ON A THEOREM OF McCOY

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Abstract. We study McCoy's theorem to the skew Hurwitz series ring (HR,ω) for some different classes of rings such as: semiprime rings, APP rings and skew Hurwitz serieswise quasi-Armendariz rings. Moreover, we establish an equivalence relationship between a right zip ring and its skew Hurwitz series ring in case when a ring R satisfies McCoy's theorem of skew Hurwitz series.

Keywords: skew Hurwitz series ring; ω-compatible ring; skew Hurwitz serieswise; quasi-Armendariz rings; zip ring; APP ring

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1. Introduction

In 1942, McCoy (see [26], Theorem 2) proved that if R is a commutative ring then $f(x)$ is a zero divisor in $R[x]$ if and only if $f(x)c=0$ for some nonzero $c\in R$, where $R[x]$ is the polynomial ring with the indeterminate x over R. Further, Jones and Weiner in [19] showed that this result fails in noncommutative rings. Nowadays, the above theorem of McCoy is popular by the particular name "McCoy's theorem". Motivated by the result of Jones and Weiner (see [19]), McCoy in [27] proved the following result.

Theorem 1.1. *Let R be a ring and A a right ideal of $S=R[x_1,x_2,\dots x_n]$. If* $\mathrm{r.ann}_S(A)\neq 0$ *then* $\mathrm{r.ann}_R(A)\neq 0$.

After long time, that is, in 2002, Hirano (see [16]) studied McCoy's theorem independently. In particular, he proved that if $f(x)\in R[x]$, $\mathrm{r.ann}_{R[x]}(f(x)R[x])\neq 0$ then $\mathrm{r.ann}_R(f(x)R[x])\neq 0$. On the other hand, McCoy's theorem fails in the formal

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power series $R[[x]]$ over a commutative ring R by [11], Example 3 in general. However, Gilmar and Parker in [13] proved McCoy's theorem for several ring structures. Such ring structures include reducedness and the von Neumann regularity of the total quotient ring, etc.

Inspired by above development, several authors are working in this direction. Recently, Hong et al. (see [18]) extended McCoy's theorem to Ore extensions, the skew power series rings and so on. Faith in [9] called a ring R right zip provided that if the right annihilator $\mathrm{r.ann}_R(X)$ of a subset X of R is zero, then there exists a finite subset $Y \subseteq X$ such that $\mathrm{r.ann}_R(Y) = 0$; equivalently, for a left ideal L of R with $\mathrm{r.ann}_R(L) = 0$, there exists a finitely generated left ideal $L_1 \subseteq L$ such that $\mathrm{r.ann}_R(L_1) = 0$. The ring R is zip if it is both right and left zip.

The concept of zip rings was initiated by Zelmanowitz (see [39]) and appeared in various papers, see e.g. [4], [7], [9], [10] and references therein. Zelmanowitz stated that any ring satisfying the descending chain conditions on right annihilators is a right zip ring, but the converse does not hold. Beachy and Blair in [4] studied rings that satisfy the condition that every faithful right ideal I of a ring R (a right ideal I of a ring R is faithful if $\mathrm{r.ann}_R(I) = 0$) is cofaithful (a right ideal I of a ring R is cofaithful if there exists a finite subset $I_1 \subseteq I$ such that $\mathrm{r.ann}_R(I_1) = 0$). Right zip rings have this property and conversely for commutative ring R. Properties of zip rings and their extensions were studied by several authors.

Following [31], [32], [34], [35], a ring R is called Armendariz if whenever for any $f(x) = \sum\limits_{i=0}^{m} a_i x^i$, $g(x) = \sum\limits_{j=0}^{n} b_j x^j \in R[x]$, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for each $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$. Hirano in [16] generalized the concept of Armendariz ring and coined the structure of quasi-Armendariz ring. A ring R is called quasi-Armendariz if whenever for any $f(x) = \sum\limits_{i=0}^{m} a_i x^i$, $g(x) = \sum\limits_{j=0}^{n} b_j x^j \in R[x]$, $f(x)R[x]g(x) = 0$ implies $a_i R b_j = 0$ for each $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$.

Further, Hashemi and Moussavi (see [14]) provided skew polynomial and power series versions of quasi-Armendariz rings, introduced by conditions (SQA1) and (SQA2), respectively, and defined: For any automorphism ω, a ring R satisfies the (i) condition (SQA1) if whenever for any $f(x) = \sum\limits_{i=0}^{m} a_i x^i$, $g(x) = \sum\limits_{j=0}^{n} b_j x^j \in R[x,\omega]$, $f(x)R[x,\omega]g(x) = 0$ implies $a_i R b_j = 0$ for each $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$; (ii) for any $f(x) = \sum\limits_{i=0}^{\infty} a_i x^i$, $g(x) = \sum\limits_{j=0}^{\infty} b_j x^j \in R[[x,\omega]]$, $f(x)R[[x,\omega]]g(x) = 0$ implies $a_i R b_j = 0$ for each $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$.

Throughout this article R denotes an associative ring with identity. In this paper, we study McCoy's theorem to the skew Hurwitz series ring (HR,ω) for various classes of rings such as: semiprime rings, APP rings (for the definition see page 6) and skew Hurwitz serieswise quasi-Armendariz rings (for the definition see page 7). Moreover, we establish an equivalence relationship between a right zip ring and its skew Hurwitz series ring in case when a ring satisfies McCoy's theorem of the skew Hurwitz series ring.

2. Skew Hurwitz series rings with the right Beachy-Blair condition

The concept of Hurwitz series ring was proposed by Keigher in [21], as a variant of the ring of formal power series. He also studied some of its properties especially the categorical properties. He and later with Pritchard in [20], [22] demonstrated that Hurwitz series have many interesting applications in differential algebra and in the discussion of weak normalization. The elements of Hurwitz series HR are sequences of the form $a = (a_n) = (a_0, a_1, a_2, a_3, \ldots) \in R$, for each $n \in \mathbb{Z}^+$ or $f(n) = a_n$, where $f\colon \mathbb{Z}^+ \to R$ and $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ is a set of nonnegative integers. The addition in HR is pointwise and the product of two sequences uses binomial coefficients. This was studied by Fliess (see [12]) and Taft (see [37]).

Number of authors have studied the properties of some abstract ring structures in the skew Hurwitz series and the Hurwitz series ring. Now, recall the construction of skew Hurwitz series ring (HR, ω) (see [5], [15], [28], [29]). Let $\omega\colon R \to R$ be an endomorphism of R and $\omega(1) = 1$. The ring (HR, ω) of skew Hurwitz series over a ring R is defined as follows: the elements of (HR, ω) are functions $\omega\colon \mathbb{Z}^+ \to R$, where $\mathbb{Z}^+$ is the set of positive integers with zero. The operation of addition in (HR, ω) is component wise and the multiplication is defined for every $f, g \in (\mathrm{HR}, \omega)$ by

$$fg(p) = \sum_{k=0}^{p} C_k^p f(k)\omega^k(g(p-k))$$

for all $p, k \in \mathbb{Z}^+$, where $C_k^p = p!\,/(k!\,(p-k)!\,)$. It can be easily shown that (HR, ω) is a ring with identity h_1, defined by

$$h_1(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 1, \end{cases}$$

where $n \in \mathbb{Z}^+$. It is clear that R is canonically embedded as a subring of (HR, ω) via $a \to h_a \in (\mathrm{HR}, \omega)$, where

$$h_a(n) = \begin{cases} a & \text{if } n = 0, \\ 0 & \text{if } n \geqslant 1. \end{cases}$$

For any function $f \in (\mathrm{HR}, \omega)$, $\operatorname{supp}(f) = \{n \in \mathbb{Z}^+ \colon\ f(n) \neq 0\}$ denotes the support of f and $\pi(f)$ denotes the minimal element of $\operatorname{supp}(f)$. For any nonempty subset X of R, we put

$$(HX, \omega) = \{f \in (\mathrm{HR}, \omega) \colon\ f(n) \in X \cup \{0\},\ n \in \mathbb{Z}^+\}.$$

Due to Krempa (see [23]), a monomorphism ω of a ring R is said to be rigid if $a\omega(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is called ω-rigid if there exists a rigid endomorphism ω of R. In [2], Annin stated a ring R is ω-compatible if for every $a, b \in R$, $ab = 0$ if and only if $a\omega(b) = 0$. Hashemi and Moussavi in [14] gave some examples of nonrigid ω-compatible rings and proved the following lemma.

Lemma 2.1. *Let ω be an endomorphism of a ring R. Then*

(1) *if ω is compatible, then ω is injective;*
(2) *ω is compatible if and only if for all $a, b \in R$, $\omega(a)b = 0 \Leftrightarrow ab = 0$;*
(3) *the following conditions are equivalent:*
  (a) *ω is rigid;*
  (b) *ω is compatible and R is reduced;*
  (c) *for every $a \in R$, $\omega(a)a = 0$ implies that $a = 0$.*

Now, we prove the following lemma which is required to prove next theorem.

Lemma 2.2. *Let R be a semiprime ring and ω be an endomorphism of R. If R is an ω-compatible and torsion-free as a $\mathbb{Z}$-module then for any $f, g \in (\mathrm{HR}, \omega)$, $f(\mathrm{HR}, \omega)g = 0$ if and only if $f(u)Rg(v) = 0$ for every $u \in \operatorname{supp}(f)$ and every $v \in \operatorname{supp}(g)$.*

P r o o f. Let for any $f, g \in (\mathrm{HR}, \omega)$, $f(\mathrm{HR}, \omega)g = 0$ with $u \in \operatorname{supp}(f)$ and $v \in \operatorname{supp}(g)$. To prove the result, we need to show $f(u)Rg(v) = 0$ for every $u \in \operatorname{supp}(f)$ and every $v \in \operatorname{supp}(g)$. Suppose that $u_0 = \pi(f)$ and $v_0 = \pi(g)$ are minimal elements of $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$, respectively. So,

$$\begin{aligned}(fh_r g)(u_0 + v_0) &= \sum_{(u,v) \in X_{u_0+v_0}(f, h_r g)} C_u^{u+v} f(u)\omega^u((h_r g)(v)) \\ &= C_{u_0}^{u_0+v_0} f(u_0)\omega^{u_0}(g(v_0)) = 0.\end{aligned}$$

Since R is torsion-free as a $\mathbb{Z}$-module and ω-reduced so $f(u_0)rg(v_0) = 0$ for all $r \in R$. Suppose $k \in \mathbb{Z}^+$ such that for every $u \in \operatorname{supp}(f)$ and every $v \in \operatorname{supp}(g)$ with $u + v < k$, $f(u)rg(v) = 0$. By applying the induction method, we prove that

$f(u)rg(v)=0$ for every $u\in\operatorname{supp}(f)$ and every $v\in\operatorname{supp}(g)$ with $u+v=k$, and for all $r\in R$. We have

$$(fh_rg)(u+v)=\sum_{(p,q)\in X_{u+v=k}(f,h_rg)} C_p^{p+q} f(p)\omega^p((h_rg)(q)),$$

where $X_{u+v=k}(f,h_rg)=\{(p,q)\colon\ p+q=k,\ p\in\operatorname{supp}(f),\ q\in\operatorname{supp}(g)\}$. Without loss of generality, we take $\{(p_i,q_i)\colon\ p_i+q_i=k,\ i=1,2,3,\ldots,s\}=X_{u+v=k}(f,h_rg)$ with $p_1<p_2<p_3<\ldots<p_s$. We obtain

$$(2.1)\qquad (fh_rg)(u+v)=\sum_{i=1}^{s} C_{p_i}^{p_i+q_i} f(p_i)\omega^{p_i}(rg(q_i))=0.$$

Since $p_i+q_1<p_i+q_i=k$ for every $i\geqslant 2$, $C_{p_i}^{p_i+q_1}f(p_i)\omega^{p_i}(h_rg(q_1))=0$ for every $i\geqslant 2$. It follows that $f(p_i)\omega^{p_i}(rg(q_1))=0$ for all $i\geqslant 2$ since R is torsion-free as a $\mathbb{Z}$-module. Replace r by $rg(q_1)t$ in (2.1), where $r,s\in R$. Then, we obtain

$$(2.2)\qquad \sum_{i=1}^{s} C_{p_i}^{p_i+q_i} f(p_i)\omega^{p_i}(r(g(q_1)t)g(q_i))=0.$$

It follows that $C_{p_1}^{p_1+q_1}f(p_1)\omega^{p_1}(r(g(q_1))tg(q_1))=0$. Since R is torsion-free as a $\mathbb{Z}$-module, $f(p_1)\omega^{p_1}(rg(q_1)tg(q_1))=0$. Thus, we have

$$f(p_1)R\omega^{p_1}(g(q_1)R\omega^{p_1}g(q_1))=0.$$

It follows that $f(p_1)rg(q_1)=0$ for all $r\in R$ since R is a semiprime ring and an ω-compatible ring. Therefore from (2.1), we have

$$(2.3)\qquad \sum_{i=2}^{s} C_{p_i}^{p_i+q_i} f(p_i)\omega^{p_i}(g(q_i))=0.$$

Using the same logic of proof and replacing r by $rg(q_2)t$ in (2.3), where $r,s\in R$, we get $f(p_2)rg(q_2)=0$ for all $r\in R$. Thus from (2.3), we get

$$(2.4)\qquad \sum_{i=3}^{s} C_{p_i}^{p_i+q_i} f(p_i)\omega^{p_i}(g(q_i))=0.$$

Similarly, we get $f(p_i)rg(q_i)=0$ for every $p_i\in\operatorname{supp}(f)$ and every $q_i\in\operatorname{supp}(g)$ with $p_i+q_i=k$, and for all $r\in R$. Thus $f(u)rg(v)=0$ for every $u\in\operatorname{supp}(f)$ and every $v\in\operatorname{supp}(g)$, and for all $r\in R$. $\square$

Now, we prove the following result.

Theorem 2.3. *Let R be a semiprime ring, torsion-free as a $\mathbb{Z}$-module and an ω-compatible ring. If for any right ideal A of the skew Hurwitz series ring* (HR, ω), $\mathrm{r.ann}_{(\mathrm{HR},\omega)}(A) \neq 0$ *then* $\mathrm{r.ann}_R(A) \neq 0$.

Proof. Let A be a right ideal of (HR, ω) with $\mathrm{r.ann}_{(\mathrm{HR},\omega)}(A) \neq 0$ and $g \in \mathrm{r.ann}_{(\mathrm{HR},\omega)}(A)$ with $\mathrm{supp}(g) \neq 0$. Then $Ag = 0$ which implies that $f(\mathrm{HR}, \omega)g = 0$ for all $f \in A$. Thus $f(n)Rg(m) = 0$ for all $n \in \mathrm{supp}(f)$ and all $m \in \mathrm{supp}(g)$ by Lemma 2.2. Since $m \in \mathrm{supp}(g)$, then $g(m) \neq 0$. It follows that $f(\mathrm{HR}, \omega)c = 0$, where $c = g(m) \neq 0$. Hence $\mathrm{r.ann}_R(A) \neq 0$. □

According to Tominaga (see [38]) an ideal U is called right s-unital if there exists $a \in U$ such that $ua = u$ for any $a \in U$. A submodule L_1 of a left R-module L is said to be pure submodule if $N \otimes_R L_1 \to N \otimes_R L$ is a monomorphism for any right R-module N. From [36], Proposition 11.3.13 for an U, the following statements are equivalent:

(1) U is pure as a left ideal in R;
(2) R/U is flat as a left R-module;
(3) U is right s-unital.

By [38], Definition 2.1, R is a left APP ring if for every element of R, the left annihilator $\mathrm{l.ann}_R(Ra)$ is a right s-unital as an ideal of R. Similarly, we can define a right APP ring. The class of APP ring includes well known classes of rings such as: Baer rings, p.p.-rings, quasi-Baer rings, p.q.-Baer rings, and biregular rings, for definitions, see [3], [6], [14], [30].

Lemma 2.4 ([30], Lemma 3.2). *Let R be a right APP ring, torsion free as a $\mathbb{Z}$-module and an ω-compatible ring. If for any $f, g \in (\mathrm{HR}, \omega)$, $f(\mathrm{HR}, \omega)g = 0$ then $f(n)Rg(m) = 0$ for all $n \in \mathrm{supp}(f)$ and all $m \in \mathrm{supp}(g)$.*

Lemma 2.5. *Let R be a ring satisfying descending chain condition on left and right annihilators, torsion-free as a $\mathbb{Z}$-module and ω-compatible. If R is a left APP ring then the skew Hurwitz series* (HR, ω) *is left APP.*

Proof. Let $f, g \in (\mathrm{HR}, \omega)$ such that $f \in \mathrm{l.ann}_{(\mathrm{HR},\omega)}((\mathrm{HR}, \omega)g)$. Then

$$f(\mathrm{HR}, \omega)g = 0.$$

Since R is left APP and ω-compatible, by Lemma 2.4, we have $f(n)Rg(m) = 0$ for all $n \in \mathrm{supp}(f)$ and for all $m \in \mathrm{supp}(f)$. It follows that $f(n) \in \mathrm{l.ann}_R(Rg(m))$ for all $n \in \mathrm{supp}(f)$ and for all $m \in \mathrm{supp}(f)$. Let $A = \{n_i \in \mathrm{supp}\, f\colon \mathrm{supp}(f) \neq 0$ and $n_i \in \mathbb{N}\}$, where $i = 0, 1, 2, \ldots$, and $B = \{m_j \in \mathrm{supp}\, g\colon \mathrm{supp}(g) \neq 0$ and $m_j \in \mathbb{N}\}$, where $j = 0, 1, 2, \ldots$ □

Theorem 2.6. *Let R be a right APP ring, torsion-free as a $\mathbb{Z}$-module and an ω-compatible ring. If for any right ideal A of the skew Hurwitz series ring* (HR, ω), $\mathrm{r.ann}_{(\mathrm{HR},\omega)}(A) \neq 0$ *then* $\mathrm{r.ann}_R(A) \neq 0$.

Proof. Let A be a right ideal of (HR, ω) with $\mathrm{r.ann}_{(\mathrm{HR},\omega)}(A) \neq 0$ and $g \in \mathrm{r.ann}_{(\mathrm{HR},\omega)}(A)$ with $\mathrm{supp}(g) \neq 0$. Then $Ag = 0$ which implies that $f(\mathrm{HR}, \omega)g = 0$ for all $f \in A$. Since R is a right APP ring, torsion-free as a $\mathbb{Z}$-module and an ω-compatible ring, $f(n)Rg(m) = 0$ for all $n \in \mathrm{supp}(f)$ and all $m \in \mathrm{supp}(g)$, by Lemma 2.4. Since $m \in \mathrm{supp}(g)$, $g(m) \neq 0$. It follows that $f(\mathrm{HR}, \omega)c = 0$, where $c = g(m) \neq 0$. Hence $\mathrm{r.ann}_R(A) \neq 0$. $\square$

In the following example, we show that there is no straight relation between APP ring and semiprime ring.

Example 2.7. Let

$$S = \Big\{(a_n)_{n=1}^{\infty} \in \prod F\colon\ a_n \text{ is eventually constant}\Big\}$$

be a commutative reduced ring and F be a field. So, $R = S[[x]]$ is also commutative reduced ring, see [6], Example 2.3. It follows from [25], Example 2.4 that R is a commutative semiprime ring but not a left APP ring.

From the above example it is clear that the consideration of a left APP ring in Theorem 2.6 is not trivial and provides a new class of ring structure which satisfies McCoy's theorem for the skew Hurwitz series ring (HR, ω).

The class of Armendariz rings to skew Hurwitz series rings (HR, ω), where $\omega\colon R \to R$ is an endomorphism of R was studied by Ahmedi et al. in [1]. A commutative ring R is called skew Hurwitz serieswise Armendariz (or SHA) if for every $\alpha, \beta \in (\mathrm{HR}, \omega)$, $\alpha\beta = 0$ if and only if $\alpha(n)\beta(m)$ for every $n, m \in \mathbb{N}$. Recently, Sharma and Singh (see [33]) extended the concept of skew Hurwitz serieswise Armendariz, in case of R is a noncommutative ring. A ring R is said to be skew Hurwitz serieswise Armendariz if for every skew Hurwitz series $f, g \in (\mathrm{HR}, \omega)$, $fg = 0$ implies $f(n)\omega^n g(m) = 0$ for $n, m \in \mathbb{N}$.

Motivated by this development of different versions to Armendariz and quasi-Armendariz rings, we introduce the concept of quasi-Armendariz ring to the skew Hurwitz series ring by considering a noncommutative ring R.

Definition 2.8. A ring R is called skew Hurwitz serieswise quasi-Armendariz if for every $\alpha, \beta \in (\mathrm{HR}, \omega)$, $\alpha(\mathrm{HR}, \omega)\beta = 0$ if and only if $\alpha(n)R\beta(m) = 0$ for every $n, m \in \mathbb{N}$.

It is clear that the skew Hurwitz serieswise quasi-Armendariz ring is a natural generalization of ω-rigid ring and skew Hurwitz serieswise Armendariz (or SHA).

Lemma 2.9. *Every skew Hurwitz serieswise quasi-Armendariz ring is torsion-free as a $\mathbb{Z}$-module.*

P r o o f. Proof is similar to [1], Proposition 2.2. $\square$

Lemma 2.10. *Every skew Hurwitz serieswise quasi-Armendariz ring is ω-compatible.*

P r o o f. Proof is similar to [1], Proposition 2.5. $\square$

Hirano [16], Corollary 3.8 proved that a semiprime ring is a quasi-Armendariz ring. In the following proposition, we generalize this result of Hirano [16].

Proposition 2.11. *If R is an ω-compatible semiprime ring and torsion-free as a $\mathbb{Z}$-module. Then R is a skew Hurwitz serieswise quasi-Armendariz ring.*

P r o o f. Suppose $\alpha, \beta \in (\mathrm{HR}, \omega)$ such that $\alpha(\mathrm{HR}, \omega)\beta = 0$. Since R is an ω-compatible semiprime ring and torsion-free as a $\mathbb{Z}$-module, by Lemma 2.1, $\alpha(n)R\beta(m) = 0$ for every $n, m \in \mathbb{N}$. $\square$

In the following example, we see that the assumption of an ω-compatible ring in the above proposition is not superfluous.

E x a m p l e 2.12. Let $R = Z \oplus Z$ be a commutative semiprime ring and $\omega(a, b) = (b, a)$. Then ω is an endomorphism of R. Thus for any $(1, 0), (0, 1) \in R$, $(1, 0)(0, 1) = (0, 0)$ which implies that $(1, 0)\omega(0, 1) = (1, 0)(1, 0) \neq (0, 0)$. It follows that R is not ω-compatible. Hence, by Lemma 2.10, R is not skew Hurwitz serieswise quasi-Armendariz.

In [16], Theorem 3.9, Hirano prove that every left APP ring is a quasi-Armendariz ring. Here, we show this result to skew Hurwitz serieswise quasi-Armendariz.

Proposition 2.13. *If R is an ω-compatible left APP ring and torsion-free as a $\mathbb{Z}$-module. Then R is a skew Hurwitz serieswise quasi-Armendariz ring.*

P r o o f. Suppose $\alpha, \beta \in (\mathrm{HR}, \omega)$ such that $\alpha(\mathrm{HR}, \omega)\beta = 0$. Since R is an ω-compatible left APP ring and torsion-free as a $\mathbb{Z}$-module, by Lemma 2.4,

$$\alpha(n)R\beta(m) = 0$$

for every $n, m \in \mathbb{N}$. $\square$

We get the following result.

Theorem 2.14. *Let R be a skew Hurwitz serieswise quasi-Armendariz ring. If for any right ideal A of (HR, ω), $\mathrm{r.ann}_{(\mathrm{HR},\omega)}(A) \neq 0$ then $\mathrm{r.ann}_R(A) \neq 0$.*

P r o o f. Let A be a right ideal of R and $\beta \in \mathrm{r.ann}_{(\mathrm{HR},\omega)}(A)$. It follows that $A\beta = 0$ which implies that $\alpha(\mathrm{HR},\omega)\beta = 0$ for any $\alpha \in A$. Since R is a skew Hurwitz serieswise quasi-Armendariz ring so $\alpha(n)R\beta(m) = 0$ for all $m, n \in \mathbb{N}$. Therefore $Aa = 0$, where $\beta(m) = a \neq 0$. Thus $\mathrm{r.ann}_R(A) \neq 0$. $\square$

Extensions of zip rings were studied by several authors. Beachy and Blair in [4] showed that if R is a commutative zip ring, then polynomial ring $R[x]$ over R is a zip ring. Afterward, Cedo (see [7]) studied that if R is a commutative zip ring, then the $n \times n$ full matrix ring $\mathrm{Mat}_n(R)$ over R is zip; moreover, he settled negatively the question which were posed by Faith (see [9]): Does R being a right zip ring implies $R[x]$ being a right zip? Based on the preceding results, Faith in [10] again raised the following question: When does R being a right zip ring implies $R[x]$ being a right zip? In [17], Hong et al. answered this question positively for Armendariz ring. They proved that R is a right zip ring if and only if $R[x]$ is a right zip ring when R is an Armendariz ring. Further, Cortes (see [8]) studied the relationship between the right (left) zip property of R and its skew polynomial and power series extensions over R by using the skew versions of Armendariz rings. Ahmadi et al. (see [1], Theorem 2.19) studied the same property of a zip ring to the skew Hurwitz series ring (HR,ω) for a commutative ring R. They proved that if R is a SHA-ring and ω an endomorphism of R, then R is a right zip ring if and only if (HR,ω) is a right zip ring. In [24], Leroy and Matczuk investigated the behavior of the right zip property under some ring constructions. After that, Sharma and Singh in [33] showed that if a ring R (not necessary commutative) is skew Hurwitz serieswise Armendariz and ω-compatible then R is a right zip ring if and only if (HR,ω) is a right zip ring.

Here, we establish an equivalence relationship between right zip ring and its skew Hurwitz series ring if it in case of a ring R satisfies McCoy's theorem of skew Hurwitz serieswise.

Theorem 2.15. *Let R be a ring and ω be an endomorphism of R. If R satisfies McCoy's theorem of skew Hurwitz serieswise and R is ω-compatible. Then the following statements are equivalent:*

(1) *R is right zip;*
(2) *(HR,ω) is right zip.*

P r o o f. Suppose that (HR,ω) is a right zip ring. We show that R is a right zip ring. For this, consider $Y \subseteq R$ with $\mathrm{r.ann}_R(Y) = 0$. Since $Y \subseteq R$, so it is also subset of (HR,ω). Now, we prove $\mathrm{r.ann}_{(\mathrm{HR},\omega)}(Y) = 0$. Let $f \in \mathrm{r.ann}_{(\mathrm{HR},\omega)}(Y)$ with $\mathrm{supp}(f) = \{n \in \mathbb{N} \colon f(n) \neq 0\}$. It follows that $yf = 0$ for any $y \in Y$, $0 = (h_y f)(n) = \sum\limits_{k=0}^{n} C_k^n h_y(k)\omega^k(f(n-k)) = yf(n)$. Thus $f(n) \in \mathrm{r.ann}_R(Y) = 0$

which implies $f = 0$. It follows that $\mathrm{r.ann}_{(\mathrm{HR},\omega)}(Y) = 0$. Since (HR,ω) is a right zip ring, there exists a finite subset Y_0 of Y such that $\mathrm{r.ann}_{(\mathrm{HR},\omega)}(Y_0) = 0$. Thus $\mathrm{r.ann}_R(Y_0) = \mathrm{r.ann}_{(\mathrm{HR},\omega)}(Y_0) \cap R = 0$. Hence R is right zip.

Conversely, suppose that R is a right zip ring and a subset $U \subseteq (\mathrm{HR},\omega)$ with $\mathrm{r.ann}_{(\mathrm{HR},\omega)}(U) = 0$. We put $C_U = \bigcup_{f\in U} \{f(n)\colon f \in U \text{ and } n \in \mathrm{supp}(f)\}$ which is a nonempty subset of R. Now, we show $\mathrm{r.ann}_R(C_U) = 0$. Let $a \in \mathrm{r.ann}_R(C_U)$, $f(n)a = 0$ for any $n \in \mathrm{supp}(f)$. Which gives $0 = f(n)a = f(n)h_a(0) = f(n)\omega^n(h_a(0))$ since R is ω-compatible. It follows that $h_a \in \mathrm{r.ann}_{(\mathrm{HR},\omega)}(U)$. Thus $h_a = 0$ which implies that $a = 0$. Therefore $\mathrm{r.ann}_R(C_U) = 0$. Since R is right zip, there exists a nonempty finite subset X of C_U such that $\mathrm{r.ann}_R(X) = 0$. Consider $X = \{a_1, a_2, a_3, \ldots, a_k\}$, a subset of C_U. Now, for each a_i there exists f_i for each $i = 1, 2, 3\ldots, k$ such that $f_i(n) = a_i$ for some $n \in \mathbb{N}$. Let U_0 be a minimal subset of U such that $f_i \in U_0$ for each $a_i \in X$, which implies that $X \subseteq C_{U_0}$. Thus $\mathrm{r.ann}_R(C_{U_0}) \subseteq \mathrm{r.ann}_R(X) = 0$. Now, we prove that $\mathrm{r.ann}_{(\mathrm{HR},\omega)}(U_0) = 0$. Suppose that $\mathrm{r.ann}_{(\mathrm{HR},\omega)}(U_0) \neq 0$. Then there exists $0 \neq g \in \mathrm{r.ann}_{(\mathrm{HR},\omega)}(U_0)$ and $f_i \in U_0$. Thus $f_i g = 0$. Since R satisfies McCoy's theorem of skew Hurwitz serieswise, then there exists a nonzero $r \in R$ such that $f_i(n)r = 0$. Therefore $r \in \mathrm{r.ann}_R(C_{U_0}) = 0$. It follows that $r = 0$, which is a contradiction. Thus $g = 0$. Hence (HR,ω) is right zip. □

As a direct consequence of above theorem, we obtain the following corollaries.

Corollary 2.16 ([33], Theorem 3.6)**.** *Let R be a ring and ω be an endomorphism of R. If R is skew Hurwitz serieswise Armendariz and ω-compatible. Then the following statements are equivalent:*

(1) *R is right zip;*
(2) (HR,ω) *is right zip.*

Corollary 2.17 ([28], Corollary 2.15)**.** *Let R be a ring that is torsion-free as a Z-module and ω an endomorphism of R. If R is ω-rigid, then R is zip if and only if* (HR,ω) *is zip.*

The following result was proved by Ahmadi et al. (see [1], Theorem 2.19) for commutative ring.

Corollary 2.18 ([1], Theorem 2.19)**.** *Let R be an SHA-ring and ω is an endomorphism of R. Then R is right zip if and only if* (HR,ω) *is right zip.*

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