ON WEAKENED $(\alpha, \delta)$-SKEW ARMENDARIZ RINGS

ALIREZA MAJDABADI FARAHANI, MOHAMMAD MAGHASEDI,
FARIDEH HEYDARI, HAMIDAGHA TAVALLAE, Karaj

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Abstract. In this note, for a ring endomorphism $\alpha$ and an $\alpha$-derivation $\delta$ of a ring $R$, the notion of weakened $(\alpha, \delta)$-skew Armendariz rings is introduced as a generalization of $\alpha$-rigid rings and weak Armendariz rings. It is proved that $R$ is a weakened $(\alpha, \delta)$-skew Armendariz ring if and only if $T_n(R)$ is weakened $(\bar{\alpha}, \bar{\delta})$-skew Armendariz if and only if $R[x]/(x^n)$ is weakened $(\bar{\pi}, \bar{\delta})$-skew Armendariz ring for any positive integer $n$.

Keywords: Armendariz ring; $(\alpha, \delta)$-skew Armendariz ring; weak Armendariz ring; weak $(\alpha, \delta)$-skew Armendariz ring

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1. Introduction

Throughout this paper, $R$ denotes an associative ring with unity, $\alpha: R \to R$ is an endomorphism and $\delta$ an $\alpha$-derivation of $R$, that is, $\delta$ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$ for all $a, b \in R$. We denote by $R[x; \alpha, \delta]$ the Ore extension whose elements are the polynomials over $R$, the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x + \delta(a)$ for any $a \in R$. Rege and Chhawchharia in [22] introduced the notion of an Armendariz ring. They defined a ring $R$ to be an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \ldots + a_mx^m, g(x) = b_0 + b_1x + \ldots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each $i$ and $j$. The name “Armendariz ring” was chosen because Armendariz (see [5]) had noted that every reduced ring satisfies this condition. Some properties of Armendariz rings were studied in Rege and Chhawchharia [22], Armendariz [5], Anderson and Camillo [2], Huh et al. [14], and Kim and Lee [16]. Liu and Zhao in [20] called a ring $R$ weak Armendariz if whenever polynomials $f(x) = a_0 + a_1x + \ldots + a_mx^m, g(x) = b_0 + b_1x + \ldots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j \in \text{nil}(R)$ for each $i$ and $j$.
where \( \text{nil}(R) \) denotes the set of all nilpotent elements of \( R \). For an endomorphism \( \alpha \) and an \( \alpha \)-derivation \( \delta \) of a ring \( R \), Moussavi and Hashemi (see [21]) called \( R \) an \((\alpha, \delta)\)-skew Armendariz ring if whenever polynomials \( f(x) = a_0 + a_1x + \ldots + a_mx^m \), \( g(x) = b_0 + b_1x + \ldots + b_nx^n \in R[x; \alpha, \delta] \) satisfy \( f(x)g(x) = 0 \), then \( a_ix^ib_jx^j = 0 \) for each \( i \) and \( j \), which is a generalization of \( \alpha \)-rigid rings and Armendariz rings. Alhevaz et al. in [1] called a ring \( R \) weak \((\alpha, \delta)\)-skew Armendariz if whenever polynomials \( f(x) = a_0 + a_1x + \ldots + a_mx^m \), \( g(x) = b_0 + b_1x + \ldots + b_nx^n \in R[x; \alpha, \delta] \) satisfy \( f(x)g(x) = 0 \), then \( a_ix^ib_jx^j \in \text{nil}(R)[x; \alpha, \delta] \) for each \( i \) and \( j \).

According to Krempa (see [17]), an endomorphism \( \alpha \) of a ring \( R \) is said to be rigid if \( a\alpha(a) = 0 \) implies \( a = 0 \) for \( a \in R \). Hong et al. in [13], Definition 3 called a ring \( R \) \( \alpha \)-rigid if there exists a rigid endomorphism \( \alpha \) of \( R \). Note that any rigid endomorphism of a ring \( R \) is a monomorphism and \( \alpha \)-rigid rings are reduced rings by Hong et al. (see [13]). Properties of \( \alpha \)-rigid rings have been studied in Krempa [17], Hong et al. [13], and Hirano [11].

By [4], a ring \( R \) is \( \alpha \)-compatible if for all \( a, b \in R \), \( ab = 0 \Leftrightarrow a\alpha(b) = 0 \). In [10], Hashemi and Moussavi introduced \((\alpha, \delta)\)-compatible rings and studied their properties. For an \( \alpha \)-derivation \( \delta \) of \( R \), the ring is said to be \( \delta \)-compatible if for each \( a, b \in R \), \( ab = 0 \Rightarrow a\delta(b) = 0 \). A ring \( R \) is \((\alpha, \delta)\)-compatible if it is both \( \alpha \)-compatible and \( \delta \)-compatible. In this case, clearly the endomorphism \( \alpha \) is monomorphic. Also, any \( \alpha \)-rigid ring is \((\alpha, \delta)\)-compatible, see [13], Lemma 4.

For an endomorphism \( \alpha \) and an \( \alpha \)-derivation \( \delta \) of a ring \( R \), we call \( R \) a weakened \((\alpha, \delta)\)-skew Armendariz ring if whenever polynomials \( f(x) = \sum_{i=0}^{m} a_ix^i \) and \( g(x) = \sum_{j=0}^{n} b_jx^j \in R[x; \alpha, \delta] \) satisfy \( f(x)g(x) = 0 \), then \( a_ix^ib_jx^j \in \text{nil}(R)[x; \alpha, \delta] \) for each \( i \) and \( j \). Clearly, weak Armendariz rings are weakened \((\alpha, \delta)\)-skew Armendariz. We show that weakly 2-primal \((\alpha, \delta)\)-compatible rings are weakened \((\alpha, \delta)\)-skew Armendariz and thus weakened \((\alpha, \delta)\)-skew Armendariz rings are a common generalization of \( \alpha \)-rigid rings and weak Armendariz rings. Also, we prove that \( R \) is a weakened \((\alpha, \delta)\)-skew Armendariz ring if and only if the \( n \times n \) upper triangular matrix ring \( T_n(R) \) is weakened \((\pi, \delta)\)-skew Armendariz if and only if \( R[x]/(x^n) \) is weakened \((\pi, \delta)\)-skew Armendariz ring for any positive integer \( n \).

2. WEAKENED \((\alpha, \delta)\)-SKEW ARMENDARIZ RINGS

Let \( \delta \) be an \( \alpha \)-derivation of a ring \( R \). For any \( 0 \leq u \leq v \) \((u, v \in \mathbb{N})\), \( f_u^n \in \text{End}(R, +) \) will denote the map which is the sum of all possible “words” in \( \alpha \), \( \delta \) built with \( u \) letters \( \alpha \) and \( (v-u) \) letters \( \delta \). For instance, \( f_2^4 = \alpha^2\delta^2 + \alpha\delta^2\alpha + \delta^2\alpha^2 + \alpha\delta\alpha\delta + \delta\alpha\delta\alpha \). In particular, \( f_0^0 = 1, f_0^n = \delta^n, \ldots, f_{n-1}^n = \alpha^{n-1}\delta + \alpha^{n-2}\delta\alpha + \ldots + \delta\alpha^{n-1} \).
and \( f_n^\alpha = \alpha^n \), where \( n \in \mathbb{N} \). For any positive integer \( n \) and \( r \in R \) we have \( x^n r = \sum_{i=0}^{n} f_i^\alpha(r)x^i \) in the ring \( R[x; \alpha, \delta] \) (see [18], Lemma 4.1).

**Definition 2.1.** Let \( \alpha \) be an endomorphism and \( \delta \) an \( \alpha \)-derivation of a ring \( R \). The ring \( R \) is called a weakened \((\alpha, \delta)\)-skew Armendariz ring if for each elements \( f(x) = \sum_{i=0}^{m} a_i x^i \) and \( g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta] \), \( f(x)g(x) = 0 \) implies \( a_i x^i b_j x^j \in \text{nil}(R[x; \alpha, \delta]) \) for each \( i \) and \( j \).

Note that each Armendariz (or weak Armendariz) ring is weakened \((\alpha, \delta)\)-skew Armendariz, where \( \alpha \) is the identity endomorphism of \( R \) and \( \delta \) is the zero mapping. The following example shows that there exists an endomorphism \( \alpha \) and an \( \alpha \)-derivation \( \delta \) of an Armendariz (or weak Armendariz) ring \( R \) such that \( R \) is not weakened \((\alpha, \delta)\)-skew Armendariz.

**Example 2.2.** Let \( S \) be a reduced ring and \( R = S[x] \) a polynomial ring over \( S \). Then \( R \) is reduced and so Armendariz (or weak Armendariz). Consider the endomorphism \( \alpha : R \to R \) given by \( \alpha(f(x)) = f(0) \) and \( \alpha \)-derivation \( \delta : R \to R \) by \( \delta(f(x)) = xf(x) - f(0)x \). Take \( p(y) = x - y \) and \( q(y) = x + xy \in R[y; \alpha, \delta] \). Then \( p(y)q(y) = 0 \). But \( x^2 \) is not nilpotent and hence \( R \) is not weakened \((\alpha, \delta)\)-skew Armendariz.

Clearly, every subring \( S \) with \( \alpha(S) \subseteq S \) and \( \delta(S) \subseteq S \) of a weakened \((\alpha, \delta)\)-skew Armendariz ring is also weakened \((\alpha, \delta)\)-skew Armendariz.

It will be useful to establish a criteria for transfering the weakened \((\alpha, \delta)\)-skew Armendariz condition from one ring to another.

**Proposition 2.3.** Let \( \alpha \) be an endomorphism and \( \delta \) an \( \alpha \)-derivation of a ring \( R \). Let \( S \) be a ring and \( \gamma : R \to S \) a ring isomorphism. Then \( R \) is weakened \((\alpha, \delta)\)-skew Armendariz if and only if \( S \) is weakened \((\gamma \alpha \gamma^{-1}, \gamma \delta \gamma^{-1})\)-skew Armendariz.

**Proof.** Let \( \alpha' = \gamma \alpha \gamma^{-1} \) and \( \delta' = \gamma \delta \gamma^{-1} \). Clearly, \( \alpha' \) is an endomorphism of \( S \). Also \( \delta'(ab) = \gamma \delta(\gamma^{-1}(a)\gamma^{-1}(b)) = \gamma((\delta \gamma^{-1})(a)\gamma^{-1}(b) + (\alpha \gamma^{-1})(a)(\delta \gamma^{-1))(b)) = \delta'(a)b + \alpha'(a)\delta'(b) \). Thus \( \delta' \) is an \( \alpha' \)-derivation on \( S \). Suppose that \( a' = \gamma(a) \) and \( b' = \gamma(b) \) for each \( a, b \in R \). Note that

\[
\gamma(\alpha^k \delta^t(b)) = a' \gamma(\alpha^k \delta^t(b)) = a' \gamma(\alpha^k \gamma^{-1} \delta^t \gamma^{-1}(b)) = a' \gamma(\alpha \gamma^{-1})^k(\gamma(\alpha \gamma^{-1})^t(b')) = a' \alpha^k \delta^t(b').
\]

Therefore \( \gamma(a^u f^\alpha_n(b)) = a' f^\gamma_n(b') \) for each \( a, b \in R \) and \( 0 \leq u \leq v \). Let \( g(x) = \sum_{i=0}^{m} a_i x^i \) and \( h(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta] \). According to the above argument, \( g(x)h(x) = 0 \)
Let \( \alpha \) be an endomorphism and \( \delta \) an \( \alpha \)-derivation of a ring \( R \). Recall that for an ideal \( I \) of \( R \), if \( \alpha(I) \subseteq I \), then \( \overline{\alpha} : R/I \to R/I \) defined by \( \overline{\alpha}(a + I) = \alpha(a) + I \) for \( a \in R \) is an endomorphism of a factor ring \( R/I \), and if \( \delta(I) \subseteq I \), then \( \overline{\delta} : R/I \to R/I \) defined by \( \overline{\delta}(a + I) = \delta(a) + I \) for \( a \in R \) is an \( \overline{\delta} \)-derivation of a factor ring \( R/I \). Also, for each \( f(x) = \sum_{i=0}^{m} a_i x^i \in R[x; \alpha, \delta] \), denote \( \overline{f}(x) = \sum_{i=0}^{m} \overline{a}_i x^i \in (R/I)[x; \overline{\alpha}, \overline{\delta}] \), where \( \overline{a}_i = a_i + I \) for each \( i \).

**Proposition 2.4.** Let \( \alpha \) be an endomorphism and \( \delta \) an \( \alpha \)-derivation of a ring \( R \). Let \( I \) be an ideal of \( R \) with \( \alpha(I) \subseteq I \) and \( \delta(I) \subseteq I \). If \( R/I \) is a weakened \((\overline{\alpha}, \overline{\delta})\)-skew Armendariz ring and \( I[x; \alpha, \delta] \) is nil, then \( R \) is a weakened \((\alpha, \delta)\)-skew Armendariz ring.

**Proof.** Let \( f(x) = \sum_{i=0}^{m} a_i x^i \) and \( g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta] \) satisfy \( f(x)g(x) = 0 \).

Then from canonical ring isomorphism \( R[x; \alpha, \delta]/I[x; \alpha, \delta] \cong (R/I)[x; \overline{\alpha}, \overline{\delta}] \) we have

\[
\sum_{i=0}^{m} \overline{a}_i x^i \sum_{j=0}^{n} \overline{b}_j x^j = 0.
\]

Thus, \( \overline{a}_i x^i \overline{b}_j x^j \in \mathrm{nil}((R/I)[x; \overline{\alpha}, \overline{\delta}]) \) for each \( i, j \), since \( R/I \) is weakened \((\overline{\alpha}, \overline{\delta})\)-skew Armendariz, then \( (a_i x^i b_j x^j)^{n_{ij}} \in I[x; \alpha, \delta] \) for a positive integer \( n_{ij} \). Since \( I[x; \alpha, \delta] \) is nil, \( a_i x^i b_j x^j \in \mathrm{nil}(R[x; \alpha, \delta]) \) for each \( i \) and \( j \). Therefore \( R \) is weakened \((\alpha, \delta)\)-skew Armendariz.

Recall that a ring \( R \) is an \( NI \) ring if the set of nilpotent elements, \( \mathrm{nil}(R) \), forms an ideal. In the following lemma, we determine a property for idempotents of a weakened \((\alpha, \delta)\)-skew Armendariz \( NI \) ring.

**Lemma 2.5.** Let \( R \) be a weakened \((\alpha, \delta)\)-skew Armendariz \( NI \) ring. Then \( \delta(e) \in \mathrm{nil}(R) \) for each \( e^2 = e \in R \).

**Proof.** Let \( e^2 = e \in R \). Then we have \( \delta(e) = \delta(e^2) = \delta(e)e + \alpha(e)\delta(e) \). Now suppose that \( f(x) = \delta(e) + \alpha(e)x \) and \( g(x) = (e - 1) + (e - 1)x \in R[x; \alpha, \delta] \). Then we have \( f(x)g(x) = 0 \). Since \( R \) is a weakened \((\alpha, \delta)\)-skew Armendariz ring, \( \delta(e)(e - 1) = \delta(e)e - \delta(e) \in \mathrm{nil}(R) \). On the other hand, if we take \( p(x) = \delta(e) - (1 - \alpha(e))x \) and \( q(x) = e + ex \in R[x; \alpha, \delta] \), then we have \( p(x)q(x) = 0 \). Thus, \( \delta(e)e \in \mathrm{nil}(R) \) since \( R \) is a weakened \((\alpha, \delta)\)-skew Armendariz ring. So \( \delta(e) \in \mathrm{nil}(R) \), as desired.
Recall that a ring $R$ is Abelian if every idempotent of $R$ is central. The following theorem is a characterization of an Abelian ring $R$ to be weakened $(\alpha, \delta)$-skew Armendariz in terms of its idempotents.

**Theorem 2.6.** Let $R$ be an Abelian ring, $\alpha$ an endomorphism and $\delta$ an $\alpha$-derivation of $R$. Then the following statements are equivalent:

(i) $R$ is weakened $(\alpha, \delta)$-skew Armendariz;

(ii) For each idempotent $e \in R$ such that $\alpha(e) = e$ and $\delta(e) = 0$, $eR$ and $(1-e)R$ are weakened $(\alpha, \delta)$-skew Armendariz;

(iii) For an idempotent $e \in R$ such that $\alpha(e) = e$ and $\delta(e) = 0$, $eR$ and $(1-e)R$ are weakened $(\alpha, \delta)$-skew Armendariz.

**Proof.** (i) $\Rightarrow$ (ii): It is obvious, since $eR$ and $(1-e)R$ are subrings of $R$.

(ii) $\Rightarrow$ (iii): It is clear.

(iii) $\Rightarrow$ (i): Suppose that for an idempotent $e \in R$ such that $\alpha(e) = e$ and $\delta(e) = 0$, $eR$ and $(1-e)R$ are weakened $(\alpha, \delta)$-skew Armendariz and let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta]$ with $f(x) g(x) = 0$. Then $(e f(x))(e g(x)) = 0$ and $((1-e) f(x))((1-e) g(x)) = 0$. Since $eR$ and $(1-e)R$ are weakened $(\alpha, \delta)$-skew Armendariz, there exist $m_{ij}, n_{ij} \in \mathbb{N}$ such that $(e a_i x^i e b_j x^j)^{m_{ij}} = 0$ and $((1-e)a_i x^i(1-e)b_j x^j)^{n_{ij}} = 0$. On the other hand, since $\alpha(e) = e$ and $\delta(e) = 0$, we have $\alpha(e b_j) = e \alpha(b_j)$ and $\delta(e b_j) = e \delta(b_j)$. Hence, one can see that $(e a_i x^i e b_j x^j)^{m_{ij}} = e(a_i x^i b_j x^j)^{m_{ij}} = 0$ and $((1-e)a_i x^i(1-e)b_j x^j)^{n_{ij}} = (1-e)(a_i x^i b_j x^j)^{n_{ij}} = 0$. Let $k_{ij} = \max\{m_{ij}, n_{ij}\}$. Then $e(a_i x^i b_j x^j)^{k_{ij}} = 0$ and $(1-e)(a_i x^i b_j x^j)^{k_{ij}} = 0$. Therefore $(a_i x^i b_j x^j)^{k_{ij}} = e(a_i x^i b_j x^j)^{k_{ij}} + (1-e)(a_i x^i b_j x^j)^{k_{ij}} = 0$. Hence $R$ is weakened $(\alpha, \delta)$-skew Armendariz. $\square$

For weak Armendariz rings we have the following result.

**Proposition 2.7.** If $R[x; \alpha]$ is a weak Armendariz ring, then $R$ is a weakened $\alpha$-skew Armendariz ring.

**Proof.** Suppose $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha]$ satisfy $f(x) g(x) = 0$. Then we have $c_k = a_0 b_k + a_1 \alpha(b_{k-1}) + \ldots + a_k \alpha^k(b_0)$ for each $0 \leq k \leq m + n$. Now, let

$$p(y) = a_0 + (a_1 x)y + (a_2 x^2)y^2 + \ldots + (a_m x^m)y^m,$$

$$q(y) = b_0 + (b_1 x)y + (b_2 x^2)y^2 + \ldots + (b_n x^n)y^n.$$
be polynomials in $R[x;\alpha][y]$. Thus, we have $p(y)q(y) = \sum_{k=0}^{m+n} (c_k x^k) y^k = 0$, since $c_k = 0$ for each $0 \leq k \leq m + n$. So $a_i x^i b_j x^j \in \text{nil}(R[x;\alpha])$ for each $0 \leq i \leq m$ and $0 \leq j \leq n$, since $R[x;\alpha]$ is weak Armendariz. Hence, $R$ is a weakened $\alpha$-skew Armendariz ring and the result follows.

Let $\alpha_i$ be an endomorphism and $\delta_i$ an $\alpha_i$-derivation of a ring $R_i$, $i = 1, 2, \ldots, k$. Let $R = \bigoplus_{i=1}^{k} R_i$. Then the map $\alpha: R \rightarrow R$ defined by $\alpha((a_i)) = (\alpha_i(a_i))$ is an endomorphism of $R$ and $\delta: R \rightarrow R$ defined by $\delta((a_i)) = (\delta_i(a_i))$ is an $\alpha$-derivation of $R$.

**Proposition 2.8.** Let $\alpha_i$ be an endomorphism and $\delta_i$ an $\alpha_i$-derivation of a ring $R_i$ for each $1 \leq i \leq k$. Then $R_i$ is a weakened $(\alpha_i, \delta_i)$-skew Armendariz ring if and only if $R = \bigoplus_{i=1}^{k} R_i$ is a weakened $(\alpha, \delta)$-skew Armendariz ring.

**Proof.** It is not hard to see that there exists a ring isomorphism $\varphi: R[x;\alpha, \delta] \rightarrow \bigoplus_{i=1}^{k} (R_i[x;\alpha_i, \delta_i])$, given by $\varphi\left( \sum_{s=0}^{m} A_s x^s \right) = (f_i)$, where $A_s = (a_{1s}, a_{2s}, \ldots, a_{ks})$ in $R$ and $f_i(x) = \sum_{s=0}^{m} a_{is} x^s$ in $R_i[x;\alpha_i, \delta_i]$ for each $0 \leq s \leq m$ and $1 \leq i \leq k$. Let $f(x) = \sum_{s=0}^{m} A_s x^s$ and $g(x) = \sum_{t=0}^{n} B_t x^t \in R[x;\alpha, \delta]$ satisfy $f(x)g(x) = 0$, where $A_s = (a_{1s}, a_{2s}, \ldots, a_{ks})$ and $B_t = (b_{1t}, b_{2t}, \ldots, b_{kt}) \in R$ and $a_{is}, b_{it} \in R_i$ for each $0 \leq s \leq m$ and $0 \leq t \leq n$. Then from isomorphism $R[x;\alpha, \delta] \cong \bigoplus_{i=1}^{k} (R_i[x;\alpha_i, \delta_i])$ we have that $f_i(x)g_i(x) = 0$ for each $1 \leq i \leq k$, where $f_i(x) = \sum_{s=0}^{m} a_{is} x^s$ and $g_i(x) = \sum_{t=0}^{n} b_{it} x^t \in R_i[x;\alpha_i, \delta_i]$. Since $R_i$ is weakened $(\alpha_i, \delta_i)$-skew Armendariz for every $1 \leq i \leq k$, there exists $p_{sti} \in \mathbb{N}$ such that $(a_{is} x^s b_{it} x^t)^{p_{sti}} = 0$ for each $1 \leq i \leq k$. Let $p_{st} = \max\{p_{st1}, p_{st2}, \ldots, p_{stk}\}$. Then $(A_s x^s B_t x^t)^{p_{st}} = 0$. Therefore $R = \bigoplus_{i=1}^{k} R_i$ is a weakened $(\alpha, \delta)$-skew Armendariz ring. Conversely, since $R_i$ is an invariant subring of $R$ for each $1 \leq i \leq k$, the assertion holds.

Let $R$ be a ring and $\sigma$ denotes an endomorphism of $R$ with $\sigma(1) = 1$. In [6] the authors introduced skew triangular matrix ring, denoted by $T_n(R, \sigma)$, as a set of all triangular matrices with addition point-wise and a new multiplication subject to the condition $E_{ij}r = \sigma^{j-i}(r)E_{ij}$. So $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = a_{ii} b_{ij} + a_{i,i+1} \sigma(b_{i+1,j}) + \ldots + a_{ij} \sigma^{j-i}(b_{jj})$ for each $i \leq j$.

The subring of the skew triangular matrices with constant main diagonal is denoted by $S(R, n, \sigma)$; and the subring of the skew triangular matrices with constant diagonals is denoted by $T(R, n, \sigma)$. We can denote $A = (a_{ij}) \in T(R, n, \sigma)$ by $(a_{11}, \ldots, a_{nn})$. Then $T(R, n, \sigma)$ is a ring with addition point-wise and multiplication given by:
\[(a_0, \ldots, a_{n-1})(b_0, \ldots, b_{n-1}) = (a_0b_0, a_0* b_1 + a_1 * b_0, \ldots, a_0 * b_{n-1} + \ldots + a_{n-1} * b_0),\]

with \(a_i * b_j = a_i\sigma^i(b_j)\) for each \(i\) and \(j\). Therefore, clearly one can see that \(T(R,n,\sigma) \cong \mathcal{R}[x;\sigma]/(x^n)\), where \((x^n)\) is the ideal generated by \(x^n\) in \(\mathcal{R}[x;\sigma]\).

Also, we consider the following two subrings of \(\mathcal{S}(R,n,\sigma)\):

\[
A(R,n,\sigma) = \left\{ \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{i=1}^{n-j+1} a_{j} E_{i,i+j-1} + \sum_{j=\left\lfloor \frac{n}{2} \right\rfloor+1}^{n} \sum_{i=1}^{n-j+1} a_{i,i+j-1} E_{i,i+j-1} \right\};
\]

\[
B(R,n,\sigma) = \{ A + rE_{1k} : A \in A(R,n,\sigma) \text{ and } r \in R \}, \quad n = 2k \geq 4.
\]

Let \(\sigma\) be an endomorphism of a ring \(\mathcal{R}\), \(\alpha\) an endomorphism of \(\mathcal{R}\) and \(\delta\) an \(\alpha\)-derivation of \(\mathcal{R}\) such that \(\sigma\alpha = \alpha\sigma\) and \(\delta\sigma = \sigma\delta\). The endomorphism \(\alpha\) of \(\mathcal{R}\) is extended to the endomorphism \(\sigma\): \(D \rightarrow D\) defined by \(\sigma((a_{ij})) = (\alpha(a_{ij}))\) and the \(\alpha\)-derivation \(\delta\) of \(\mathcal{R}\) is also extended to \(\delta\): \(D \rightarrow D\) defined by \(\delta((a_{ij})) = (\delta(a_{ij}))\), where \(D\) is one of the rings \(T_n(R,\sigma), S(R,n,\sigma), A(R,n,\sigma), B(R,n,\sigma)\) or \(T(R,n,\sigma)\).

Also, the map \(\bar{\sigma}: \mathcal{R}[x;\alpha,\delta] \rightarrow \mathcal{R}[x;\alpha,\delta]\) defined by \(\bar{\sigma}(\sum_{i=0}^{m} a_{i}x^{i}) = \sum_{i=0}^{m} \sigma(a_{i})x^{i}\) is an endomorphism of \(\mathcal{R}[x;\alpha,\delta]\).

Kim and Lee in [15], Example 1 showed that \(n \times n\) upper triangular matrix rings over a ring \(\mathcal{R}\) are not Armendariz when \(n \geq 2\). But we have the following result.

**Proposition 2.9.** Let \(\sigma\) and \(\alpha\) be endomorphisms of a ring \(\mathcal{R}\) and \(\delta\) an \(\alpha\)-derivation of \(\mathcal{R}\) such that \(\sigma\alpha = \alpha\sigma\), \(\delta\alpha = \sigma\\delta\) and \(n\) is a positive integer number. Then \(\mathcal{R}\) is a weakened \((\alpha,\delta)\)-skew Armendariz ring if and only if \(D\) is a weakened \((\bar{\sigma},\bar{\delta})\)-skew Armendariz ring, where \(D\) is one of the rings \(T_n(R,\sigma), S(R,n,\sigma), A(R,n,\sigma), B(R,n,\sigma)\), \(T(R,n,\sigma)\).

**Proof.** We only prove this proposition for the case \(D = T_n(R,\sigma)\). Note that any invariant subring of weakened \((\alpha,\delta)\)-skew Armendariz rings is a weakened \((\alpha,\delta)\)-skew Armendariz ring. Thus, if \(T_n(R,\sigma)\) is a weakened \((\bar{\sigma},\bar{\delta})\)-skew Armendariz ring, then \(R\) is a weakened \((\alpha,\delta)\)-skew Armendariz ring. Conversely, Let \(I = \{ A \in D : \text{ each diagonal entry of } A \text{ is zero} \}\). Then \(I[x;\bar{\sigma},\bar{\delta}]\) is a nil ideal of \(D[x;\bar{\sigma},\bar{\delta}]\). On the other hand, we can obtain \(D/I \cong \bigoplus_{i=1}^{n} R_i\), where \(R_i = R\). The proof is completed by Proposition 2.4 and Proposition 2.8. \(\square\)

**Corollary 2.10.** If \(\mathcal{R}\) is an \((\alpha,\delta)\)-skew Armendariz ring, then \(D\) is a weakened \((\bar{\sigma},\bar{\delta})\)-skew Armendariz ring, where \(D\) is one of the rings \(T_n(R,\sigma), S(R,n,\sigma), A(R,n,\sigma), B(R,n,\sigma)\), \(T(R,n,\sigma)\).

Given a ring \(\mathcal{R}\) and a bimodule \(\mathcal{R}M\), the trivial extension of \(\mathcal{R}\) by \(M\) is \(T(\mathcal{R},M) = \mathcal{R} \oplus M\) with the usual addition and the multiplication: \((r_1,m_1)(r_2,m_2) = (r_1r_2,r_1m_2 + m_1r_2)\).

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This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

**Corollary 2.11.** Let $\alpha$ be an endomorphism and $\delta$ an $\alpha$-derivation of a ring $R$. Then $R$ is a weakened $(\alpha, \delta)$-skew Armendariz ring if and only if the trivial extension $T(R, R)$ is a weakened $(\overline{\alpha}, \overline{\delta})$-skew Armendariz ring.

**Proof.** It follows from Proposition 2.9. □

Note that if $\sigma$ is an identity endomorphism of $R$, then we have the following corollary.

**Corollary 2.12.** Let $\sigma$ and $\alpha$ be endomorphisms of a ring $R$ and $\delta$ an $\alpha$-derivation of $R$ such that $\sigma \alpha = \alpha \sigma$ and $\delta \sigma = \sigma \delta$. Then we have the following statements:

(i) $R$ is a weakened $(\alpha, \delta)$-skew Armendariz ring if and only if for each positive integer $n$, $R[x; \sigma]/(x^n)$ is a weakened $(\overline{\alpha}, \overline{\delta})$-skew Armendariz ring.

(ii) $R$ is a weakened $(\alpha, \delta)$-skew Armendariz ring if and only if for each positive integer $n$, $R[x]/(x^n)$ is a weakened $(\overline{\alpha}, \overline{\delta})$-skew Armendariz ring.

Now we can give the examples of weakened $(\alpha, \delta)$-skew Armendariz rings which are not $(\alpha, \delta)$-skew Armendariz.

**Example 2.13.** Let $\alpha$ be an endomorphism and $\delta$ an $\alpha$-derivation of a field $F$ and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ be the 2-by-2 upper triangular matrix ring over $F$. Let $f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x$ and $g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x \in R[x; \sigma, \delta]$. Then it is easy to see that $f(x)g(x) = 0$, but $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x \neq 0$. So $R$ is not $(\overline{\alpha}, \overline{\delta})$-skew Armendariz. Since $F$ is a field, $F$ is an $(\alpha, \delta)$-skew Armendariz. Thus, by Corollary 2.10, $R$ is a weakened $(\overline{\alpha}, \overline{\delta})$-skew Armendariz ring.

**Example 2.14.** Let $R$ be a weakened $(\alpha, \delta)$-skew Armendariz ring. Let

$$S_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \ldots & a_{1n} \\ 0 & a & a_{23} & \ldots & a_{2n} \\ 0 & 0 & a & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a \end{pmatrix} : a, a_{ij} \in R \right\}$$

with $n \geq 4$. 

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Suppose

\[ f(x) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} x \]

and

\[ g(x) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} x \]

be polynomials in \( S_n[x; \bar{\alpha}, \bar{\delta}] \). Then it is easy to see that \( f(x)g(x) = 0 \), but

\[ \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} x \neq 0. \]

So \( S_n \) is not \((\bar{\alpha}, \bar{\delta})\)-skew Armendariz, but \( S_n \) is a weakened \((\bar{\alpha}, \bar{\delta})\)-skew Armendariz ring by Proposition 2.9, since any subring of weakened \((\bar{\alpha}, \bar{\delta})\)-skew Armendariz rings is a weakened \((\bar{\alpha}, \bar{\delta})\)-skew Armendariz ring.

From Proposition 2.9, one may suspect that if \( R \) is weakened \((\alpha, \delta)\)-skew Armendariz, then every \( n \times n \) full matrix ring \( M_n(R) \) over \( R \) is a weakened \((\bar{\alpha}, \bar{\delta})\)-skew Armendariz ring, where \( n \geq 2 \). But the following example erases this possibility.

Example 2.15. Let \( \alpha \) be an endomorphism and \( \delta \) an \( \alpha \)-derivation of a ring \( R \) and \( R \) be a weakened \((\alpha, \delta)\)-skew Armendariz ring. Let \( S = M_2(R) \). Suppose

\[ f(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x \quad \text{and} \quad g(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} x \]

be polynomials in \( S[x; \bar{\alpha}, \bar{\delta}] \). Then it is easy to see that \( f(x)g(x) = 0 \), but

\[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x \]

is not nilpotent. Thus, \( S \) is not weakened \((\bar{\alpha}, \bar{\delta})\)-skew Armendariz.
Let \( D \) be a ring and \( C \) a subring of \( D \) with \( 1_D \in C \). With addition and multiplication defined component-wise,
\[
R = \mathfrak{R}(D, C) = \{(d_1, \ldots, d_n, c, c, \ldots) : d_i \in D, \ c \in C, \ n \geq 1\}
\]
is a ring (see [7]). For an endomorphism \( \alpha \) and an \( \alpha \)-derivation \( \delta \) of \( D \) such that \( \alpha(C) \subseteq C \) and \( \delta(C) \subseteq C \), the natural extension \( \overline{\alpha} : R \rightarrow R \) defined by
\[
\overline{\alpha}((d_1, \ldots, d_n, c, c, \ldots)) = (\alpha(d_1), \ldots, \alpha(d_n), \alpha(c), \alpha(c), \ldots)
\]
for \((d_1, \ldots, d_n, c, c, \ldots) \in R\) is an endomorphism of \( R \) and \( \overline{\delta} : R \rightarrow R \) defined by
\[
\overline{\delta}((d_1, \ldots, d_n, c, c, \ldots)) = (\delta(d_1), \ldots, \delta(d_n), \delta(c), \delta(c), \ldots)
\]
for \((d_1, \ldots, d_n, c, c, \ldots) \in R\) is an \( \overline{\alpha} \)-derivation of \( R \).

**Theorem 2.16.** Let \( \alpha \) be an endomorphism and \( \delta \) an \( \alpha \)-derivation of a ring \( D \) and let \( C \) be a subring of \( D \) with \( 1_D \in C \), \( \alpha(C) \subseteq C \) and \( \delta(C) \subseteq C \). Then \( D \) is a weakened \((\alpha, \delta)\)-skew Armendariz ring if and only if \( R = \mathfrak{R}(D, C) \) is a weakened \((\overline{\alpha}, \overline{\delta})\)-skew Armendariz ring.

**Proof.** Suppose that \( D \) is a weakened \((\alpha, \delta)\)-skew Armendariz ring. Let \( f(x) = \sum_{i=0}^{p} \xi_i x^i \) and \( g(x) = \sum_{j=0}^{q} \eta_j x^j \in R[x; \overline{\alpha}, \overline{\delta}] \) and \( f(x)g(x) = 0 \). Without loss of generality, we can assume that there exists a positive integer \( n \) such that \( \xi_i = (a_{1i}, \ldots, a_{ni}, c_i, c, \ldots), \eta_j = (b_{1j}, \ldots, b_{nj}, d_j, d, \ldots) \in R \) for all \( i, j \). Let \( f_s(x) = \sum_{i=0}^{p} a_{si} x^i, \ g_s(x) = \sum_{j=0}^{q} b_{sj} x^j \) with \( 1 \leq s \leq n \) and \( f'(x) = \sum_{i=0}^{p} c_i x^i, \ g'(x) = \sum_{j=0}^{q} d_j x^j \). From \( f(x)g(x) = 0 \) we obtain \( f_s(x)g_s(x) = 0 \) and \( f'(x)g'(x) = 0 \) in \( D[x; \alpha, \delta] \) for all \( s \). Thus, \( a_{si} x^i b_{sj} x^j \in \text{nil}(D[x; \alpha, \delta]) \) and \( c_i x^i d_j x^j \in \text{nil}(D[x; \alpha, \delta]) \) for all \( i, j, s \) since \( D \) is weakened \((\alpha, \delta)\)-skew Armendariz. Hence, there exist \( t_{si}, t'_{ij} \in \mathbb{N} \) such that \( (a_{si} x^i b_{sj} x^j)^{t_{si}} = 0 \) and \( (c_i x^i d_j x^j)^{t'_{ij}} = 0 \) for \( 1 \leq s \leq n \). Let \( t_{ij} = \max\{t_{1ij}, \ldots, t_{nij}, t'_{ij}\} \). Then we have \((\xi_i x^i \eta_j x^j)^{t_{ij}} = 0 \) for all \( i, j \). Therefore \( R \) is weakened \((\overline{\alpha}, \overline{\delta})\)-skew Armendariz. Conversely, since \( D \) is an invariant subring of \( R \), the assertion holds. \( \square \)
3. WEAKLY 2-PRIMAL \((\alpha , \delta )\)-COMPATIBLE RINGS AND WEAKENED \((\alpha , \delta )\)-SKEW ARMENDARIZ RINGS

A ring \(R\) is \textit{semicommutative} if the right annihilator of each element of \(R\) is an ideal (equivalently, if for all \(a, b \in R\) we have \(ab = 0 \Rightarrow aRb = 0\)). A ring \(R\) is \textit{symmetric} if for all \(a, b, c \in R\) we have \(abc = 0 \Rightarrow bac = 0\). A ring \(R\) is called \textit{reversible} if for all \(a, b \in R\) we have \(ab = 0 \Rightarrow ba = 0\). Recall that a ring \(R\) is \textit{2-primal} if \(\text{nil}(R) = \text{Nil}_s(R)\), where \(\text{Nil}_s(R)\) denotes the prime radical of \(R\). Hong et al. (see [12]) called a ring \(R\) to be \textit{locally 2-primal} if each finite subset generates a 2-primal ring. Chen and Cui (see [8]) called a ring \(R\) \textit{weakly 2-primal} if the set of nilpotent elements in \(R\) coincides with its locally nilpotent radical. Note that every reduced ring is symmetric by [3], Theorem 1.3, every symmetric ring is reversible, every reversible ring is semicommutative by [19], Proposition 1.3, every semicommutative ring is 2-primal by [23], Theorem 1.5, every 2-primal ring is locally 2-primal and every locally 2-primal ring is weakly 2-primal.

The following example shows that there exists a semicommutative ring with an endomorphism \(\alpha\) and an \(\alpha\)-derivation \(\delta\) which is not weakened \((\alpha, \delta)\)-skew Armendariz.

\textbf{Example 3.1.} Let \(S\) be a reduced ring and \(R = S[x]\) a polynomial ring over \(S\). Then \(R\) is reduced and so semicommutative. Consider the endomorphism \(\alpha: R \to R\) given by \(\alpha(f(x)) = f(0)\) and \(\alpha\)-derivation \(\delta: R \to R\) by \(\delta(f(x)) = xf(x) - f(0)x\). Take \(p(y) = x - y\) and \(q(y) = x + xy\in R[y; \alpha, \delta]\). Then \(p(y)q(y) = 0\). But \(x^2\) is not nilpotent and hence \(R\) is not weakened \((\alpha, \delta)\)-skew Armendariz.

The following example shows that weakened \((\alpha, \delta)\)-skew Armendariz rings may not be semicommutative.

\textbf{Example 3.2.} Let \(\alpha\) be an endomorphism and \(\delta\) an \(\alpha\)-derivation of a division ring \(F\) and \(R = \left( \begin{array}{cc} F & F \\ 0 & F \end{array} \right)\) be the 2-by-2 upper triangular matrix ring over \(F\). Then \(R\) is not semicommutative by [14], Example 5. But by Corollary 2.10, \(R\) is a weakened \((\bar{\alpha}, \bar{\delta})\)-skew Armendariz ring.

Habibi and Moussavi (see [9]) called a ring \(R\) \textit{nil (\(\alpha, \delta\))-skew Armendariz} if whenever polynomials \(f(x) = a_0 + a_1 x + \ldots + a_m x^m\), \(g(x) = b_0 + b_1 x + \ldots + b_n x^n \in R[x; \alpha, \delta]\) satisfy \(f(x)g(x) \in \text{nil}(R)[x; \alpha, \delta]\), then \(a_i x^i b_j x^j \in \text{nil}(R)[x; \alpha, \delta]\) for each \(i\) and \(j\).

\textbf{Proposition 3.3.} Let \(R\) be an \(\alpha\)-compatible ring such that \(\text{nil}(R[x; \alpha, \delta]) = \text{nil}(R)[x; \alpha, \delta]\). Then \(R\) is a weakened \((\alpha, \delta)\)-skew Armendariz ring.

\textbf{Proof.} Let \(f(x) = \sum_{i=0}^{m} a_i x^i\) and \(g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta]\) such that \(f(x)g(x) = 0\). Since \(R\) is an \(\alpha\)-compatible and \(\text{nil}(R[x; \alpha, \delta]) = \text{nil}(R)[x; \alpha, \delta]\), \(R\) is
nil \((\alpha, \delta)\)-skew Armendariz by [9], Proposition 2.9. Then \(a_i x^i b_j x^j \in \text{nil}(R)[x; \alpha, \delta]\) for each \(i, j\). Hence \(a_i x^i b_j x^j \in \text{nil}(R[x; \alpha, \delta])\) for each \(i, j\). Therefore \(R\) is a weakened \((\alpha, \delta)\)-skew Armendariz ring. \(\square\)

Wang et al. in [24], Corollary 2.1 proved that if \(R\) is a weakly 2-primal \((\alpha, \delta)\)-compatible ring, then \(\text{nil}(R[x; \alpha, \delta]) = \text{nil}(R)[x; \alpha, \delta]\). So we have the following result.

**Proposition 3.4.** Every weakly 2-primal \((\alpha, \delta)\)-compatible ring is weakened \((\alpha, \delta)\)-skew Armendariz.

**Corollary 3.5.** \(\alpha\)-rigid rings are weakened \((\alpha, \delta)\)-skew Armendariz rings.

The following example shows that the converse of Corollary 3.5 is not true in general.

**Example 3.6.** Let \(\delta\) be an \(\alpha\)-derivation of a ring \(R\) and \(R\) be an \(\alpha\)-rigid ring. Then by Corollary 3.5, \(R\) is a weakened \((\alpha, \delta)\)-skew Armendariz ring. Consider the following subring of \(T_3(R)\):

\[
R_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in R \right\}.
\]

The endomorphism \(\alpha\) of \(R\) is extended to the endomorphism \(\overline{\alpha}\): \(R_3 \rightarrow R_3\) defined by \(\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))\) and the \(\alpha\)-derivation \(\delta\) of \(R\) is also extended to \(\overline{\delta}\): \(R_3 \rightarrow R_3\) defined by \(\overline{\delta}((a_{ij})) = (\delta(a_{ij}))\). By Proposition 2.9, \(R_3\) is weakened \((\overline{\alpha}, \overline{\delta})\)-skew Armendariz. But it is not \(\overline{\alpha}\)-rigid, by [10], Example 1.2.

**Lemma 3.7.** Let \(\alpha\) be an endomorphism and \(\delta\) an \(\alpha\)-derivation of a ring \(R\). Then \(R\) is \((\alpha, \delta)\)-compatible and reduced if and only if \(R[x]\) is \((\alpha, \delta)\)-compatible and reduced.

**Proof.** We know, \(R\) is reduced if and only if \(R[x]\) is reduced. Let \(R\) be \((\alpha, \delta)\)-compatible and reduced. Let \(f(x) = \sum_{i=0}^{m} a_i x^i\) and \(g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]\) with \(f(x)g(x) = 0\). Since \(R\) is Armendariz, \(a_i b_j = 0\) for all \(0 \leq i \leq m\) and \(0 \leq j \leq n\). Then \(a_i \alpha(b_j) = 0, a_i \delta(b_j) = 0\) because \(R\) is \((\alpha, \delta)\)-compatible. Thus

\[
f(x)\alpha(g(x)) = \sum_{i=0}^{m} a_i x^i \sum_{j=0}^{n} \alpha(b_j) x^j = \sum_{k=0}^{m+n} \left( \sum_{i+j=k} a_i \alpha(b_j) \right) x^k = 0
\]

and

\[
f(x)\delta(g(x)) = \sum_{i=0}^{m} a_i x^i \sum_{j=0}^{n} \delta(b_j) x^j = \sum_{k=0}^{m+n} \left( \sum_{i+j=k} a_i \delta(b_j) \right) x^k = 0.
\]
Now assume that \( f(x)\alpha(g(x)) = 0 \). Then we have

\[
f(x)\alpha(g(x)) = \sum_{i=0}^{m} a_i x^i \sum_{j=0}^{n} \alpha(b_j) x^j = 0.
\]

Since \( R \) is Armendariz, \( a_i\alpha(b_j) = 0 \) for all \( 0 \leq i \leq m \) and \( 0 \leq j \leq n \). So \( a_i b_j = 0 \) because \( R \) is \((\alpha, \delta)\)-compatible. Hence

\[
f(x)g(x) = \sum_{i=0}^{m} a_i x^i \sum_{j=0}^{n} b_j x^j = \sum_{k=0}^{m+n} \left( \sum_{i+j=k} a_i b_j x^k \right) = 0.
\]

Therefore \( R[x] \) is an \((\alpha, \delta)\)-compatible ring. Conversely, it is clear. \( \square \)

**Proposition 3.8.** Let \( R \) be an \((\alpha, \delta)\)-compatible and reduced ring. Then \( R[x] \) is a weakened \((\alpha, \delta)\)-skew Armendariz ring.

**Proof.** Let \( R \) be an \((\alpha, \delta)\)-compatible and reduced ring. Then \( R[x] \) is \((\alpha, \delta)\)-compatible and reduced, by Lemma 3.7. But every reduced ring is weakly 2-primal. Thus, \( R[x] \) is a weakly 2-primal \((\alpha, \delta)\)-compatible ring. Therefore \( R[x] \) is a weakened \((\alpha, \delta)\)-skew Armendariz ring, by Proposition 3.4. \( \square \)

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Authors’ address: Alireza Majdabadi Farahani, Mohammad Maghasedi, Farideh Heydari, Hamidagha Tavallaee, Department of Mathematics, Karaj Branch, Islamic Azad University, Imam Ali Complex, Moazen Blvd, 3149968111 Karaj, Iran, e-mail: a.majdabadi@kiau.ac.ir, f-heydari@kiau.ac.ir, tavallaee@iust.ac.ir, maghasedi@kiau.ac.ir.