ENTIRE FUNCTION SHARING TWO POLYNOMIALS WITH ITS kth DERIVATIVE

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Abstract. We investigate the uniqueness problem of entire functions that share two polynomials with their kth derivatives and obtain some results which improve and generalize the recent result due to Lü and Yi (2011). Also, we exhibit some examples to show that the conditions of our results are the best possible.

Keywords: meromorphic function; derivative; Nevanlinna theory; uniqueness

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1. Introduction, definitions and main results

Let $\mathscr{M}(\mathbb{C})$ be the family of non-constant functions which are meromorphic in $\mathbb{C}$, whereas $\mathscr{E}(\mathbb{C})$ denotes the family of non-constant entire functions. On the other hand, we denote by $\mathscr{M}_T(\mathbb{C})$ and $\mathscr{E}_T(\mathbb{C})$ the families of transcendental meromorphic and entire functions, respectively. In the paper for $f \in \mathscr{M}(\mathbb{C})$ we shall use the standard notations of Nevanlinna's value distribution theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $S(r,f)$, ... (see, e.g., [2], [11]). Throughout the paper we denote by $\varrho(f)$ the order of $f \in \mathscr{M}(\mathbb{C})$. Let $f \in \mathscr{M}(\mathbb{C})$. A meromorphic function a is said to be a small function of f if $T(r,a) = S(r,f)$.

Let $k \in \mathbb{N}$ and $a \in \mathbb{C}$. We use $N_{(k}(r, 1/(f-a))$ to denote the counting function of 0-points of $f-a$ with multiplicity greater than or equal to k, whereas $\overline{N}_{(k}(r, 1/(f-a))$ is its reduced counting function.

Let $f, g \in \mathscr{M}(\mathbb{C})$ and Q be a polynomial or a finite complex number. If $g - Q = 0$ whenever $f - Q = 0$, we write $f = Q \Rightarrow g = Q$. If $f = Q \Rightarrow g = Q$ and $g = Q \Rightarrow f = Q$, we then write $f = Q \Leftrightarrow g = Q$ and we say that f and g share Q IM. If $f - Q$ and $g - Q$ have the same zeros with the same multiplicities, we write $f = Q \rightleftharpoons g = Q$ and we say that f and g share Q CM.

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Let $f \in \mathscr{E}(\mathbb{C})$. We know that f can be expressed by the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We denote

$$\mu(r,f) = \max_{\substack{n\in\mathbb{N}\\ |z|=r}}\{|a_n z^n|\} \quad \text{and} \quad \nu(r,f) = \sup\{n\colon\ |a_n| r^n = \mu(r,f)\}.$$

Clearly, for a polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$, $a_n \neq 0$ we have

$$\mu(r,P) = |a_n| r^n \quad \text{and} \quad \nu(r,P) = n$$

for all r sufficiently large. In the general case, $|a_n| r^n \leqslant \mu(r,f)$ for all $n \geqslant 0$ and $|a_n| r^n < \mu(r,f)$ for all $n > \nu(r,f)$.

Here it is enough to recall that (see [10]):

(1) $\mu(r,f)$ is strictly increasing for all r sufficiently large, continuous and tends to ∞ as $r \to \infty$;

(2) $\nu(r,f)$ is increasing, piecewise constant, right-continuous and also tends to ∞ as $r \to \infty$;

(3) $\nu(r,F) = O(\log r)$ if $\varrho(f) < \infty$.

Rubel and Yang (see [9]) considered the uniqueness of an entire function when it shares two values CM with its first derivative. In 1977, the authors proved the following well-known theorem.

Theorem A ([9]). *Let $a,b \in \mathbb{C}$ such that $b \neq a$ and let $f \in \mathscr{E}(\mathbb{C})$. If $f = a \rightleftharpoons f' = a$ and $f = b \rightleftharpoons f' = b$, then $f \equiv f'$.*

Mues and Steinmetz (see [8]) have generalized Theorem A in view of relaxing the sharing values from CM to IM and obtained the following result.

Theorem B ([8]). *Let $a,b \in \mathbb{C}$ such that $b \neq a$ and let $f \in \mathscr{E}(\mathbb{C})$. If $f = a \Leftrightarrow f' = a$ and $f = b \Leftrightarrow f' = b$, then $f \equiv f'$.*

Since then, shared value problems, especially the case of f and f' sharing two values, have undergone various extensions and improvements (see [11]).

In 2006, Li and Yi in [5] improved Theorem A with the idea of "partially" sharing values. In the following, we recall their result.

Theorem C ([5]). *Let $a,b \in \mathbb{C}$ such that $b \neq a, 0$ and let $f \in \mathscr{E}(\mathbb{C})$. If $f = a \Rightarrow f' = a$ and $f = b \rightleftharpoons f' = b$, then one of the following cases must occur:*

(1) $f \equiv f'$,

(2) $f(z) = c\exp((b/(b-a))z) + a$, *where* $c \in \mathbb{C} \setminus \{0\}$.

Since $b \neq a$, one may assume that $b \neq 0$ in Theorem A. So Theorem C improves Theorem A with the idea of "partially" sharing values.

In 2009, Lü et al. (see [6]) generalized Theorem C with the idea of sharing polynomials. They proved the following result.

Theorem D ([6]). *Let Q_1 and $Q_2\,(\not\equiv 0)$ be two distinct polynomials and let $f \in \mathscr{E}_T(\mathbb{C})$. If $f = Q_1 \Rightarrow f' = Q_1$ and $f = Q_2 \rightleftharpoons f' = Q_2$, then one of the following cases must occur:*

(1) $f \equiv f'$,

(2) $f(z) = Q_1(z) + A\exp(\lambda z)$ *and* $(\lambda - 1)Q_2 = \lambda Q_1 - Q_1'$, *where* $A, \lambda \in \mathbb{C} \setminus \{0\}$ *such that* $\lambda \neq 1$.

In 2011, Lü and Yi in [7] asked the following question.

Question A. What will happen if the first derivative f' in Theorem D is replaced by the general derivative $f^{(k)}$?

By considering the above question, Lü and Yi obtained the following result, which is an improvement of Theorem D.

Theorem E ([7]). *Let Q_1 and $Q_2\,(\not\equiv 0)$ be two distinct polynomials, $k \in \mathbb{N}$ and let $f \in \mathscr{E}_T(\mathbb{C})$. If*

(i) *all the zeros of $f - Q_1$ have multiplicity at least k,*

(ii) $f = Q_1 \Rightarrow f^{(k)} = Q_1$ *and* $f = Q_2 \rightleftharpoons f^{(k)} = Q_2$,

then one of the following cases must occur:

(1) $f \equiv f^{(k)}$,

(2) $f(z) = Q_1(z) + A\exp(\lambda z)$ *and* $(\mu - 1)Q_2 = \mu Q_1 - Q_1^{(k)}$, *where* $A, \lambda, \mu \in \mathbb{C} \setminus \{0\}$ *such that* $\lambda^k = \mu \neq 1$.

Remark 1.1. If a is a Picard exceptional value of f, then one can easily conclude that the zeros of $f - a$ have multiplicity ∞. Therefore, Theorem E holds even when $f - Q_1$ has no zeros. But it is to be noted that if $f - Q_1$ has a zero at z_0, say, then the multiplicity of z_0 must be at least k. On the other hand, if we add the condition that $f - Q_1$ has at least one zero in Theorem E, then conclusion (2) does not occur.

We now explain the notion of weighted sharing of values as introduced in [3].

Definition 1.1 ([3]). Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f and g share a with weight k.

We write f and g share (a,k) to mean that f and g share a with weight k. Also we note that f and g share a IM or CM if and only if f and g share $(a,0)$ or (a,∞), respectively.

After considering Theorem E, one may ask whether the conclusion of Theorem E remains valid if the hypothesis "$f = Q_2 \rightleftharpoons f^{(k)} = Q_2$" is replaced by "$f - Q_2$ and $f^{(k)} - Q_2$ share $(0,1)$"? In the paper, we give an affirmative answer to this question by proving the following result.

Theorem 1.1. *Let Q_1 and Q_2 $(\not\equiv 0)$ be two distinct polynomials, $k \in \mathbb{N}$ and let $f \in \mathscr{E}_T(\mathbb{C})$. Suppose*

(i) *all the zeros of $f - Q_1$ have multiplicity at least k,*

(ii) *$f = Q_1 \Rightarrow f^{(k)} = Q_1$ and $f - Q_2$ and $f^{(k)} - Q_2$ share $(0,1)$.*

Now one of the following cases must occur:

(1) *$f \equiv f^{(k)}$,*

(2) *if $\deg(Q_1) < \deg(Q_2)$, then $f(z) = Q_1(z) + P(z)\exp(\lambda z)$, where P is a nonzero polynomial, $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda^k = 1$ and*

$$\frac{Q_1^{(k)} - Q_1}{Q_1 - Q_2} = \sum_{i=1}^{k} \binom{k}{i} \lambda^{k-i} \frac{P^{(i)}}{P}, \tag{1.1}$$

(3) *if $\deg(Q_1) = \deg(Q_2)$ and $\lim\limits_{z\to\infty} Q_1(z)/Q_2(z) \neq 1$, then $f(z) = Q_1(z) + P(z)\exp(\lambda z)$, where P is a nonzero polynomial, $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda^k = \lim\limits_{z\to\infty} (Q_1^{(k)}(z) - Q_2(z))/(Q_1(z) - Q_2(z))$ and*

$$\frac{Q_1^{(k)} - Q_2}{Q_1 - Q_2} = \lambda^k + \sum_{i=1}^{k} \binom{k}{i} \lambda^{k-i} \frac{P^{(i)}}{P}, \tag{1.2}$$

(4) *if $\deg(Q_1) = \deg(Q_2)$ and $\lim\limits_{z\to\infty} Q_1(z)/Q_2(z) = 1$, then $f = Q_1 + P\exp(Q)$, where P is a nonzero polynomial and Q is a non-constant polynomial such that $k\deg(Q') = \deg(Q_1^{(k)} - Q_2) - \deg(Q_1 - Q_2)$ and*

$$\frac{Q_1^{(k)} - Q_2}{Q_1 - Q_2} = \frac{(P\exp(Q))^{(k)}}{P\exp(Q)}. \tag{1.3}$$

We now make the following observations on the conclusions of Theorem 1.1:

(1) If $\deg(Q_1) > \deg(Q_2)$, then we deduce immediately from Theorem 1.1 that $f \equiv f^{(k)}$. It is to be noted that $\deg(Q_1) = -\infty$ if $Q_1 \equiv 0$.

(2) If $Q_1 \equiv 0$, then since $Q_2 \not\equiv 0$, we must have $\deg(Q_1) < \deg(Q_2)$. Thereby from (1.1) we observe that P is a nonzero constant. In this case, we must have $f(z) = c\exp(\lambda z)$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$, which implies that $f \equiv f^{(k)}$. Consequently, from Theorem 1.1 we solely have $f \equiv f^{(k)}$.

(3) If we assume that $f-Q_1$ has no zeros and $\deg(Q_1)<\deg(Q_2)$, then P is a nonzero constant and so from (1.1) we deduce that $Q_1\equiv 0$. In this case, we also have $f(z)=c\exp(\lambda z)$, where $c\in\mathbb{C}\setminus\{0\}$ and $\lambda^k=1$ and so $f\equiv f^{(k)}$. Consequently, from Theorem 1.1 we solely have $f\equiv f^{(k)}$.

(4) If we assume that $f-Q_1$ has no zeros, $\deg(Q_1)=\deg(Q_2)$ and $\lim\limits_{z\to\infty} Q_1(z)/Q_2(z)\neq 1$, then P is a nonzero constant and so from (1.2) we deduce that $(\lambda^k-1)Q_2=\lambda^kQ_1-Q_1^{(k)}$, where $\lambda^k\neq 1$. Therefore, we have $f(z)=Q_1(z)+c\exp(\lambda z)$ and $(\lambda^k-1)Q_2=\lambda^kQ_1-Q_1^{(k)}$, where $c,\lambda\in\mathbb{C}\setminus\{0\}$ such that $\lambda^k\neq 1$.

(5) If we assume that $f-Q_1$ has no zeros, $\deg(Q_1)=\deg(Q_2)$ and $\lim\limits_{z\to\infty} Q_1(z)/Q_2(z)=1$, then P is a nonzero constant and so from (1.3) we deduce that $(Q_1^{(k)}-Q_2)/(Q_1-Q_2)=(\exp(Q))^{(k)}/\exp(Q)$.

(6) If $Q_1,Q_2\in\mathbb{C}\setminus\{0\}$, then $(Q_1^{(k)}-Q_2)/(Q_1-Q_2)\in\mathbb{C}\setminus\{0\}$ and $\lim\limits_{z\to\infty} Q_1(z)/Q_2(z)\neq 1$. Now from (1.2) we conclude that $\lambda^k=(Q_1^{(k)}-Q_2)/(Q_1-Q_2)$ and P is a nonzero constant. Therefore we have $f(z)=Q_1(z)+c\exp(\lambda z)$ and $(\lambda^k-1)Q_2=\lambda^kQ_1$, where $c,\lambda\in\mathbb{C}\setminus\{0\}$ such that $\lambda^k\neq 1$.

(7) If $Q_2\in\mathbb{C}\setminus\{0\}$, then from Theorem 1.1 one can easily deduce that $Q_1\in\mathbb{C}$.

(8) If $f-Q_1$ has infinitely many zeros, then from Theorem 1.1 we solely have $f\equiv f^{(k)}$.

Remark 1.2. The following example shows that conclusion (2) in Theorem 1.1 cannot be deleted.

Example 1.1. Let $f(z)=1-(1/e-1)(z+1)+z\exp(z)$, $k=1$, $Q_1(z)=1-(1/e-1)(z+1)$ and $Q_2(z)=1-(1/e-1)(z+1)-(1/e-1)z^2+z$. Then $\deg(Q_1)<\deg(Q_2)$ and $f(z)-Q_1(z)=z\exp(z)$ has only one zero at $z=0$. It is easy to derive that $z=0$ is also a zero of $f'-Q_1$, which implies that $f=Q_1\Rightarrow f'=Q_1$. We also have

$$f(z)-Q_2(z)=z(\exp(z)+(1/e-1)z-1),$$
$$f'(z)-Q_2(z)=(z+1)(\exp(z)+(1/e-1)z-1).$$

Clearly, $f-Q_2$ and $f'-Q_2$ share 0 CM except for the zero of $z(z+1)$ and $(Q_1'-Q_2)/(Q_1-Q_2)=Q'$, where $P(z)=z$ and $Q(z)=z$.

Remark 1.3. The following example shows that conclusion (3) in Theorem 1.1 cannot be deleted.

Example 1.2. Let $f(z)=\frac{1}{2}\exp(\frac{1}{2}z)+z$, $k=1$, $Q_1(z)=z$ and $Q_2(z)=2-z$. Clearly, $\deg(Q_1)=\deg(Q_2)$ and $\lim\limits_{z\to\infty} Q_1(z)/Q_2(z)=-1$. Also we see that $f(z)-Q_1(z)=\frac{1}{2}\exp(\frac{1}{2}z)$ has no zero and so $f=Q_1\Rightarrow f'=Q_1$. Note that

$$f(z)-Q_2(z)=\frac{1}{2}\Big(\exp\Big(\frac{1}{2}z\Big)+4z-4\Big) \text{ and } f'(z)-Q_2(z)=\frac{1}{4}\Big(\exp\Big(\frac{1}{2}z\Big)+4z-4\Big).$$

Clearly, $f - Q_2$ and $f' - Q_2$ share 0 CM and $(Q_1' - Q_2)/(Q_1 - Q_2) = Q'$, where $P(z) = \frac{1}{2}$ and $Q(z) = \frac{1}{2}z$.

R e m a r k 1.4. The following example shows that conclusion (4) in Theorem 1.1 cannot be deleted.

E x a m p l e 1.3. Let $f(z) = \exp(\frac{1}{2}(z-1)^2) + z^2$, $k = 1$, $Q_1 = z^2$ and $Q_2(z) = z^2 + z$. Clearly, $\deg(Q_1) = \deg(Q_2)$ and $\lim\limits_{z\to\infty} Q_1(z)/Q_2(z) = 1$. Also we see that $f(z) - Q_1(z) = \exp(\frac{1}{2}(z-1)^2)$ has no zero and so $f = Q_1 \Rightarrow f' = Q_1$. Note that

$$f(z) - Q_2(z) = \exp\frac{(z-1)^2}{2} - z \quad \text{and} \quad f'(z) - Q_2(z) = (z-1)\Big(\exp\frac{(z-1)^2}{2} - z\Big).$$

Clearly, $f - Q_2$ and $f' - Q_2$ share 0 CM except for $z = 1$ and $(Q_1' - Q_2)/(Q_1 - Q_2) = Q'$, where $P(z) = 1$ and $Q(z) = \frac{1}{2}(z-1)^2$.

R e m a r k 1.5. The following examples show that conditions "$f = Q_1 \Rightarrow f^{(k)} = Q_1$" and "all the zeros of $f - Q_1$ have multiplicity at least k" in Theorem 1.1 are sharp.

E x a m p l e 1.4. Let $f(z) = \frac{1}{4}\exp(2z) + \frac{3}{4}z$, $k = 2$, $Q_1(z) = 1$ and $Q_2(z) = z$. Note that $f(z) - Q_1(z) = \frac{1}{4}\exp(2z) + \frac{3}{4}z - 1$, $f''(z) - Q_1(z) = \exp(2z) - 1$ and so $f - Q_1$ has only simple zeros and $f = Q_1 \not\Rightarrow f'' = Q_1$. On the other hand, we have

$$f(z) - Q_2(z) = \frac{1}{4}(\exp(2z) - z) \quad \text{and} \quad f''(z) - Q_2(z) = \exp(2z) - z$$

and so $f - Q_2$ and $f'' - Q_2$ share 0 CM, but f does not satisfy any case of Theorem 1.1.

E x a m p l e 1.5. Let $f(z) = z + 2\exp(\frac{1}{2}z)$, $k = 1$, $Q_1(z) = 1$ and $Q_2(z) = 2 - z$. Then $f - Q_1$ has simple zeros and $f' - Q_1$ has no zeros and so $f = Q_1 \not\Rightarrow f' = Q_1$. On the other hand, we have

$$f(z) - Q_2(z) = 2\Big(\exp\Big(\frac{1}{2}z\Big) + z - 1\Big) \quad \text{and} \quad f'(z) - Q_1(z) = \exp\Big(\frac{1}{2}z\Big) + z - 1$$

and so $f - Q_2$ and $f' - Q_2$ share 0 CM, f does not satisfy any case of Theorem 1.1.

R e m a r k 1.6. The following example shows that Theorem 1.1 does not hold when $f \in \mathscr{M}_T(\mathbb{C})$.

E x a m p l e 1.6. Let $f(z) = z/(1 - \exp(-z))$, $k = 1$, $Q_1(z) = 0$ and $Q_2(z) = 1$. Clearly, $f - Q_1$ has no zero and so $f = Q_1 \Rightarrow f' = Q_1$. Note that

$$f(z) - Q_2(z) = \frac{z - 1 + \exp(-z)}{1 - \exp(-z)} \quad \text{and} \quad f'(z) - Q_2(z) = -\exp(-z)\frac{z - 1 + \exp(-z)}{(1 - \exp(-z))^2}.$$

Clearly, $f - Q_2$ and $f' - Q_2$ share 0 CM, but f does not satisfy any case of Theorem 1.1.

If one of the polynomials Q_1 and Q_2 is a constant, then we immediately obtain the following result.

Corollary 1.1. *Let Q_1 and $Q_2\,(\not\equiv 0)$ be two distinct polynomials such that one of them is a constant, $k\in\mathbb{N}$ and let $f\in\mathscr{E}_T(\mathbb{C})$. If*

(i) *all the zeros of $f-Q_1$ have multiplicity at least k,*

(ii) *$f=Q_1\Rightarrow f^{(k)}=Q_1$ and $f-Q_2$ and $f^{(k)}-Q_2$ share $(0,1)$,*

then one of the following cases must occur:

(1) *$f\equiv f^{(k)}$,*

(2) *$f(z)=Q_1+A\exp(\lambda z)$, where $A,\lambda\in\mathbb{C}\setminus\{0\}$ such that $\lambda^k=Q_2/(Q_2-Q_1)$ and both Q_1 and Q_2 are constants.*

2. Auxiliary lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 2.1 ([7], Theorem 1.3)**.** *Let $f\in\mathscr{M}(\mathbb{C})$ such that $N(r,f)=O(\log r)$. Suppose that $\alpha=Q_1\exp(Q)$ and $\beta=Q_2\exp(Q)$, where Q_1, $Q_2\,(\not\equiv Q_1)$ and Q are three polynomials. If for $k\in\mathbb{N}$, all the zeros of $f-\alpha$ have multiplicities at least k, $f=\alpha\Rightarrow f^{(k)}=\alpha$ and $f=\beta\Leftrightarrow f^{(k)}=\beta$, then $\varrho(f)<\infty$.*

Lemma 2.2 ([4], Corollary 2.3.4)**.** *Let $f\in\mathscr{M}_T(\mathbb{C})$ and $k\in\mathbb{N}$. If $\varrho(f)<\infty$, then*

$$m\Big(r,\frac{f^{(k)}}{f}\Big)=O(\log r)\quad\text{as } r\to\infty.$$

Lemma 2.3 ([2], Lemma 3.5)**.** *Let F be meromorphic in a domain D and $n\in\mathbb{N}$. Then*

$$\frac{F^{(n)}}{F}=f^n+\frac{n(n-1)}{2}f^{n-2}f'+a_nf^{n-3}f''+b_nf^{n-4}(f')^2+P_{n-3}(f),$$

where $f=F'/F$, $a_n=\frac{1}{6}n(n-1)(n-2)$, $b_n=\frac{1}{8}n(n-1)(n-2)(n-3)$ and $P_{n-3}(f)$ is a differential polynomial with constant coefficients, which vanishes identically for $n\leqslant 3$ and has degree $n-3$ when $n>3$.

Lemma 2.4 ([4], Theorem 3.2)**.** *Let $f\in\mathscr{E}_T(\mathbb{C})$. Then there exists a set $E\subset(1,\infty)$ with finite logarithmic measure; we choose z satisfying $|z|=r\notin[0,1]\cup E$ and $|f(z)|=M(r,f)$, such that*

$$\frac{f^{(j)}(z)}{f(z)}=\Big(\frac{\nu(r,f)}{z}\Big)^j(1+o(1))\quad\text{for } j\in\mathbb{N}.$$

3. Proofs of the main results

Proof of Theorem 1.1. Set

$$\Phi = \frac{L(f)(f-f^{(k)})}{(f-Q_1)(f-Q_2)}, \tag{3.1}$$

where $L(f) = (Q_1'-Q_2')(f-Q_1)-(Q_1-Q_2)(f'-Q_1') = (Q_1'-Q_2')(f-Q_2)-(Q_1-Q_2)(f'-Q_2')$. If possible, suppose $L(f)\equiv 0$. Then we have

$$\frac{f'-Q_1'}{f-Q_1} \equiv \frac{Q_1'-Q_2'}{Q_1-Q_2}.$$

On integration we have $f-Q_1 \equiv d(Q_1-Q_2)$, where $d\in\mathbb{C}$, i.e., $f\equiv Q_1+d(Q_1-Q_2)$. This shows that f is a polynomial, which is impossible as $f\in\mathscr{E}_T(\mathbb{C})$. Hence $L(f)\not\equiv 0$.

We now divide the proof considering the following two possible cases.

Case 1. Suppose $\Phi\not\equiv 0$. Then $f\not\equiv f^{(k)}$. Now from (3.1), we have

$$\begin{aligned}\Phi &= \frac{1}{Q_1-Q_2}\Big(Q_1\frac{L(f)}{f-Q_1}-Q_2\frac{L(f)}{f-Q_2}\Big)\Big(1-\frac{f^{(k)}}{f}\Big)\\ &=\Big(\frac{f'-Q_2'}{f-Q_2}Q_2-\frac{f'-Q_1'}{f-Q_1}Q_1+Q_1'-Q_2'\Big)\Big(1-\frac{f^{(k)}}{f}\Big).\end{aligned} \tag{3.2}$$

Therefore, applying Lemma 2.1, we deduce that $\varrho(f)<\infty$. Consequently, from Lemma 2.2 and (3.2), we conclude that $m(r,\Phi)=O(\log r)$ as $r\to\infty$.

Next we want to prove that Φ has no poles. For this let z_0 be a zero of $f-Q_1$ of multiplicity p_0. Since $f=Q_1\Rightarrow f^{(k)}=Q_1$, it follows that z_0 must be a zero of $f^{(k)}-Q_1$ of multiplicity q_0. Clearly, z_0 is a zero of $L(f)$ and $f-f^{(k)}$ of multiplicities p_0-1 and $t_0=\min\{p_0,q_0\}$ ($\geqslant 1$), respectively, and so from (3.1) we have

$$\Phi(z)=O((z-z_0)^{t_0-1}). \tag{3.3}$$

This shows that Φ is holomorphic at z_0.

Let z_1 be a zero of $f-Q_2$ of multiplicity p_1. Since $f-Q_2$ and $f^{(k)}-Q_2$ share $(0,1)$, it follows that z_1 is also a zero of $f^{(k)}-Q_2$ of multiplicity q_1. Then in some neighbourhood of z_1 we get by Taylor's expansion

$$\begin{aligned}f(z)-Q_2(z) &= a_{p_1}(z-z_1)^{p_1}+a_{p_1+1}(z-z_1)^{p_1+1}+\dots, \quad a_{p_1}\neq 0,\\ f^{(k)}(z)-Q_2(z) &= b_{q_1}(z-z_1)^{q_1}+b_{q_1+1}(z-z_1)^{q_1+1}+\dots, \quad b_{q_1}\neq 0.\end{aligned}$$

Clearly,

$$f'(z)-Q_2'(z)=p_1a_{p_1}(z-z_1)^{p_1-1}+(p_1+1)a_{p_1+1}(z-z_1)^{p_1}+\dots$$

Note that

$$f(z) - f^{(k)}(z) = \begin{cases} a_{p_1}(z-z_1)^{p_1} + \dots & \text{if } p_1 < q_1, \\ -b_{q_1}(z-z_1)^{q_1} - \dots & \text{if } p_1 > q_1, \\ (a_{p_1} - b_{p_1})(z-z_1)^{p_1} + \dots & \text{if } p_1 = q_1. \end{cases}$$

Clearly, from (3.1) we get

$$\Phi(z) = O((z-z_1)^{t_1-1}), \tag{3.4}$$

where $t_1 \geqslant \min\{p_1, q_1\} \geqslant 1$. Now from (3.4) it follows that Φ is holomorphic at z_1. Consequently, Φ has no poles, i.e., $N(r,\Phi) = 0$ and so $T(r,\Phi) = m(r,\Phi) = O(\log r)$ as $r \to \infty$. This means that Φ is a polynomial.

If z_1 is a zero of $f - Q_1$ and $f^{(k)} - Q_2$ of multiplicities $p_1\,(\geqslant 2)$ and $q_1\,(\geqslant 2)$, respectively, then from (3.1) and (3.4) we see that z_2 is a zero of Φ. Since $T(r,\Phi) = O(\log r)$, it follows that

$$\overline{N}(r, Q_2; f \mid\geqslant 2) = O(\log r) \text{ and } \overline{N}(r, Q_2; f^{(k)} \mid\geqslant 2) = O(\log r) \quad \text{as } r \to \infty. \tag{3.5}$$

Consequently, $f - Q_2$ and $f^{(k)} - Q_2$ have finitely many multiple zeros. Let

$$\alpha = \frac{f^{(k)} - Q_2}{f - Q_2}.$$

Clearly, $\alpha \not\equiv 0$. Since $f - Q_2$ and $f^{(k)} - Q_2$ share $(0,1)$ and $f - Q_2$, $f^{(k)} - Q_2$ have finitely many multiple zeros, we deduce that α has finitely many zeros and poles. Therefore, by Hadamard's factorization theorem, we can assume that $\alpha = \beta \exp(\gamma)$, where $\beta\,(\not\equiv 0)$ is a rational function and γ is a polynomial. Hence

$$\frac{f^{(k)} - Q_2}{f - Q_2} = \beta \exp(\gamma). \tag{3.6}$$

Now we want to prove that γ is a constant. If not, suppose $\deg(\gamma) \geqslant 1$. Then from (3.6) we have

$$\beta \exp(\gamma) = \frac{f^{(k)}/f - Q_2/f}{1 - Q_2/f}, \quad \text{i.e.,} \ \gamma = \log \frac{1}{\beta} \frac{f^{(k)}/f - Q_2/f}{1 - Q_2/f},$$

where $\log h$ is the principle branch of the logarithm. Therefore, we have

$$|\gamma(z)| = \left| \log \frac{1}{\beta(z)} \frac{f^{(k)}(z)/f(z) - Q_2(z)/f(z)}{1 - Q_2(z)/f(z)} \right|. \tag{3.7}$$

Since $f \in \mathscr{E}_T(\mathbb{C})$, it follows that $M(r,f) \to \infty$ as $r \to \infty$, where $M(r,f) = \max_{|z|=r} |f(z)|$. Again we let

$$M(r,f) = |f(z_r)|, \quad \text{where } z_r = r\mathrm{e}^{\mathrm{i}\theta} \text{ and } \theta \in [0, 2\pi). \tag{3.8}$$

Now from (3.8) and Lemma 2.4, there exists a subset $E \subset (1, \infty)$ with finite logarithmic measure such that for a point $z_r = r\mathrm{e}^{\mathrm{i}\theta}$ ($\theta \in [0, 2\pi)$) satisfying $|z_r| = r \notin E$ and $M(r,f) = |f(z_r)|$, we have

$$\frac{f^{(k)}(z)}{f(z)} = \Big(\frac{\nu(r,f)}{z}\Big)^k (1 + o(1)) \quad \text{as } r \to \infty. \tag{3.9}$$

Since $f \in \mathscr{E}_T(\mathbb{C})$ and $M(r,f)$ increases faster than $M(r,Q_2)$, it follows from (3.8) that

$$\lim_{r\to\infty} \Big|\frac{Q_2(z_r)}{f(z_r)}\Big| \leqslant \lim_{r\to\infty} \frac{M(r, Q_2(z_r))}{M(r, f(z_r))} = 0. \tag{3.10}$$

Also, we know that if $\varrho(f) < \infty$, then

$$\log \nu(r,f) = O(\log r). \tag{3.11}$$

Therefore, from (3.7)–(3.11) we conclude that $|\gamma(z_r)| = O(\log r)$ for $|z_r| = r \notin E$, which is impossible. Hence, γ is a constant. Without loss of generality we assume that

$$f^{(k)} - Q_2 \equiv \beta(f - Q_2), \quad \text{i.e., } f^{(k)} \equiv \beta f + (1-\beta)Q_2. \tag{3.12}$$

Since $f \not\equiv f^{(k)}$, from (3.12) we conclude that $\beta \not\equiv 1$.

Next we want to prove that $f - Q_1$ has only finitely many zeros. Let z_0 be a zero of $f - Q_1$ such that $\beta(z_0) \neq 0, \infty$. Then $f(z_0) = Q_1(z_0)$. Since $f = Q_1 \Rightarrow f^{(k)} = Q_1$, it follows that $f^{(k)}(z_0) = Q_1(z_0)$. Putting z_0 into (3.12), we get $Q_1(z_0) = Q_2(z_0)$. Since $Q_1 \not\equiv Q_2$, it follows that z_0 is a zero of $Q_1 - Q_2$. Therefore

$$\overline{N}(r, 1/(f-Q_1)) \leqslant N(r, 1/(Q_1 - Q_2)) + N(r, 1/\beta) + N(r, \beta) = O(\log r),$$

i.e., $\overline{N}(r, 1/(f-Q_1)) = O(\log r)$ as $r \to \infty$. This shows that $f - Q_1$ has only finitely many zeros.

We now consider the following two possible sub-cases.

Sub-case 1.1. Suppose $\varrho(f) < 1$. Then clearly, $f \not\equiv Q_1$ and $\varrho(f - Q_1) < 1$. Since $f - Q_1$ is an entire function having finitely many zeros, by Hadamard's factorization theorem we may assume that $f - Q_1 = \mathcal{P}$, where $\mathcal{P}$ is a nonzero polynomial. Therefore, $f = Q_1 + \mathcal{P}$, which is a contradiction as $f \in \mathscr{E}_T(\mathbb{C})$.

Sub-case 1.2. Suppose $\varrho(f) \geqslant 1$. Since $f - Q_1$ has only finitely many zeros, by Hadamard's factorization theorem, we can express $f = Q_1 + P\exp(Q)$, where P is a nonzero polynomial and Q is a non-constant polynomial. Clearly, $\deg(Q) = \varrho(f) \geqslant 1$. Now differentiating k-times, we get

$$f^{(k)} = Q_1^{(k)} + (P(Q')^k + P_1)\exp(Q),$$

where P_1 is a polynomial such that $\deg(P_1) < \deg(P(Q')^k)$. Putting them into (3.12), we get

$$(P(Q')^k + P_1 - \beta P)\exp(Q) = \beta Q_1 - Q_1^{(k)} + (1-\beta)Q_2,$$

which implies that

$$P(Q')^k + P_1 \equiv \beta P, \tag{3.13}$$

$$Q_1^{(k)} \equiv \beta Q_1 + (1-\beta)Q_2. \tag{3.14}$$

We now consider the following three possible sub-cases.

Sub-case 1.2.1. Suppose $\deg(Q_1) > \deg(Q_2)$. Then clearly, $\deg(Q_1 - Q_2) > \deg(Q_1^{(k)} - Q_2)$. Now from (3.13) and (3.14) we deduce that

$$\frac{Q_1^{(k)} - Q_2}{Q_1 - Q_2} = \frac{(P\exp(Q))^{(k)}}{P\exp(Q)}. \tag{3.15}$$

Let $F = (P\exp(Q))'/(P\exp(Q))$. Then $F = Q' + P'/P$ and so by Lemma 2.3, we have

$$\frac{(P\exp(Q))^{(k)}}{P\exp(Q)} = \Big(Q' + \frac{P'}{P}\Big)^k + P_{k-1}\Big(Q' + \frac{P'}{P}\Big), \tag{3.16}$$

where $P_{k-1}(Q' + P'/P)$ is a differential polynomial with constant coefficients of degree at most $k-1$ in $Q' + P'/P$.

Now from (3.15) and (3.16) we obtain

$$\frac{Q_1^{(k)} - Q_2}{Q_1 - Q_2} = \Big(Q' + \frac{P'}{P}\Big)^k + P_{k-1}\Big(Q' + \frac{P'}{P}\Big). \tag{3.17}$$

Letting $|z| \to \infty$, from (3.17) we see that $k\deg(Q') = 0$, i.e., $Q' \in \mathbb{C}$. We claim that $Q' \equiv 0$. If not, suppose that $Q' = c \in \mathbb{C} \setminus \{0\}$. Note that

$$\frac{(P\exp(Q))^{(k)}}{P\exp(Q)} = c^k + \sum_{i=1}^{k} \binom{k}{i} c^{k-i} \frac{P^{(i)}}{P} \tag{3.18}$$

and so from (3.15) we have

$$\frac{Q_1^{(k)} - Q_2}{Q_1 - Q_2} = c^k + \sum_{i=1}^{k} \binom{k}{i} c^{k-i} \frac{P^{(i)}}{P}. \tag{3.19}$$

Letting $|z| \to \infty$, we arrive at a contradiction from (3.19). Hence $Q' \equiv 0$ and so $Q \in \mathbb{C}$. Thus, it follows from $f = Q_1 + P\exp(Q)$ that f is a polynomial, a contradiction.

Sub-case 1.2.2. Suppose $\deg(Q_1) < \deg(Q_2)$. Then clearly $\deg(Q_1 - Q_2) = \deg(Q_1^{(k)} - Q_2)$ and so from (3.17) we have

$$1 + \frac{Q_1^{(k)} - Q_1}{Q_1 - Q_2} = \Big(Q' + \frac{P'}{P}\Big)^k + P_{k-1}\Big(Q' + \frac{P'}{P}\Big). \tag{3.20}$$

Note that $\deg(Q_1^{(k)} - Q_1) < \deg(Q_1 - Q_2)$. Letting $|z| \to \infty$, from (3.20) we see that $k\deg(Q') = 0$, i.e., $Q' \in \mathbb{C}$. Since $f \in \mathscr{E}_T(\mathbb{C})$, it follows that $Q' \in \mathbb{C} \setminus \{0\}$. Again from (3.16) and (3.19), we have

$$1 + \frac{Q_1^{(k)} - Q_1}{Q_1 - Q_2} = (Q')^k + \sum_{i=1}^{k} \binom{k}{i} (Q')^{k-i} \frac{P^{(i)}}{P}. \tag{3.21}$$

This shows that $(Q')^k = 1$ and

$$\frac{Q_1^{(k)} - Q_1}{Q_1 - Q_2} = \sum_{i=1}^{k} \binom{k}{i} (Q')^{k-i} \frac{P^{(i)}}{P},$$

which implies that $\deg(P) \geqslant 1$ if $Q_1 \not\equiv 0$. Let $Q' = \lambda$. Clearly, $\lambda^k = 1$. In this case, we have $f(z) = Q_1(z) + P(z)\exp(\lambda z)$, where P is a nonzero polynomial, $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda^k = 1$ and

$$\frac{Q_1^{(k)} - Q_1}{Q_1 - Q_2} = \sum_{i=1}^{k} \binom{k}{i} \lambda^{k-i} \frac{P^{(i)}}{P}.$$

Sub-case 1.2.3. Suppose $\deg(Q_1) = \deg(Q_2)$. In this sub-case, we consider two possibilities: (1) $\lim\limits_{z\to\infty} (Q_1(z)/Q_2(z)) \neq 1$ and (2) $\lim\limits_{z\to\infty} (Q_1(z)/Q_2(z)) = 1$.

First we assume that $\lim\limits_{z\to\infty} (Q_1(z)/Q_2(z)) \neq 1$. Then clearly, $\deg(Q_1 - Q_2) = \deg(Q_1^{(k)} - Q_2)$. Let

$$\lim_{z\to\infty} \frac{Q_1^{(k)}(z) - Q_2(z)}{Q_1(z) - Q_2(z)} = \mu.$$

Then from (3.17) we have

$$\mu + \frac{Q_1^{(k)} - \mu Q_1 + (\mu - 1)Q_2}{Q_1 - Q_2} = \Big(Q' + \frac{P'}{P}\Big)^k + P_{k-1}\Big(Q' + \frac{P'}{P}\Big), \tag{3.22}$$

where $\deg(Q_1^{(k)} - \mu Q_1 + (\mu - 1)Q_2) < \deg(Q_1 - Q_2)$.

Now letting $|z| \to \infty$, from (3.22) we conclude that $(Q')^k = \mu$. Let $Q' = \lambda$. Clearly, $\lambda^k = \mu$. In this case, we have $f(z) = Q_1(z) + P(z)\exp(\lambda z)$, where P is a nonzero polynomial, $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda^k = \lim\limits_{z\to\infty} (Q_1^{(k)}(z) - Q_2(z))/(Q_1(z) - Q_2(z))$ and

$$\frac{Q_1^{(k)} - Q_2}{Q_1 - Q_2} = \lambda^k + \sum_{i=1}^{k} \binom{k}{i} \lambda^{k-i} \frac{P^{(i)}}{P}.$$

Next we assume that $\lim\limits_{z\to\infty} Q_1(z)/Q_2(z) = 1$. Then clearly, $\deg(Q_1 - Q_2) < \deg(Q_1^{(k)} - Q_2)$. Now letting $|z| \to \infty$, from (3.17) we conclude that $k\deg(Q') = \deg(Q_1^{(k)} - Q_2) - \deg(Q_1 - Q_2)$ and so $\deg(Q) \geqslant 2$. In this case, we have $f = Q_1 + P\exp(Q)$, where P is a nonzero polynomial and Q is a non-constant polynomial such that $k\deg(Q') = \deg(Q_1^{(k)} - Q_2) - \deg(Q_1 - Q_2)$ and

$$\frac{Q_1^{(k)} - Q_2}{Q_1 - Q_2} = \frac{(P\exp(Q))^{(k)}}{P\exp(Q)},$$

which is immediately obtained from (3.13) and (3.14).

Case 2. Suppose $\Phi \equiv 0$. Since $L(f) \not\equiv 0$, it follows that $f \equiv f^{(k)}$.

This completes the proof. □

P r o o f of Corollary 1.1. We prove Corollary 1.1 with the line of proof of Theorem 1.1 with some necessary modifications. Here we use the same auxiliary function Φ given by (3.1). Note that if $\Phi \equiv 0$, then since $L(f) \not\equiv 0$, we have $f \equiv f^{(k)}$. Next we suppose that $\Phi \not\equiv 0$ and so $f \not\equiv f^{(k)}$.

Now we divide the proof considering the following two possible cases.

Case 1. Suppose $Q_1 \notin \mathbb{C}$. Then by the given condition, we must have $Q_2 \in \mathbb{C}\setminus\{0\}$ and so $\deg(Q_1) > \deg(Q_2)$. Then by Theorem 1.1, we have $f \equiv f^{(k)}$, which is a contradiction.

Case 2. Suppose $Q_1 \in \mathbb{C}$. Then one of the conclusions of Theorem 1.1 must hold except conclusion (1).

We now consider the following two sub-cases.

Sub-case 2.1. Suppose $Q_2 \notin \mathbb{C}$. Clearly, $\deg(Q_1) < \deg(Q_2)$ and so conclusion (2) of Theorem 1.1 must hold. Therefore, we have $f(z) = Q_1(z) + P(z)\exp(\lambda z)$, where P is a nonzero polynomial, $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda^k = 1$ and

$$\frac{Q_1^{(k)} - Q_1}{Q_1 - Q_2} = \sum_{i=1}^{k} \binom{k}{i} \lambda^{k-i} \frac{P^{(i)}}{P}. \tag{3.23}$$

First we suppose that $Q_1 \equiv 0$. Then from (3.23) we deduce that P is a nonzero constant. Consequently, we have $f(z) = A\exp(\lambda z)$, where $A \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$, which implies that $f \equiv f^{(k)}$, a contradiction. Next we suppose that $Q_1 \in \mathbb{C} \setminus \{0\}$. Therefore, from (3.23) we have

$$\frac{Q_1}{Q_2 - Q_1} = \sum_{i=1}^{k} \binom{k}{i} \lambda^{k-i} \frac{P^{(i)}}{P},$$

i.e.,

$$\frac{PQ_1}{Q_2 - Q_1} = \sum_{i=1}^{k} \binom{k}{i} \lambda^{k-i} P^{(i)}, \tag{3.24}$$

which implies that $\deg(Q_2) = 1$ and $\deg(P) \geqslant 1$.

We now want to prove that P has only one zero. If possible, suppose that P has at least two zeros. For the sake of simplicity we assume that $P(z) = a(z-z_1)^m(z-z_2)^n$, where $a \in \mathbb{C} \setminus \{0\}$.

Since $f(z) = Q_1 + P(z)\exp(\lambda z)$ and all the zeros of $f - Q_1$ have multiplicity at least k, it follows that $m \geqslant k$ and $n \geqslant k$. Also from (3.24), without loss of generality we may assume that $Q_1/(Q_2(z) - Q_1) = (z - z_1)/b$, where $b \in \mathbb{C} \setminus \{0\}$. Note that

$$P^{(i)}(z) = a(z - z_1)^{m-i}(z - z_2)^{n-i}\varphi_i(z),$$

where φ_i is a polynomial such that $\deg(\varphi_i) = i$ and $\varphi_i(z_j) \neq 0$ for $i = 1, 2, \ldots, k$ and $j = 1, 2$. Now from (3.24) we see that

$$b(z - z_1)^{m-1}(z - z_2)^n = \sum_{i=1}^{k} \binom{k}{i} \lambda^{k-i}(z - z_1)^{m-i}(z - z_2)^{n-i}\varphi_i(z). \tag{3.25}$$

Cancelling the term $(z - z_2)^{n-k}$ from both sides of (3.25), we get

$$b(z - z_1)^{m-1}(z - z_2)^k = \sum_{i=1}^{k} \binom{k}{i} \lambda^{k-i}(z - z_1)^{m-i}(z - z_2)^{k-i}\varphi_i(z),$$

which again implies that $\varphi_k(z_2) = 0$, a contradiction. Therefore, P just has one zero, say z_1 and we may assume that $P(z) = a(z - z_1)^m$, where $m \geqslant k$. Again from (3.24) we see that

$$b(z - z_1)^{m-1} = \sum_{i=1}^{k} \binom{k}{i} \lambda^{k-i}(z - z_1)^{m-i}\psi_i(z), \tag{3.26}$$

where ψ_i is a polynomial such that $\deg(\psi_i) = i$ and $\psi_i(z_1) \neq 0$ for $i = 1, 2, \ldots, k$.

If $k \geqslant 2$, then by simple calculation, one can easily arrive at a contradiction from (3.26). Hence, the only possibility is that $k = 1$. Therefore, $P(z) = a(z - z_1)^m$, where $m \geqslant 1$.

Consequently, from (3.24) we have $Q_1/(Q_2 - Q_1) = P'/P = m/(z - z_1)$, i.e., $Q_2(z) = Q_1(z - z_1)/m + Q_1$. On the other hand, we have $f(z) = Q_1 + a(z - z_1)^m \exp(z)$. Also, since $k = 1$, by the given condition, we have $f = Q_1 \Rightarrow f' = Q_1$. Note that $f'(z) = a(z - z_1)^{m-1}(z - z_1 + m)\exp(z)$.

Since $f(z_1) = Q_1$, it follows that $f'(z_1) = Q_1$. Therefore, we conclude that $m = 1$ and $a\exp(z_1) = Q_1$. Also we see that

$$f(z) - Q_2(z) = (z - z_1)(a\exp(z) - Q_1) \text{ and } f'(z) - Q_2(z) = (z - z_1 + 1)(a\exp(\lambda z) - Q_1).$$

Since $f - Q_2$ and $f' - Q_2$ share $(0, 1)$, it follows that $z = z_1 - 1$ must be a zero of $f(z) - Q_2(z)$ and so

$$a\exp(z_1 - 1) = Q_1,$$
$$\text{i.e., } a\exp(z_1)\exp(-1) = Q_1, \text{ i.e., } Q_1\exp(-1) = Q_1, \text{ i.e., } \exp(-1) = 1,$$

which is impossible.

Sub-case 2.2. Suppose $Q_2 \in \mathbb{C} \setminus \{0\}$. Then clearly, we have

$$\frac{Q_1^{(k)} - Q_2}{Q_1 - Q_2} = \frac{-Q_2}{Q_1 - Q_2} \in \mathbb{C} \setminus \{0\}.$$

If $Q_1 \equiv 0$, then obviously $\deg(Q_1) < \deg(Q_2)$. Therefore, from (3.23) we conclude that P is a nonzero constant. In this case, we must have $f \equiv f^{(k)}$, which is a contradiction.

If $Q_1 \in \mathbb{C} \setminus \{0\}$, then obviously $\deg(Q_1) = \deg(Q_2)$. Since $Q_1 \not\equiv Q_2$, it follows that $\lim_{z\to\infty}(Q_1(z)/Q_2(z)) \neq 1$. Then conclusion (3) of Theorem 1.1 must occur. Consequently, from (1.2) we deduce that P is a nonzero constant and $\lambda^k = Q_2/(Q_2 - Q_1)$. Therefore, in this case, we have $f(z) = Q_1 + A\exp(\lambda z)$, where $A, \lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda^k = Q_2/(Q_2 - Q_1)$. This completes the proof. □

4. An application for Brück conjecture

What can be the relationship between f and f' if $f \in \mathscr{E}(\mathbb{C})$ shares only one value CM with its first derivative f'?

In 1996, Brück (see [1]) first discussed the possible relationship between f and f' when $f \in \mathscr{E}(\mathbb{C})$ and its derivative f' share only one finite value CM. In this direction, a still open and interesting problem is the following conjecture proposed by Brück.

Conjecture A ([1]). *Let $f \in \mathscr{E}(\mathbb{C})$ such that $\varrho_2(f) \notin \mathbb{N} \cup \{\infty\}$, where $\varrho_2(f)$ is the hyper-order of f. If f and f' share $a \in \mathbb{C}$ CM, then $f' - a \equiv c(f - a)$, where $c \in \mathbb{C} \setminus \{0\}$.*

Though the conjecture is not settled in its full generality, it gives rise to a long course of research on the uniqueness of entire and meromorphic functions sharing a single value with its derivatives. Specially, it was observed by Yang and Zhang in [12] that Brück's conjecture holds if instead of an entire function one considers its suitable power. They proved the following theorem.

Theorem F ([12]). *Let $f \in \mathscr{E}(\mathbb{C})$ and $n \in \mathbb{N}$ such that $n \geqslant 7$. Suppose that f^n and $(f^n)'$ share 1 CM, then $f^n \equiv (f^n)'$ and $f(z) = c\exp(z/n)$, where $c \in \mathbb{C} \setminus \{0\}$.*

In 2010, Zhang and Yang (see [13]) improved and generalised Theorem F by considering higher order derivatives and by lowering the power of the entire function.

Theorem G ([13]). *Let $f \in \mathscr{E}(\mathbb{C})$ and $k, n \in \mathbb{N}$ such that $n \geqslant k + 1$. If f^n and $(f^n)^{(k)}$ share 1 CM, then $f^n \equiv (f^n)^{(k)}$ and $f(z) = c\exp(\lambda z/n)$, where $c, \lambda \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$.*

In 2011, Lü and Yi (see [7]) replaced the sharing value 1 by sharing a polynomial in Theorem G and obtained the following result.

Theorem H ([7]). *Let $f \in \mathscr{E}_T(\mathbb{C})$, $k, n \in \mathbb{N}$ such that $n \geqslant k + 1$ and let $Q \not\equiv 0$ be a polynomial. If $f^n - Q$ and $(f^n)^{(k)} - Q$ share 0 CM, then the conclusion of Theorem G holds.*

Naturally, one may ask whether the conclusion of Theorem H still holds if $f^n - Q$ and $(f^n)^{(k)} - Q$ share $(0, 1)$. In the following we give an affirmative answer.

Theorem 4.1. *Let $f \in \mathscr{E}_T(\mathbb{C})$, $k, n \in \mathbb{N}$ such that $n \geqslant k + 1$ and let $Q \not\equiv 0$ be a polynomial. If $f^n - Q$ and $(f^n)^{(k)} - Q$ share $(0, 1)$, then the conclusion of Theorem G holds.*

Proof. Let us take $Q_1 = 0$ and $Q_2 = Q$. By the given condition, we see that $f^n - Q_2$ and $(f^n)^{(k)} - Q_2$ share $(0, 1)$ and $f^n = Q_1 \Rightarrow (f^n)^{(k)} = Q_1$. Note that if $\Phi \equiv 0$, then since $L(f^n) \not\equiv 0$, we have $f^n \equiv (f^n)^{(k)}$ and so the conclusion of Theorem G holds. Next we suppose that $\Phi \not\equiv 0$ and so $f^n \not\equiv (f^n)^{(k)}$.

Now we consider the following two possible cases.

Case 1. Suppose $Q \in \mathbb{C} \setminus \{0\}$. Clearly, $\deg(Q_1) < \deg(Q_2)$ and so conclusion (2) of theorem must hold. Thereby from (1.1) we observe that P is a nonzero constant. Consequently, we must have $f^n(z) = c\exp(\lambda z)$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$, which implies that $f^n \equiv (f^n)^{(k)}$, a contradiction.

Case 2. Suppose $Q \notin \mathbb{C}$. Here we see that $f^n - Q_1 = f^n$ has zeros of multiplicities at least $k+1$. Therefore, from Corollary 1.1 we conclude that $f^n \equiv (f^n)^{(k)}$, a contradiction. This completes the proof. $\square$

5. Concluding remark

Keeping other conditions intact, can the sharing condition in Theorem 1.1 be relaxed to $(0,0)$ so that the conclusions remain the same?

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