertex of $G - v$ is dominated by all the four colors except the vertices of $N_G(v)$.*

2. Bounds

We first present a bound for the domination number.

Theorem 4. *If G is an n-vertex triangulated disc such that $\mathrm{In}(G)$ is a unicyclic graph and $C(G)$ is an induced cycle of G, then $\gamma(G) \leqslant \frac{1}{4}(n+1)$. This bound is sharp.*

Proof. Let G be an n-vertex triangulated disc such that $\mathrm{In}(G)$ is a unicyclic graph and $C(G)$ is an induced cycle of G. Clearly, $\mathrm{In}(G)$ contains a triangle, since all faces are triangles except for the outer (infinite) face. Let abc be a triangle in $\mathrm{In}(G)$, and G^* be the graph obtained from G by contraction of the edge ab. Then G^* is an $(n-1)$-vertex triangulated disc such that $\mathrm{In}(G)$ is a tree, namely T. Clearly, $C(G^*) = C(G)$ and $\{a,b,c\} \cap V(C) = \emptyset$. Let v be a vertex of $C(G^*)$ with $\deg_{G^*}(v) = 3$. Clearly, $\deg_{G^*}(v) = \deg_G(v) = 3$. Let $N_{G^*}(v) = \{u,w,x\}$, where $N_C(v) = \{u,w\}$ and x is the unique vertex of T, which is adjacent to v in G^*. Now we follow the proof of Theorem 2 given in [18]. By Lemma 3, $G^* - v$ has a proper 4-coloring f such that each vertex of $G^* - v$ is dominated by all the four colors except the vertices of $N_{G^*}(v)$. Let G' be the $(n+1)$-vertex graph such that $V(G') = V(G^*) \cup \{p,q\}$ and $E(G') = E(G^*) \cup \{pu, pv, pw, qu, qv, qw\}$.

We define a 4-coloring f' on G^* as follows. Define $f'(y) = f(y)$ if $y \in V(G^*) - \{v\}$, and let $f'(v)$ be a color different from $f'(x)$, $f'(u)$ and $f'(w)$. Now assign $f'(p)$ and $f'(q)$ such that $\{f'(x), f'(p), f'(q), f'(v)\} = \{1,2,3,4\}$. Then each vertex of G^* is dominated by all the four colors. By renaming the colors, if necessary, we may assume that $|\{y \in V(G^*)\colon\ f'(y) = f'(c) = f(c)\}| \leqslant \frac{1}{4}(n+1)$. Let $S = \{y \in V(G^*)\colon f'(y) = f'(c) = f(c)\}$. Now, we form a set S' defined by $S' = S$ if $S \cap \{p,q\} = \emptyset$, $S' = (S - \{p\}) \cup \{v\}$ if $p \in S$, and $S' = (S - \{q\}) \cup \{v\}$ if $q \in S$. Then S' is a dominating set for G^* of cardinality at most $\frac{1}{4}(n+1)$. Since $c \in S'$, we deduce that S' is a dominating set for G, and the proof is completed.

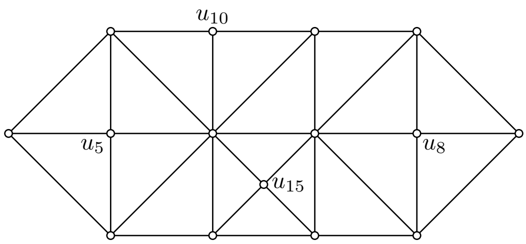

Figure 1. The graph H.

For the sharpness, see the graph H depicted in Figure 1. It is easy to see that the set $\{u_5, u_8, u_{10}, u_{15}\}$ is a minimum dominating set. □

Next apply the above arguments for double domination, total domination, and double total domination. Following the proofs of Theorem 2, we can assume that $|\{y\colon f'(y) \in \{1,2\}\}| \leqslant \frac{1}{2}(n+2)$. Then we form a double dominating set for G of cardinality at most $\frac{1}{2}(n+2)$. Thus, we have the following.

Corollary 5. *If G is an n-vertex triangulated disc such that* $\mathrm{In}(G)$ *is a tree and* $C(G)$ *is an induced cycle of G, then* $\gamma_{\times 2}(G) \leqslant \frac{1}{2}(n+2)$.

We next prove an upper bound for the double domination number for G when $\mathrm{In}(G)$ is a unicyclic graph.

Theorem 6. *If G is an n-vertex triangulated disc such that* $\mathrm{In}(G)$ *is a unicyclic graph and* $C(G)$ *is an induced cycle of G, then* $\gamma_{\times 2}(G) \leqslant \frac{1}{2}(n+1)$.

Proof. We follow the proof of Theorem 4. Let G be an n-vertex triangulated disc such that $\mathrm{In}(G)$ is a unicyclic graph and $C(G)$ is an induced cycle of G. Let abc be a triangle in $\mathrm{In}(G)$, and G^* be the graph obtained from G by contraction of the edge ab. Then G^* is an $(n-1)$-vertex triangulated disc such that $\mathrm{In}(G)$ is a tree. Clearly, a and b have a common neighbor $d \neq c$ in G, since G is a triangulated disc. Now, following the proof, we may assume that $|\{y\colon f'(y) \in \{f'(c), f'(d)\}\}| \leqslant \frac{1}{2}(n+1)$. Let $S = \{y\colon f'(y) \in \{f'(c), f'(d)\}\}$, and form S' as described in the proof of Theorem 4. Since $c, d \in S'$, we obtain that S' is a dominating set for G, and the proof is completed. □

It is evident that $\gamma_t(G) \leqslant \gamma_{\times 2}(G)$ for any graph G with no isolated vertex. Thus, the bounds given in Corollary 5 and Theorem 6 are also valid for total domination. We next prove upper bounds for the double total domination number.

Theorem 7. *Let G be an n-vertex triangulated disc such that* $C(G)$ *is an induced cycle of G. If* $\mathrm{In}(G)$ *is a tree, then* $\gamma_{\times 2,t}(G) \leqslant \frac{3}{4}(n+2)$, *and if* $\mathrm{In}(G)$ *is a unicyclic graph, then* $\gamma_{\times 2,t}(G) \leqslant \frac{3}{4}(n+1)$.

P r o o f. We follow the proof of Theorems 2 and 4. First assume that $\mathrm{In}(G)$ is a tree. Let f' be the given 4-coloring in the proof of Theorem 2, and assume that $|\{y\colon f'(y) \neq 1\}| \leqslant \frac{3}{4}(n+2)$. Let $S = \{y\colon f'(y) \neq 1\}$ and S' be formed from S as described in the proof of Theorem 2. Then S' is a double total dominating set for G of cardinality at most $\frac{3}{4}(n+2)$. Next assume that $\mathrm{In}(G)$ is a unicyclic graph. We follow the proof of Theorem 4. Let y^* be the vertex formed by contraction of the edge ab. Let f' be the given 4-coloring in the proof of Theorem 4, and assume that $|\{y\colon f'(y) \neq f'(y^*)\}| \leqslant \frac{3}{4}(n+1)$. Let $S = \{y\colon f'(y) \neq f'(y^*)\}$ and S' be formed from S as described in the proof of Theorem 4. Then S' is a double total dominating set for G of cardinality at most $\frac{3}{4}(n+1)$. □

We close with the following conjecture.

C o n j e c t u r e 8. If G is an n-vertex triangulated disc, then $\gamma(G) \leqslant \frac{1}{4}(n+2-t)$, $\gamma_{\times 2}(G) \leqslant \frac{1}{2}(n+2-t)$ and $\gamma_{\times 2,t}(G) \leqslant \frac{3}{4}(n+2-t)$, where t is the number of vertex-disjoint triangles in $\mathrm{In}(G)$.

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