RECURRENCE AND MIXING RECURRENCE OF MULTIPLICATION OPERATORS

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Abstract. Let X be a Banach space, $\mathcal{B}(X)$ the algebra of bounded linear operators on X and $(J, \|\cdot\|_J)$ an admissible Banach ideal of $\mathcal{B}(X)$. For $T \in \mathcal{B}(X)$, let $L_{J,T}$ and $R_{J,T} \in \mathcal{B}(J)$ denote the left and right multiplication defined by $L_{J,T}(A) = TA$ and $R_{J,T}(A) = AT$, respectively. In this paper, we study the transmission of some concepts related to recurrent operators between $T \in \mathcal{B}(X)$, and their elementary operators $L_{J,T}$ and $R_{J,T}$. In particular, we give necessary and sufficient conditions for $L_{J,T}$ and $R_{J,T}$ to be sequentially recurrent. Furthermore, we prove that $L_{J,T}$ is recurrent if and only if $T \oplus T$ is recurrent on $X \oplus X$. Moreover, we introduce the notion of a mixing recurrent operator and we show that $L_{J,T}$ is mixing recurrent if and only if T is mixing recurrent.

Keywords: hypercyclicity; recurrent operator; left multiplication operator; right multiplication operator; tensor product; Banach ideal of operators

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1. Introduction and preliminaries

Throughout the paper, let X be a Banach space with $\dim X > 1$, such that X^*, its dual, is separable, and use $\mathcal{B}(X)$ and $\mathcal{K}(X)$ to represent the algebra of all bounded linear operators on X and the algebra of all compact operators on X, respectively. For $T \in \mathcal{B}(X)$ the orbit of a vector $x \in X$ under T is the set

$$\operatorname{Orb}(T, x) := \{T^n x \colon n \in \mathbb{N}\}.$$

One of the most important and studied concepts in the linear dynamical systems is that of hypercyclicity. An operator $T \in \mathcal{B}(X)$ is said to be hypercyclic if there is some vector $x \in X$ such that $\operatorname{Orb}(T, x)$ is dense in X. In this case, x is called a hypercyclic vector for T. Note that on the Banach space setting, Rolewicz (see [24])

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in 1969 gave the first examples of hypercyclic operators. Birkhoff in [4] introduced the notion of topological transitivity and he showed that $T \in \mathcal{B}(X)$ is hypercyclic if and only if it is topologically transitive. Recall that $T \in \mathcal{B}(X)$ is said to be topologically transitive, if for each pair (U, V) of nonempty open subsets of X, there exists a positive integer n such that

$$T^n(U) \cap V \neq \emptyset.$$

For more information about hypercyclicity and their related properties in a linear dynamical system, we refer to the books [19] by Grosse-Erdmann and Peris, and [3] by Bayart and Matheron, and the survey article [18] by Grosse-Erdmann.

The concept of recurrence was introduced by Henri Poincaré in 1890 (see [23]). Recently, in the context of linear dynamical system, recurrent operators have been studied by Costakis et al. in [9]. An operator $T \in \mathcal{B}(X)$ is called recurrent if for every nonempty open subset $U \subset X$, there exists a strictly positive integer n such that

$$T^n(U) \cap U \neq \emptyset.$$

A vector $x \in X \setminus \{0\}$ is called a recurrent vector for T, if there exists a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $T^{k_n}x \to x$ as $n \to \infty$. In the remainder of this paper, we say that T is sequentially recurrent if it has a recurrent vector. Obviously, hypercyclicity implies recurrence. For more information about this class of operators, we refer to [1], [6], [10], [11], [12], [16], [26].

Let Y be a Banach space, we recall that $T \in \mathcal{B}(X)$ is said to be quasi-similar or quasi-conjugate to $S \in \mathcal{B}(Y)$ if there exists a continuous map $\varphi\colon\ X \to Y$ with a dense range such that $S \circ \varphi = \varphi \circ T$. Moreover, T and S are called similar or conjugate, if φ is a homeomorphism. Furthermore, a property P is said to be invariant under similarity or conjugate, if it holds: $T \in \mathcal{B}(X)$ has the property P, then every operator $S \in \mathcal{B}(Y)$ that is similar or conjugate to T has also the property P, see [19], Definition 1.5 and Definition 1.7.

For $T \in \mathcal{B}(X)$, we denote by L_T and R_T the left multiplication operator defined by $L_T(S) = TS$ for $S \in \mathcal{B}(X)$ and the right multiplication operator defined by $R_T(S) = ST$ for $S \in \mathcal{B}(X)$, respectively. From [15], we see that $(J, \|\cdot\|_J)$ is a Banach ideal of $\mathcal{B}(X)$ if the following conditions hold:

(i) $J \subset \mathcal{B}(X)$ is a linear subspace.
(ii) The norm $\|\cdot\|_J$ is complete in J and $\|S\| \leqslant \|S\|_J$ for all $S \in J$.
(iii) For all $S \in J$, for all $A, B \in \mathcal{B}(X)$, $ASB \in J$ and $\|ASB\|_J \leqslant \|A\|\|S\|_J\|B\|$.
(iv) The rank-one operators $x \otimes x^* \in J$ and $\|x \otimes x^*\|_J = \|x\|\|x^*\|$ for all $x \in X$ and $x^* \in X^*$.

The rank-one operator $x \otimes x^*$ is defined on X by $(x \otimes x^*)(z) = \langle z, x^* \rangle x = x^*(z)x$ for all $x \in X$, $x^* \in X^*$ and any $z \in X$. The space of all finite rank operators $\mathcal{F}(X)$ is defined as the linear span of rank-one operators. We recall that a Banach ideal $(J, \|\cdot\|_J)$ of $\mathcal{B}(X)$ is said to be admissible if $\mathcal{F}(X)$ is dense in J with respect to the norm $\|\cdot\|_J$.

Let $T \in \mathcal{B}(X)$, if $(J, \|\cdot\|_J)$ is an admissible Banach ideal of $\mathcal{B}(X)$, we denote by $L_{J,T}$ and $R_{J,T}$ the left multiplication operator defined by $L_{J,T}(S) = TS$ for $S \in J$ and the right multiplication operator defined by $R_{J,T}(S) = ST$ for $S \in J$, respectively.

The hypercyclicity of elementary operators has long been considered by several authors. For instance, Chan in 1999 (see [7]) investigated the hypercyclicity in operator algebras, he proved that hypercyclicity can occur on the operator algebra $\mathcal{B}(H)$ in the strong operator topology, when H is a separable Hilbert space. Also, he studied the hypercyclicity of the left multiplication L_T defined on $\mathcal{B}(H)$. Subsequently his idea was used and developed by several authors, see, for example, [5], [8], [20], [22], [27], [28]. Recently, Gilmore et al. (see [13], [14]) have been investigated the hypercyclic properties of the commutator maps $L_T - R_T$ and the generalised derivations $L_A - R_B$. On the other hand, Bonet et al. (see [5], [21]) use tensor product techniques to characterize the hypercyclicity of $L_{J,T}$ and $R_{J,T}$. This result has been extended to the supercyclic case, see [2]. This prompted us to study the recurrence of $L_{J,T}$ and $R_{J,T}$.

In the present work, we introduce the concept of mixing recurrent operators, and we study the transmission of being recurrent, sequentially recurrent, and mixing recurrent between operators $T \in \mathcal{B}(X)$ and their multiplier operators $L_{J,T}$ and $R_{J,T}$. In Section 2, we prove some necessary and sufficient conditions for $L_{J,T}$ and $R_{J,T}$ to be sequentially recurrent. Moreover, we establish a necessary and sufficient condition for the left multiplication $L_{J,T}$ to be recurrent on an admissible Banach ideal $(J, \|\cdot\|_J)$ of $\mathcal{B}(X)$. In particular, we show that $L_{J,T}$ has a recurrent property if and only if $T \oplus T$ has it, too. In Section 3, we introduce the concept of mixing recurrent and we show that it is preserved under quasi-conjugate. Furthermore, we show that T is mixing recurrent if and only if $T \oplus T$ is mixing recurrent. Finally, we prove that $L_{J,T}$ has a mixing recurrent property if and only if T has it, too.

2. Recurrent and sequential recurrent property of the left and right multiplication of operators

In this section, we characterize the recurrence and the sequentially recurrence of the left and right multiplication on an admissible Banach ideal $(J, \|\cdot\|_J)$ of $\mathcal{B}(X)$.

Theorem 2.1. *Let $T \in \mathcal{B}(X)$. Then the following are equivalent:*

(i) *T is sequentially recurrent on X.*

(ii) *$L_{J,T}$ is sequentially recurrent.*

(iii) *$T \oplus T$ is sequentially recurrent on $X \bigoplus X$.*

P r o o f. (i) $\Rightarrow$ (ii) Suppose that T is sequentially recurrent on X. Then there exist a vector $x \in X \setminus \{0\}$ and a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $T^{k_n}x \to x$ as $n \to \infty$. Let $A = x \otimes \varphi \in J$ where $\varphi \in X^*$, then

$$\|(L_{J,T})^{k_n}A - A\|_J \to 0 \quad \text{as } n \to \infty.$$

Thus $L_{J,T}$ is sequentially recurrent.

(ii) $\Rightarrow$ (iii) Assume that $L_{J,T}$ is sequentially recurrent. Then, there exist a vector $A \in J \setminus \{0\}$ and a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $\|(L_{J,T})^{k_n}(A) - A\|_J \to 0$ as $n \to \infty$. We pick $x, y \in X$ such that $\{x, y\}$ is linearly independent and put

$$\varphi\colon\ J \to X \bigoplus X,\ \ S \mapsto Sx \oplus Sy.$$

For $B \in J$, we have

$$(\varphi \circ L_{J,T})(B) = (T \oplus T) \circ \varphi(B).$$

Therefore, $\varphi \circ L_{J,T} = (T \oplus T) \circ \varphi$ on J. Hence, for all integers $m \in \mathbb{N}$, $\varphi \circ L^m_{J,T} = (T \oplus T)^m \circ \varphi$ on J. Then

$$\|(T \oplus T)^{k_n} \circ \varphi(A) - \varphi(A)\| \to 0 \quad \text{as } n \to \infty.$$

This implies that $T \oplus T$ is sequentially recurrent on $X \bigoplus X$.

(iii) $\Rightarrow$ (i) Suppose that $T \oplus T$ is sequentially recurrent on $X \bigoplus X$, then there exist a nonzero vector $x \oplus y \in X \bigoplus X$ and a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that

$$(T \oplus T)^{k_n}(x \oplus y) = T^{k_n}x \oplus T^{k_n}y \to x \oplus y \quad \text{as } n \to \infty.$$

Hence $T^{k_n}x \to x$ and $T^{k_n}y \to y$ as $n \to \infty$. Consequently, T is sequentially recurrent on X. $\square$

Theorem 2.2. *Let $T \in \mathcal{B}(X)$. Then the following are equivalent:*

(i) *T^* is sequentially recurrent on X^*.*

(ii) *$R_{J,T}$ is sequentially recurrent.*

(iii) *$T^* \oplus T^*$ is sequentially recurrent on $X^* \bigoplus X^*$.*

P r o o f. (i) $\Rightarrow$ (ii) Suppose that T^* is sequentially recurrent on X^*. Then, there exist a vector $\varphi \in X^* \setminus \{0\}$ and a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $T^{k_n}\varphi \to \varphi$ as $n \to \infty$. Let $A = x \otimes \varphi \in J$ where $x \in X$, then

$$\|(R_{J,T})^{k_n} A - A\|_J \to 0 \quad \text{as } n \to \infty.$$

It follows that $R_{J,T}$ is sequentially recurrent.

(ii) $\Rightarrow$ (iii) Assume that $R_{J,T}$ is sequentially recurrent. Thus, there exist a vector $A \in J \setminus \{0\}$ and a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $\|(R_{J,T})^{k_n}(A) - A\|_J \to 0$ as $n \to \infty$. We pick $x_1^*, x_2^* \in X^*$ such that $\{x_1^*, x_2^*\}$ is linearly independent and put

$$\varphi\colon\ J \to X^* \bigoplus X^*,\ S \mapsto S^* x_1^* \oplus S^* x_2^*.$$

For $B \in J$, we have

$$(\varphi \circ R_{J,T})(B) = (T^* \oplus T^*) \circ \varphi(B).$$

Therefore, $\varphi \circ R_{J,T} = (T^* \oplus T^*) \circ \varphi$ on J. Consequently, for all integers $m \in \mathbb{N}$, $\varphi \circ R_{J,T}^m = (T^* \oplus T^*)^m \circ \varphi$ on J. Thus

$$\|(T^* \oplus T^*)^{k_n} \circ \varphi(A) - \varphi(A)\| \to 0 \quad \text{as } n \to \infty.$$

It follows that $T^* \oplus T^*$ is sequentially recurrent on $X^* \bigoplus X^*$.

(iii) $\Rightarrow$ (i) Suppose that $T^* \oplus T^*$ is sequentially recurrent on $X^* \bigoplus X^*$, then there exist a nonzero vector $\varphi_1 \oplus \varphi_2 \in X^* \bigoplus X^*$ and a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $(T^* \oplus T^*)^{k_n}(\varphi_1 \oplus \varphi_2) = (T^*)^{k_n}\varphi_1 \oplus (T^*)^{k_n}\varphi_2 \to \varphi_1 \oplus \varphi_2$ as $n \to \infty$. Thus $(T^*)^{k_n}\varphi_1 \to \varphi_1$ and $(T^*)^{k_n}\varphi_2 \to \varphi_2$ as $n \to \infty$. Hence, T^* is sequentially recurrent on X^*. $\square$

In the following lemma, we prove that the notion of recurrence is preserved under quasi-conjugate. This lemma will be useful in the sequel.

Lemma 2.1. *Let X and Y be two Banach spaces and let $T \in \mathcal{B}(X)$ be quasi-conjugate to $S \in \mathcal{B}(Y)$. If T is recurrent on X, then S is recurrent on Y.*

P r o o f. Let U be a nonempty open subset of Y. Since T is quasi-conjugate to S, it follows that $\varphi^{-1}(U)$ is a nonempty open subset of X. On the other hand, T is recurrent on X, so there exists a positive integer $n \in \mathbb{N}^*$ such that

$$T^n(\varphi^{-1}(U)) \cap \varphi^{-1}(U) \neq \emptyset.$$

Hence, there exists a vector $x \in \varphi^{-1}(U)$ such that $T^n x \in \varphi^{-1}(U)$. This implies that $\varphi(x) \in U$ and $S^n\varphi(x) = \varphi(T^n x) \in U$. Consequently, $S^n(U) \cap U \neq \emptyset$. It follows that S is recurrent on Y. $\square$

The following theorem gives a necessary and sufficient condition for $L_{J,T}$ to be recurrent.

Theorem 2.3. *Let $T \in \mathcal{B}(X)$. Then the following are equivalent:*

(i) *$L_{J,T}$ is recurrent.*

(ii) *$T \oplus T$ is recurrent on $X \bigoplus X$.*

As a consequence, if $L_{J,T}$ is recurrent, then T is recurrent on X.

Proof. (i) $\Rightarrow$ (ii) Assume that $L_{J,T}$ is recurrent. We pick $x, y \in X$ such that $\{x, y\}$ is linearly independent and put

$$\varphi\colon\ J \to X \bigoplus X,\ \ S \mapsto Sx \oplus Sy,$$

φ is surjective and for $A \in J$, we have

$$(\varphi \circ L_T)(A) = ((T \oplus T) \circ \varphi)(A).$$

Therefore, $\varphi \circ L_T = (T \oplus T) \circ \varphi$ on J. Now, consider the commutative diagram

$$\begin{tikzcd} J \arrow[r, "L_{J,T}"] \arrow[d, "\varphi"'] & J \arrow[d, "\varphi"] \\ X \bigoplus X \arrow[r, "T\oplus T"'] & X \bigoplus X \end{tikzcd}$$

Lemma 2.1, implies that $T \oplus T$ is recurrent on $X \bigoplus X$. Consequently, T is recurrent on X by [9], Proposition 2.10.

(ii) $\Rightarrow$ (i) Suppose that $T \oplus T$ is recurrent on $X \bigoplus X$ and let U be a nonempty open subset of J. Put

$$\varphi\colon\ X \bigoplus X \to J,\ y_1 \oplus y_2 \mapsto y_1 \otimes \varphi_1 + y_2 \otimes \varphi_2$$

where $\varphi_1, \varphi_2 \in X^*$. φ is continuous and so there is a nonempty open subset $U_1 \times U_2$ of $X \oplus X$ such that $\varphi(U_1 \times U_2) \subset U$. Thus, there exists an integer $n \in \mathbb{N}^*$ such that

$$((T \oplus T)^n(U_1 \times U_2)) \cap (U_1 \times U_2) \neq \emptyset.$$

Hence, there exists $z_1 \oplus z_2 \in U_1 \times U_2$ such that $T^n z_1 \oplus T^n z_2 \in U_1 \times U_2$. Therefore

$$\begin{aligned} T^n(\varphi(z_1 \oplus z_2)) &= T^n(z_1 \otimes \varphi_1 + z_2 \otimes \varphi_2) = T^n z_1 \otimes \varphi_1 + T^n z_2 \otimes \varphi_2 \\ &= \varphi(T^n z_1 \oplus T^n z_2) \in \varphi(U_1 \times U_2) \subset U. \end{aligned}$$

On the other hand, $T^n(\varphi(U_1 \times U_2)) \subset T^n(U)$. Hence

$$T^n(\varphi(z_1 \oplus z_2)) \in T^n(U) \cap U.$$

It proves that $(L_{J,T})^n(U) \cap U = L_{J,T^n}(U) \cap U = T^n(U) \cap U$ is nonempty. Consequently, $L_{J,T}$ is recurrent. $\square$

R e m a r k 2.1. Costakis et al. in [9], Proposition 2.10 proved that if $T \oplus T$ is recurrent, then T is recurrent. The problem here is whether $T \oplus T$ is recurrent whenever T is recurrent (see [9], Question 9.6). In Theorem 2.3, we have proved that if $L_{J,T}$ is recurrent on $(J, \|\cdot\|_J)$, then T is recurrent on X. For the converse, it is an open question.

3. Mixing recurrent property of the left multiplication of operators

In this section, we investigate mixing recurrent operators and prove that it is preserved under quasi-conjugate. On the other hand, we characterize the mixing recurrent property of the left multiplication on an admissible Banach ideal $(J, \|\cdot\|_J)$ of $\mathcal{B}(X)$.

Motivated by the concept of mixing operators and their connection to hypercyclicity, see [3], [17], [25], we introduce the notion of mixing recurrent operators.

Definition 3.1. A dynamical system $T\colon X \to X$ is called mixing recurrent if for any nonempty open subset U of X there exists an integer $N \in \mathbb{N}$ such that

$$T^n(U) \cap U \neq \emptyset \quad \forall\, n \geqslant N.$$

In the following lemma, we show that the class of mixing recurrent operators is preserved under quasi-conjugate.

Lemma 3.1. *Let X and Y be Banach spaces and let $T \in \mathcal{B}(X)$ be quasi-conjugate to $S \in \mathcal{B}(Y)$. If T is mixing recurrent on X, then S is mixing recurrent on Y.*

P r o o f. Let U be a nonempty open subset of Y. Since T is quasi-conjugate to S, it follows that $\varphi^{-1}(U)$ is a nonempty open subset of X. But T is mixing recurrent on X, so there exists a positive integer $N \in \mathbb{N}$ such that

$$T^n(\varphi^{-1}(U)) \cap \varphi^{-1}(U) \neq \emptyset \quad \forall\, n \geqslant N.$$

Hence, there exists a vector $x \in \varphi^{-1}(U)$ such that $T^n x \in \varphi^{-1}(U)$ for all $n \geqslant N$. Then $\varphi(x) \in U$ and $S^n\varphi(x) = \varphi(T^n x) \in U$ for all $n \geqslant N$. Consequently, $S^n(U) \cap U \neq \emptyset$ for all $n \geqslant N$. Thus, T is mixing recurrent on Y. □

Lemma 3.2. *Let X_i be Banach spaces and $T_i \in \mathcal{B}(X_i)$, $1 \leqslant i \leqslant m$. The following are equivalent:*

(i) *T_i is mixing recurrent on X_i, $1 \leqslant i \leqslant m$.*

(ii) *$\bigoplus_{i=1}^{m} T_i$ is mixing recurrent on $\bigoplus_{i=1}^{m} X_i$.*

P r o o f. (i) $\Rightarrow$ (ii) Assume that T_i is mixing recurrent on X_i ($1 \leqslant i \leqslant m$) and let U be a nonempty open subset of $\bigoplus_{i=1}^{m} X_i$, then there exist nonempty open subsets U_i of X_i ($1 \leqslant i \leqslant m$) such that $U_1 \times U_2 \times \ldots \times U_m \subset U$. Therefore there exists $N_i \in \mathbb{N}$ such that $T_i^n(U_i) \cap U_i \neq \emptyset$ for all $n \geqslant N_i$ ($1 \leqslant i \leqslant m$). Let $N = \max_{1 \leqslant i \leqslant m} N_i$, thus

$$\Big(\bigoplus_{i=1}^{m} T_i\Big)^k (U_1 \times \ldots \times U_m) \cap (U_1 \times \ldots \times U_m) = \bigoplus_{i=1}^{m} (T_i^k(U_i) \cap U_i) \neq \emptyset \quad \forall\, k \geqslant N.$$

Consequently

$$\Big(\bigoplus_{i=1}^{m} T_i\Big)^k (U) \cap U \neq \emptyset \quad \forall\, k \geqslant N.$$

Then $\bigoplus_{i=1}^{m} T_i$ is mixing recurrent on $\bigoplus_{i=1}^{m} X_i$.

(ii) $\Rightarrow$ (i) Suppose that $\bigoplus_{i=1}^{m} T_i$ is mixing recurrent on $\bigoplus_{i=1}^{m} X_i$ and let U_i be a nonempty open subset of X_i ($1 \leqslant i \leqslant m$), then there exists an integer $N \in \mathbb{N}$ such that

$$\Big(\bigoplus_{i=1}^{m} T_i\Big)^n (U_1 \times \ldots \times U_m) \cap (U_1 \times \ldots \times U_m) = \bigoplus_{i=1}^{m} (T_i^n(U_i) \cap U_i) \neq \emptyset \quad \forall\, n \geqslant N.$$

Thus $T_i^n(U_i) \cap U_i \neq \emptyset$ for all $n \geqslant N$ ($1 \leqslant i \leqslant m$). Therefore, T_i is mixing recurrent on X_i for every $i \in \{1, 2, \ldots, m\}$. $\square$

Theorem 3.1. *Let $(J, \|\cdot\|_J)$ be an admissible Banach ideal of $\mathcal{B}(X)$ and $T \in \mathcal{B}(X)$. Then the following are equivalent:*

(i) *T is mixing recurrent on X.*

(ii) *$L_{J,T}$ is mixing recurrent.*

P r o o f. (i) $\Rightarrow$ (ii) Suppose that T is mixing recurrent on X and let U be a nonempty open subset of J. Note that D and Φ are countable dense subsets of X and X^*, respectively. Then

$$\mathcal{X} = \Big\{\sum_{i=1}^{m} x_i \otimes \varphi_i \colon\, x \in D,\, \varphi_i \in \Phi\Big\}$$

is a dense subset of J with respect to $\|\cdot\|_J$-topology, see [2], Lemma 2.1. Therefore there exist $x_1, x_2, \ldots, x_m \in X$ and $\varphi_1, \varphi_2, \ldots, \varphi_m \in X^*$ such that $\sum_{i=1}^{m} x_i \otimes \varphi_i \in U$. Put

$$\varphi\colon\, \bigoplus_{i=1}^{m} X \to J,\ \bigoplus_{i=1}^{m} y_i \mapsto \sum_{i=1}^{m} y_i \otimes \varphi_i.$$

φ is continuous and so there are nonempty open subsets $U_1 \times U_2 \times \ldots \times U_m$ of $\bigoplus_{i=1}^{m} X$ such that $\varphi(U_1 \times U_2 \times \ldots \times U_m) \subset U$. On the other hand, Lemma 3.2 implies that $\bigoplus_{i=1}^{m} T$ is mixing recurrent on $\bigoplus_{i=1}^{m} X$ and so there exists an integer $N \in \mathbb{N}$ such that

$$(T^n(U_1) \times T^n(U_2) \times \ldots \times T^n(U_m)) \cap (U_1 \times U_2 \times \ldots \times U_m) \neq \emptyset \quad \forall n \geqslant N.$$

Hence, there exists $z_1 \oplus z_2 \oplus \ldots \oplus z_m \in U_1 \times U_2 \times \ldots \times U_m$ such that $(T^n z_1 \oplus T^n z_2 \oplus \ldots \oplus T^n z_m) \in U_1 \times U_2 \times \ldots \times U_m$. Therefore

$$\begin{aligned} & T^n(\varphi(z_1 \oplus z_2 \oplus \ldots \oplus z_m)) \\ & \quad = T^n\left(\sum_{i=1}^{m} z_i \otimes \varphi_i\right) = \sum_{i=1}^{m} T^n z_i \otimes \varphi_i \\ & \quad = \varphi(T^n z_1 \oplus T^n z_2 \oplus \ldots \oplus T^n z_m) \in \varphi(U_1 \times U_2 \times \ldots \times U_m) \subset U. \end{aligned}$$

Furthermore, $T^n(\varphi(U_1 \times U_2 \times \ldots \times U_m)) \subset T^n(U)$. Hence

$$T^n(\varphi(z_1 \oplus z_2 \oplus \ldots \oplus z_m)) \in T^n(U) \cap U.$$

It follows that $(L_{J,T})^n(U) \cap U = L_{J,T^n}(U) \cap U = T^n(U) \cap U$ is nonempty for all $n \geqslant N$. Therefore, $L_{J,T}$ is mixing recurrent.

(ii) $\Rightarrow$ (i) Assume that $L_{J,T}$ is mixing recurrent. We pick $x, y \in X$ such that $\{x, y\}$ is linearly independent and put

$$\varphi\colon\ J \to X \bigoplus X,\ S \mapsto Sx \oplus Sy.$$

φ is surjective and for $A \in J$, we have

$$\varphi \circ L_{J,T}(A) = (T \oplus T) \circ \varphi(A).$$

Therefore, $\varphi \circ L_{J,T} = (T \oplus T) \circ \varphi$ on J. Now, consider the commutative diagram

$$\begin{tikzcd} J \arrow[r, "L_{J,T}"] \arrow[d, "\varphi"'] & J \arrow[d, "\varphi"] \\ X \bigoplus X \arrow[r, "T \oplus T"'] & X \bigoplus X \end{tikzcd}$$

Lemma 3.1 implies that $T \bigoplus T$ is mixing recurrent on $X \bigoplus X$. Now, by Lemma 3.2, T is mixing recurrent on X. $\square$

We have the following corollary.

Corollary 3.1. *If $\mathcal{K}(X)$ is an admissible Banach ideal of $\mathcal{B}(X)$, then for all $T \in \mathcal{B}(X)$, the following are equivalent:*

(i) *T is mixing recurrent on X.*
(ii) *L_T is mixing recurrent on $\mathcal{K}(X)$.*
(iii) *L_T is mixing recurrent on $\mathcal{B}(X)$ in the strong operator topology.*

Proof. (i) $\Leftrightarrow$ (ii) A consequence of Theorem 3.1, since $\mathcal{K}(X)$ is an admissible Banach ideal of $\mathcal{B}(X)$.

(i) $\Rightarrow$ (iii) Assume that T is mixing recurrent on X and let U be a nonempty open subset of $\mathcal{B}(X)$ in the strong operator topology. Since $\mathcal{K}(X)$ is dense in $\mathcal{B}(X)$ with the strong operator topology (see [8], Corollary 3), then $\mathcal{K}(X) \cap U$ is nonempty. Let $A \in \mathcal{K}(X) \cap U$, we can find $x \in X \setminus \{0\}$, $\varepsilon > 0$ such that

$$\{V \in \mathcal{B}(X)\colon \|(V - A)x\| < \varepsilon\} \subseteq U.$$

Let

$$U' = \Big\{V \in \mathcal{K}(X)\colon \|(V - A)\| < \frac{\varepsilon}{\|x\|}\Big\}.$$

U' is a nonempty open subset of $\mathcal{K}(X)$ with the norm operator topology. By Theorem 3.1 with $J = \mathcal{K}(X)$, there exists an integer $N \in \mathbb{N}$ such that $L_{T^n}(U') \cap U'$ is nonempty for all $n \geqslant N$. It follows that $L_{T^n}(U) \cap U \neq \emptyset$ for all $n \geqslant N$. Consequently, L_T is mixing recurrent on $\mathcal{B}(X)$ in the strong operator topology.

(iii) $\Rightarrow$) (i) By the same technique as in the proof of Theorem 3.1. $\square$

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