

EQUIVALENCE BUNDLES OVER A FINITE GROUP
AND STRONG MORITA EQUIVALENCE FOR UNITAL
INCLUSIONS OF UNITAL C^* -ALGEBRAS

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Abstract. Let $\mathcal{A} = \{A_t\}_{t \in G}$ and $\mathcal{B} = \{B_t\}_{t \in G}$ be C^* -algebraic bundles over a finite group G . Let $C = \bigoplus_{t \in G} A_t$ and $D = \bigoplus_{t \in G} B_t$. Also, let $A = A_e$ and $B = B_e$, where e is the unit element in G . We suppose that C and D are unital and A and B have the unit elements in C and D , respectively. In this paper, we show that if there is an equivalence \mathcal{A} - \mathcal{B} -bundle over G with some properties, then the unital inclusions of unital C^* -algebras $A \subset C$ and $B \subset D$ induced by \mathcal{A} and \mathcal{B} are strongly Morita equivalent. Also, we suppose that \mathcal{A} and \mathcal{B} are saturated and that $A' \cap C = \mathbf{C}1$. We show that if $A \subset C$ and $B \subset D$ are strongly Morita equivalent, then there are an automorphism f of G and an equivalence bundle \mathcal{A} - \mathcal{B}^f -bundle over G with the above properties, where \mathcal{B}^f is the C^* -algebraic bundle induced by \mathcal{B} and f , which is defined by $\mathcal{B}^f = \{B_{f(t)}\}_{t \in G}$. Furthermore, we give an application.

Keywords: C^* -algebraic bundle; equivalence bundles; inclusions of C^* -algebra; strong Morita equivalence

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1. INTRODUCTION

Let $\mathcal{A} = \{A_t\}_{t \in G}$ be a C^* -algebraic bundle over a finite group G . Let $C = \bigoplus_{t \in G} A_t$ and $A_e = A$, where e is the unit element in G . We suppose that C is unital and that A has the unit element in C . Then we obtain a unital inclusion of unital C^* -algebras, $A \subset C$. We call it the *unital inclusion of unital C^* -algebras induced by a C^* -algebraic bundle $\mathcal{A} = \{A_t\}_{t \in G}$* . Let E^A be the canonical conditional expectation from C onto A defined by

$$E^A(x) = x_e \quad \text{for all } x = \sum_{t \in G} x_t \in C.$$

Definition 1.1. Let $\mathcal{A} = \{A_t\}_{t \in G}$ be a C^* -algebraic bundle over a finite group G . We say that \mathcal{A} is *saturated* if $\overline{A_t A_t^*} = A$ for all $t \in G$.

Since A is unital, in our case we do not need to take the closure in Definition 1.1. If \mathcal{A} is saturated, by [9], Corollary 3.2, E^A is of index-finite type and its Watatani index $\text{Ind}_W(E^A) = |G|$, where $|G|$ is the order of G .

Let $\mathcal{B} = \{B_t\}_{t \in G}$ be another C^* -algebraic bundle over G . Let $D = \bigoplus_{t \in G} B_t$ and $B = B_e$. Also, we suppose that \mathcal{B} has the same conditions as \mathcal{A} . Let $B \subset D$ be the unital inclusion of unital C^* -algebras induced by \mathcal{B} .

Let $\mathcal{X} = \{X_t\}_{t \in G}$ be an $\mathcal{A} - \mathcal{B}$ -equivalence bundle defined by Abadie and Ferraro (see [1], Definition 2.2). Moreover, we suppose that

$${}_C \langle X_t, X_s \rangle = A_{ts^{-1}}, \quad \langle X_t, X_s \rangle_D = B_{t^{-1}s}$$

for any $t, s \in G$, where ${}_C \langle X_t, X_s \rangle$ means the linear span of the set

$$\{{}_C \langle x, y \rangle \in A_{ts^{-1}} : x \in X_t, y \in X_s\}$$

and $\langle X_t, X_s \rangle_D$ means the linear span of the similar set to the above. The above two properties are stronger than properties (7R) and (7L) in [1], Definition 2.1.

In the present paper, we show that if there is an $\mathcal{A} - \mathcal{B}$ -equivalence bundle $\mathcal{X} = \{X_t\}_{t \in G}$ such that ${}_C \langle X_t, X_s \rangle = A_{ts^{-1}}$ and $\langle X_t, X_s \rangle_D = B_{t^{-1}s}$ for any $t, s \in G$, then the unital inclusions of unital C^* -algebras $A \subset C$ and $B \subset D$ induced by \mathcal{A} and \mathcal{B} are strongly Morita equivalent. Also, we suppose that \mathcal{A} and \mathcal{B} are saturated and that $A' \cap C = \mathbf{C}1$. We show that if $A \subset C$ and $B \subset D$ are strongly Morita equivalent, then there are an automorphism f of G and an $\mathcal{A} - \mathcal{B}^f$ -equivalence bundle $\mathcal{X} = \{X_t\}_{t \in G}$ such that ${}_C \langle X_t, X_s \rangle = A_{ts^{-1}}$ and $\langle X_t, X_s \rangle_D = B_{f(t^{-1}s)}$ for any $t, s \in G$, where \mathcal{B}^f is the C^* -algebraic bundle induced by $\mathcal{B} = \{B_t\}_{t \in G}$ and f , which is defined by $\mathcal{B}^f = \{B_{f(t)}\}_{t \in G}$.

Let A and B be unital C^* -algebras and X an $A - B$ -equivalence bimodule. Then we denote its left A -action and right B -action on X by $a \cdot x$ and $x \cdot b$ for any $a \in A$, $b \in B$ and $x \in X$, respectively. Also, we mean by the words ‘‘Hilbert C^* -bimodules’’ Hilbert C^* -bimodules in the sense of Brown, Mingo and Shen, see [3].

2. EQUIVALENCE BUNDLES OVER A FINITE GROUP

Let $\mathcal{A} = \{A_t\}_{t \in G}$ and $\mathcal{B} = \{B_t\}_{t \in G}$ be C^* -algebraic bundles over a finite group G . Let e be the unit element in G . Let $C = \bigoplus_{t \in G} A_t$, $D = \bigoplus_{t \in G} B_t$ and $A = A_e$, $B = B_e$. We suppose that C and D are unital and that A and B have the unit elements in C and D , respectively. Let $\mathcal{X} = \{X_t\}_{t \in G}$ be an $\mathcal{A} - \mathcal{B}$ -equivalence bundle over G such that

$${}_C \langle X_t, X_s \rangle = A_{ts^{-1}}, \quad \langle X_t, X_s \rangle_D = B_{t^{-1}s}$$

for any $t, s \in G$. Let $Y = \bigoplus_{t \in G} X_t$ and $X = X_e$. Then Y is a $C - D$ -equivalence bimodule by Abadie and Ferraro (see [1], Definitions 2.1 and 2.2). Also, X is an $A - B$ -equivalence bimodule since ${}_C \langle X, X \rangle = A$ and $\langle X, X \rangle_D = B$.

Proposition 2.1. *Let $\mathcal{A} = \{A_t\}_{t \in G}$ and $\mathcal{B} = \{B_t\}_{t \in G}$ be C^* -algebraic bundles over a finite group G . Let $C = \bigoplus_{t \in G} A_t$ and $D = \bigoplus_{t \in G} B_t$. Also, let $A = A_e$ and $B = B_e$, where e is the unit element in G . We suppose that C and D are unital and that A and B have the unit elements in C and D , respectively. Also, we suppose that there is an $\mathcal{A} - \mathcal{B}$ -equivalence bundle $\mathcal{X} = \{X_t\}_{t \in G}$ over G such that*

$${}_C \langle X_t, X_s \rangle = A_{ts^{-1}}, \quad \langle X_t, X_s \rangle_D = B_{t^{-1}s}$$

for any $t, s \in G$. Then the unital inclusions of unital C^* -algebras $A \subset C$ and $B \subset D$ are strongly Morita equivalent.

Proof. Let $Y = \bigoplus_{t \in G} X_t$ and $X = X_e$. By the above discussions and [10], Definition 2.1, we only have to show that

$${}_C \langle Y, X \rangle = C, \quad \langle Y, X \rangle_D = D.$$

Let $x \in X$ and $y = \sum_{t \in G} y_t \in Y$, where $y_t \in X_t$ for any $t \in G$. Then

$${}_C \langle y, x \rangle = \sum_{t \in G} {}_C \langle y_t, x \rangle, \quad \langle y, x \rangle_D = \sum_{t \in G} \langle y_t, x \rangle_D.$$

We note that ${}_C \langle y_t, x \rangle \in A_t$ and $\langle y_t, x \rangle_D \in B_t$ for any $t \in G$. Since ${}_D \langle X_t, X_s \rangle = A_{ts^{-1}}$ and $\langle X_t, X_s \rangle_D = B_{t^{-1}s}$ for any $t, s \in G$, by the above computations, we can see that

$${}_C \langle Y, X \rangle = C, \quad \langle Y, X \rangle_D = D.$$

Therefore we obtain the conclusion. □

Next, we give an example of an equivalence bundle $\mathcal{X} = \{X_t\}_{t \in G}$ over G satisfying the above properties. In order to do this, we prepare a lemma. Let $\mathcal{A} = \{A_t\}_{t \in G}$ and $\mathcal{B} = \{B_t\}_{t \in G}$ be as above. Let $\mathcal{X} = \{X_t\}_{t \in G}$ be a complex Banach bundle over G with the maps defined by

$$\begin{aligned} (y, d) \in Y \times D &\mapsto y \cdot d \in Y, & (y, z) \in Y \times Y &\mapsto \langle y, z \rangle_D \in D, \\ (c, y) \in C \times Y &\mapsto c \cdot y \in Y, & (y, z) \in Y \times Y &\mapsto {}_C \langle y, z \rangle \in C, \end{aligned}$$

where $Y = \bigoplus_{t \in G} X_t$.

Lemma 2.2. *With the above notation, we suppose that by the above maps, Y is a $C - D$ -equivalence bimodule satisfying that*

$${}_C \langle X_t, X_s \rangle = A_{ts^{-1}}, \quad \langle X_t, X_s \rangle_D = B_{t^{-1}s}$$

for any $t, s \in G$. If \mathcal{X} satisfies Conditions (1R)–(3R) and (1L)–(3L) in [1], Definition 2.1, then \mathcal{X} is an $\mathcal{A} - \mathcal{B}$ -equivalence bundle.

Proof. Since Y is a $C - D$ -equivalence bimodule, \mathcal{X} has Conditions (4R)–(6R) and (4L)–(6L) in [1], Definition 2.1 except that X_t is complete with the norms $\|\langle \cdot, \cdot \rangle_D\|^{1/2} = \|{}_C \langle \cdot, \cdot \rangle\|^{1/2}$ for any $t \in G$. But we know that if Y is complete with two different norms, then the two norms are equivalent. Hence, X_t is complete with the norms $\|\langle \cdot, \cdot \rangle_D\|^{1/2} = \|{}_C \langle \cdot, \cdot \rangle\|^{1/2}$ for any $t \in G$. Furthermore, since

$${}_C \langle X_t, X_s \rangle = A_{ts^{-1}}, \quad \langle X_t, X_s \rangle_D = B_{t^{-1}s}$$

for any $t, s \in G$, \mathcal{X} has Conditions (7R) and (7L) in [1], Definition 2.1. Therefore we obtain the conclusion. \square

We give an example of an $\mathcal{A} - \mathcal{B}$ -equivalence bundle $\mathcal{X} = \{X_t\}_{t \in G}$ such that

$${}_C \langle X_t, X_s \rangle = A_{ts^{-1}}, \quad \langle X_t, X_s \rangle_D = B_{t^{-1}s}$$

for any $t, s \in G$.

Example 2.3. Let G be a finite group. Let α be an action of G on a unital C^* -algebra A . Let u_t be implementing unitary elements of α , that is, $\alpha_t = \text{Ad}(u_t)$ for any $t \in G$. Then the crossed product of A by α , $A \rtimes_\alpha G$ is

$$A \rtimes_\alpha G = \left\{ \sum_{t \in G} a_t u_t : a_t \in A \text{ for any } t \in G \right\}.$$

Let $A_t = Au_t$ for any $t \in G$. By routine computations, we see that $\mathcal{A}_\alpha = \{A_t\}_{t \in G}$ is a C^* -algebraic bundle over G . We call \mathcal{A}_α the C^* -algebraic bundle over G induced by an action α . Let β be an action of G on a unital C^* -algebra B and let $\mathcal{B}_\beta = \{B_t\}_{t \in G}$

induced by β , where $B_t = Bv_t$ for any $t \in G$ and v_t are implementing unitary elements of β . We suppose that α and β are strongly Morita equivalent with respect to an action λ of G on an $A - B$ -equivalence bimodule X . Let $X \rtimes_\lambda G$ be the crossed product of X by λ defined by Kajiwara and Watatani (see [5], Definition 1.4), that is, the direct sum of n -copies of X as a vector space, where n is the order of G . And its elements are written as formal sums so that

$$X \rtimes_\lambda G = \left\{ \sum_{t \in G} x_t w_t : x_t \in X \text{ for any } t \in G \right\},$$

where w_t are indeterminates for all $t \in G$. Let $C = A \rtimes_\alpha G$, $D = B \rtimes_\beta G$ and $Y = X \rtimes_\lambda G$. Then by [5], Proposition 1.7, Y is a $C - D$ -equivalence bimodule, where we define the left C -action and the right D -action on Y by

$$(au_t) \cdot (xw_s) = (a \cdot \lambda_t(x))w_{ts}, \quad (xw_s) \cdot (bv_t) = (x \cdot \beta_s(b))v_{st}$$

for any $a \in A$, $b \in B$, $x \in X$ and $t, s \in G$ and we define the left C -valued inner product and the right D -valued inner product on Y by extending linearly the following:

$${}_C \langle xw_t, yw_s \rangle = {}_A \langle x, \lambda_{ts^{-1}}(y) \rangle u_{ts^{-1}}, \quad \langle xw_t, yw_s \rangle_D = \beta_{t^{-1}}(\langle x, y \rangle_B) v_{t^{-1}s}$$

for any $x, y \in X$, $t, s \in G$. Let $X_t = Xw_t$ for any $t \in G$ and $\mathcal{X}_\lambda = \{X_t\}_{t \in G}$. Then $Y = \bigoplus_{t \in G} X_t$. Also, \mathcal{X}_λ has Conditions (1R)–(3R) and (1L)–(3L) in [1], Definition 2.1.

Furthermore, X is an $A - B$ -equivalence bimodule and \mathcal{X}_λ satisfies

$${}_C \langle X_t, X_s \rangle = A_{ts^{-1}}, \quad \langle X_t, X_s \rangle_D = B_{t^{-1}s}$$

for any $t, s \in G$. Therefore \mathcal{X}_λ is an $\mathcal{A}_\alpha - \mathcal{A}_\beta$ -equivalence bundle by Lemma 2.2.

3. SATURATED C^* -ALGEBRAIC BUNDLES OVER A FINITE GROUP

Let $\mathcal{A} = \{A_t\}_{t \in G}$ be a saturated C^* -algebraic bundle over a finite group G . Let e be the unit element in G . Let $C = \bigoplus_{t \in G} A_t$ and $A = A_e$. We suppose that C is unital and that A has the unit element in C . Let E^A be the canonical conditional expectation from C onto A defined in Section 1, which is of Watatani index-finite type. Let C_1 be the C^* -basic construction of C and e_A the Jones' projection for E^A . By [9], Lemma 3.7, there is an action α^A of G on C_1 induced by \mathcal{A} defined as follows: Since \mathcal{A} is saturated and A is unital, there is a finite set $\{x_i^t\}_{i=1}^{n_t} \subset A_t$ such that $\sum_{i=1}^{n_t} x_i^t x_i^{t*} = 1$ for any $t \in G$. Let $e_t = \sum_{i=1}^{n_t} x_i^t e_A x_i^{t*}$ for all $t \in G$. Then by [9],

Lemmas 3.3, 3.5 and Remark 3.4, $\{e_t\}_{t \in G}$ are mutually orthogonal projections in $A' \cap C_1$, which are independent of the choice of $\{x_i^t\}_{i=1}^{n_t}$, with $\sum_{t \in G} e_t = 1$ such that C and e_t generate the C^* -algebra C_1 for all $t \in G$. We define α^A by $\alpha_t^A(c) = c$ and $\alpha_t^A(e_A) = e_{t^{-1}}$ for any $t \in G, c \in C$. Let $\mathcal{A}_1 = \{Y_{\alpha_t^A}\}_{t \in G}$ be the C^* -algebraic bundle over G induced by the action α^A of G which is defined in [9], Sections 5, 6, that is, let $Y_{\alpha_t^A} = e_A C_1 \alpha_t^A(e_A) = e_A C_1 e_{t^{-1}}$ for any $t \in G$. The product \bullet and the involution \sharp in \mathcal{A}_1 are defined as follows:

$$\begin{aligned} (x, y) \in Y_{\alpha_t^A} \times Y_{\alpha_s^A} &\mapsto x \bullet y = x \alpha_t^A(y) \in Y_{\alpha_{ts}^A}, \\ x \in Y_{\alpha_t^A} &\mapsto x^\sharp = \alpha_{t^{-1}}^A(x^*) \in Y_{\alpha_{t^{-1}}^A}. \end{aligned}$$

Lemma 3.1. *With the above notation, \mathcal{A} and \mathcal{A}_1 are isomorphic as C^* -algebraic bundles over G .*

Proof. Since $C_1 = C e_A C$, for any $t \in G$

$$Y_{\alpha_t^A} = e_A C e_A C e_{t^{-1}} = e_A A C e_{t^{-1}} = e_A C e_{t^{-1}}.$$

Let x be any element in C . Then we can write that $x = \sum_{s \in G} x_s$, where $x_s \in A_s$. Hence

$$\begin{aligned} e_A x e_{t^{-1}} &= \sum_{s,i} e_A x_s x_i^{t^{-1}} e_A x_i^{t^{-1}*} = \sum_{s,i} E^A(x_s x_i^{t^{-1}}) e_A x_i^{t^{-1}*} \\ &= \sum_i x_t x_i^{t^{-1}} e_A x_i^{t^{-1}*} = e_A x_t \sum_i x_i^{t^{-1}} x_i^{t^{-1}*} = e_A x_t. \end{aligned}$$

Thus, $Y_{\alpha_t^A} = e_A C e_{t^{-1}} = e_A A_t$ for any $t \in G$. Let π_t be the map from A_t to $Y_{\alpha_t^A}$ defined by $\pi_t(x) = e_A x$ for any $x \in A_t$ and $t \in G$. By the above discussions π_t is a linear map from A_t onto $Y_{\alpha_t^A}$. Then

$$\|\pi_t(x)\|^2 = \|e_A x x^* e_A\| = \|E^A(x x^*) e_A\| = \|E^A(x x^*)\| = \|x x^*\| = \|x\|^2.$$

Hence, π_t is injective for any $t \in G$. Thus, $A_t \cong e_A C_1 \alpha_t^A(e_A)$ as Banach spaces for any $t \in G$. Also, for any $x \in A_t, y \in A_s, t, s \in G$,

$$\begin{aligned} \pi_t(x) \bullet \pi_s(y) &= e_A x \alpha_t^A(e_A y) = e_A x e_{t^{-1}} y = e_A \sum_i x x_i^{t^{-1}} e_A x_i^{t^{-1}*} y \\ &= e_A \sum_i x x_i^{t^{-1}} x_i^{t^{-1}*} y = e_A x y = \pi_{ts}(xy), \\ \pi_t(x)^\sharp &= \alpha_{t^{-1}}^A(\pi_t(x)^*) = \alpha_{t^{-1}}^A((e_A x)^*) = \alpha_{t^{-1}}^A(x^* e_A) = x^* e_t \\ &= \sum_i x^* x_i^t e_A x_i^{t*} = e_A \sum_i x^* x_i^t x_i^{t*} = e_A x^* = \pi_{t^{-1}}(x^*). \end{aligned}$$

Therefore $\mathcal{A} = \{A_t\}_{t \in G}$ and $\mathcal{A}_1 = \{Y_{\alpha_t^A}\}_{t \in G}$ are isomorphic as C^* -algebraic bundles over G . \square

4. STRONG MORITA EQUIVALENCE FOR UNITAL INCLUSIONS OF UNITAL
 C^* -ALGEBRAS

Let $\mathcal{A} = \{A_t\}_{t \in G}$ and $\mathcal{B} = \{B_t\}_{t \in G}$ be saturated C^* -algebraic bundles over a finite group G . Let e be the unit element in G . Let $C = \bigoplus_{t \in G} A_t$, $D = \bigoplus_{t \in G} B_t$ and $A = A_e$, $B = B_e$. We suppose that C and D are unital and that A and B have the unit elements in C and D , respectively. Let E^A and E^B be the canonical conditional expectations from C and D onto A and B defined in Section 1, respectively. They are of Watatani index-finite type. Let $A \subset C$ and $B \subset D$ be the unital inclusions of unital C^* -algebras induced by \mathcal{A} and \mathcal{B} , respectively. We suppose that $A \subset C$ and $B \subset D$ are strongly Morita equivalent with respect to a $C - D$ -equivalence bimodule Y and its closed subspace X . Also, we suppose that $A' \cap C = \mathbf{C}1$. Then by [10], Lemma 10.3, $B' \cap D = \mathbf{C}1$ and by [7], Lemma 4.1 and its proof, there is a unique conditional expectation E^X from Y onto X with respect to E^A and E^B .

Let C_1 and D_1 be the C^* -basic constructions of C and D and e_A and e_B the Jones' projections for E^A and E^B , respectively. Let α^A and α^B be actions of G on C_1 and D_1 induced by \mathcal{A} and \mathcal{B} , respectively. Furthermore, let C_2 and D_2 be the C^* -basic constructions of C_1 and D_1 for the dual conditional expectations E^C of E^A and E^D of E^B , which are isomorphic to $C_1 \rtimes_{\alpha^A} G$ and $D_1 \rtimes_{\alpha^B} G$, respectively. We identify C_2 and D_2 with $C_1 \rtimes_{\alpha^A} G$ and $D_1 \rtimes_{\alpha^B} G$, respectively. By [10], Corollary 6.3, the unital inclusions $C_1 \subset C_2$ and $D_1 \subset D_2$ are strongly Morita equivalent with respect to a $C_2 - D_2$ -equivalence bimodule Y_2 and its closed subspace Y_1 , where Y_1 and Y_2 are the $C_1 - D_1$ -equivalence bimodule and the $C_2 - D_2$ -equivalence bimodule defined in [10], Section 6, respectively, and Y_1 is regarded as a closed subspace of Y_2 in the same way as in [10], Section 6. Also, $C_1' \cap C_2 = \mathbf{C}1$ by the proof of Watatani (see [13], Proposition 2.7.3) since $A' \cap C = \mathbf{C}1$. Hence, by [11], Corollary 6.5, there are an automorphism f of G , a $C_1 - D_1$ -equivalence bimodule Z and an action λ of G on Z such that α^A and β , the action of G on D_1 defined by $\beta_t(d) = \alpha_{f(t)}^B(d)$ for any $t \in G$, $d \in D$, are strongly Morita equivalent with respect to λ .

Let $\mathcal{A}_1 = \{Y_{\alpha_t^A}\}_{t \in G}$ and $\mathcal{B}_1 = \{Y_{\alpha_t^B}\}_{t \in G}$ be the C^* -algebraic bundles over G induced by the actions α^A and α^B , which are defined in Section 3. Furthermore, let $\mathcal{B}^f = \{B_{f(t)}\}_{t \in G}$ be the C^* -algebraic bundle over G induced by \mathcal{B} and f and let $\mathcal{B}_1^f = \{Y_{\beta_t}\}_{t \in G}$ be the C^* -algebraic bundle over G induced by the action β , which is defined in Section 3. We construct an $\mathcal{A}_1 - \mathcal{B}_1^f$ -equivalence bundle $\mathcal{Z} = \{Z_t\}_{t \in G}$ over G .

Let $Z_t = e_A \cdot Z \cdot \beta_t(e_B)$ for any $t \in G$ and let $W = \bigoplus_{t \in G} Z_t$. Also, by Lemma 3.1 and its proof, $\bigoplus_{t \in G} Y_{\alpha_t^A} \cong C$ and $\bigoplus_{t \in G} Y_{\beta_t} \cong D$ as C^* -algebras. We identify $\bigoplus_{t \in G} Y_{\alpha_t^A}$ and $\bigoplus_{t \in G} Y_{\beta_t}$ with C and D , respectively. We define the left C -action \diamond and the left

C -valued inner product ${}_C\langle \cdot, \cdot \rangle$ on W by

$$\begin{aligned} e_A x \alpha_t^A(e_A) \diamond [e_A \cdot z \cdot \beta_s(e_B)] &\stackrel{\text{def}}{=} e_A x \alpha_t^A(e_A) \cdot \lambda_t(e_A \cdot z \cdot \beta_s(e_B)) \\ &= e_A \cdot [x \alpha_t^A(e_A) \cdot \lambda_t(z)] \cdot \beta_{ts}(e_B), \\ {}_C\langle e_A \cdot w \cdot \beta_t(e_B), e_A \cdot z \cdot \beta_s(e_B) \rangle &\stackrel{\text{def}}{=} C_1 \langle e_A \cdot w \cdot \beta_t(e_B), \lambda_{ts^{-1}}(e_A \cdot z \cdot \beta_s(e_B)) \rangle \\ &= e_{AC_1} \langle w \cdot \beta_t(e_B), \lambda_{ts^{-1}}(z) \cdot \beta_t(e_B) \rangle \alpha_{ts^{-1}}^A(e_A), \end{aligned}$$

where $e_A x \alpha_t^A(e_A) \in e_A C_1 \alpha_t^A(e_A)$, $e_A \cdot z \cdot \beta_s(e_B) \in Z_s$, $e_A \cdot w \cdot \beta_t(e_B) \in Z_t$. Also, we define the right D -action, which is also denoted by the same symbol \diamond and the D -valued inner product $\langle \cdot, \cdot \rangle_D$ on W by

$$\begin{aligned} [e_A \cdot z \cdot \beta_t(e_B)] \diamond e_B x \beta_s(e_B) &\stackrel{\text{def}}{=} e_A \cdot z \cdot \beta_t(e_B) \beta_t(x) \beta_{ts}(e_B) \\ &= e_A \cdot [z \cdot \beta_t(e_B) \beta_t(x)] \cdot \beta_{ts}(e_B), \\ \langle e_A \cdot z \cdot \beta_t(e_B), e_A \cdot w \cdot \beta_s(e_B) \rangle_D &\stackrel{\text{def}}{=} \beta_{t^{-1}}(\langle e_A \cdot z \cdot \beta_t(e_B), e_A \cdot w \cdot \beta_s(e_B) \rangle_{D_1}) \\ &= e_B \beta_{t^{-1}}(\langle e_A \cdot z, e_A \cdot w \rangle_{D_1}) \beta_{t^{-1}s}(e_B), \end{aligned}$$

where $e_B x \beta_s(e_B) \in e_B D_1 \beta_s(e_B)$, $e_A \cdot z \cdot \beta_t(e_B) \in Z_t$, $e_A \cdot w \cdot \beta_s(e_B) \in Z_s$. By the above definitions, \mathcal{Z} has Conditions (1R)–(3R) and (1L)–(3L) in [1], Definition 2.1. We show that \mathcal{Z} has Conditions (4R) and (4L) in [1], Definition 2.1 and that \mathcal{Z} is an $\mathcal{A}_1 - \mathcal{B}_1^f$ -bundle in the same way as in Example 2.3.

Lemma 4.1. *With the above notation, \mathcal{Z} has Conditions (4R) and (4L) in [1], Definition 2.1.*

Proof. Let $e_A \cdot z \cdot \beta_t(e_B) \in Z_t$, $e_A \cdot w \cdot \beta_s(e_B) \in Z_s$ and $e_B x \beta_r(e_B) \in e_B D_1 \beta_r(e_B)$, where $t, s, r \in G$. Then by routine computations, we can see that

$$\begin{aligned} \langle e_A \cdot z \cdot \beta_t(e_B), [e_A \cdot w \cdot \beta_s(e_B)] \diamond e_B x \beta_r(e_B) \rangle_D \\ = \langle e_A \cdot z \cdot \beta_t(e_B), e_A \cdot w \cdot \beta_s(e_B) \rangle_D \bullet e_B x \beta_r(e_B) \end{aligned}$$

and that

$$\langle e_A \cdot z \cdot \beta_t(e_B), e_A \cdot w \cdot \beta_s(e_B) \rangle_D^\# = \langle e_A \cdot w \cdot \beta_s(e_B), e_A \cdot z \cdot \beta_t(e_B) \rangle_D.$$

Hence, \mathcal{Z} has Condition (4R) in [1], Definition 2.1. Next, let $e_A \cdot z \cdot \beta_t(e_B) \in Z_t$, $e_A \cdot w \cdot \beta_s(e_B) \in Z_s$ and $e_A x \alpha_r^A(e_A) \in e_A C_1 \alpha_r^A(e_A)$, where $t, s, r \in G$. Then by routine computations, we can see that

$$\begin{aligned} {}_C\langle e_A x \alpha_r^A(e_A) \diamond [e_A \cdot z \cdot \beta_t(e_B)], e_A \cdot w \cdot \beta_s(e_B) \rangle \\ = e_A x \alpha_r^A(e_A) \bullet {}_C\langle e_A \cdot z \cdot \beta_t(e_B), e_A \cdot w \cdot \beta_s(e_B) \rangle, \\ {}_C\langle e_A \cdot z \cdot \beta_t(e_B), e_A \cdot w \cdot \beta_s(e_B) \rangle^\# = {}_C\langle e_A \cdot w \cdot \beta_s(e_B), e_A \cdot z \cdot \beta_t(e_B) \rangle. \end{aligned}$$

Hence, \mathcal{Z} has Condition (4L) in [1], Definition 2.1. □

By Lemma 4.1, W is a C – D -bimodule having Properties (1)–(6) in [5], Lemma 1.3. In order to prove that \mathcal{Z} has Conditions (5R), (6R) and (5L), (6L) in [1], Definition 2.1 using [5], Lemma 1.3, we show that W has Properties (7)–(10) in [5], Lemma 1.3.

Lemma 4.2. *With the above notation, W has the following:*

- (1) $(e_A x \alpha_t^A(e_A) \diamond [e_A \cdot z \cdot \beta_s(e_B)]) \diamond e_B y \beta_r(e_B) = e_A x \alpha_t^A(e_A) \diamond ([e_A \cdot z \cdot \beta_s(e_B)] \diamond e_B y \beta_r(e_B))$,
- (2) $\langle e_A x \alpha_t^A(e_A) \diamond [e_A \cdot z \cdot \beta_s(e_B)], e_A \cdot w \cdot \beta_r(e_B) \rangle_D = \langle e_A \cdot z \cdot \beta_s(e_B), (e_A x \alpha_t^A(e_A))^\# \diamond [e_A \cdot w \cdot \beta_r(e_B)] \rangle_D$,
- (3) ${}_C \langle e_A \cdot z \cdot \beta_s(e_B), [e_A \cdot w \cdot \beta_r(e_B)] \diamond e_B y \beta_t(e_B) \rangle = {}_C \langle [e_A \cdot z \cdot \beta_s(e_B)] \diamond (e_B y \beta_t(e_B))^\#, e_A \cdot w \cdot \beta_r(e_B) \rangle$,

where $x \in C_1$, $y \in D_1$, $z, w \in Z$, $t, s, r \in G$.

Proof. We can show the lemma by routine computations. □

By Lemma 4.2, W has Properties (7), (8) in [5], Lemma 1.3.

Lemma 4.3. *With the above notation, there are finite subsets $\{u_i\}_i$ and $\{v_j\}_j$ of W such that*

$$\sum_i u_i \diamond \langle u_i, x \rangle_D = x = \sum_j {}_C \langle x, v_j \rangle \diamond v_j \quad \text{for any } x \in W.$$

Proof. Since Z is a C_1 – D_1 -equivalence bimodule, there are finite subsets $\{z_i\}_i$ and $\{w_j\}_j$ of Z such that

$$\sum_i z_i \cdot \langle z_i, z \rangle_{D_1} = z = \sum_j {}_{C_1} \langle z, w_j \rangle \cdot w_j$$

for any $z \in Z$. Then for any $z \in Z$, $s \in G$,

$$\begin{aligned} & \sum_{i,t} [e_A \cdot z_i \cdot \beta_t(e_B)] \diamond \langle e_A \cdot z_i \cdot \beta_t(e_B), e_A \cdot z \cdot \beta_s(e_B) \rangle_D \\ &= \sum_{i,t} e_A \cdot z_i \cdot \beta_t(e_B) \diamond \beta_{t-1}(\langle e_A \cdot z_i \cdot \beta_t(e_B), e_A \cdot z \cdot \beta_s(e_B) \rangle_{D_1}) \\ &= \sum_{i,t} e_A \cdot z_i \cdot \beta_t(e_B) \diamond e_B \beta_{t-1}(\langle e_A \cdot z_i, e_A \cdot z \rangle_{D_1}) \beta_{t-1s}(e_B) \\ &= \sum_{i,t} e_A \cdot z_i \cdot \beta_t(e_B) \langle e_A \cdot z_i, e_A \cdot z \rangle_{D_1} \beta_s(e_B) \\ &= \sum_{i,t} e_A \cdot [z_i \cdot \langle z_i \cdot \beta_t(e_B), e_A \cdot z \rangle_{D_1}] \cdot \beta_s(e_B) \\ &= \sum_i e_A \cdot [z_i \cdot \langle z_i, e_A \cdot z \rangle_{D_1}] \cdot \beta_s(e_B) = e_A \cdot z \cdot \beta_s(e_B) \end{aligned}$$

since $\sum_{t \in G} \beta_t(e_B) = 1$ by [9], Remark 3.4. Also, by the same way and the same reason, for any $z \in Z$, $s \in G$,

$$\sum_{j,t} {}_C \langle e_A \cdot z \cdot \beta_s(e_B), e_A \cdot \lambda_t(w_j) \cdot \beta_t(e_B) \rangle \diamond [e_A \cdot \lambda_t(w_j) \cdot \beta_t(e_B)] = e_A \cdot z \cdot \beta_s(e_B).$$

Therefore we obtain the conclusion. \square

Remark 4.4. By Lemma 4.2, $\{e_A \cdot z_i \cdot \beta_t(e_B)\}_{i,t}$ is a right D -basis and $\{e_A \cdot \lambda_t(w_j) \cdot \beta_t(e_B)\}_{j,t}$ is a left C -basis of W in the sense of Kajiwara and Watatani (see [6]).

By Lemma 4.2, W has Properties (9), (10) in [5], Lemma 1.3. Hence, by [5], Lemma 1.3, W is a Hilbert $C - D$ -bimodule in the sense of [5], Definition 1.1. Thus, Z has Conditions (5R), (6R) and (5L), (6L) in [1], Definition 2.1.

Proposition 4.5. *With the above notation, Z is an $\mathcal{A}_1 - \mathcal{B}_1^f$ -equivalence bundle over G such that*

$${}_{\mathcal{A}_1} \langle Z_t, Z_s \rangle = Y_{\alpha_{ts}^{\mathcal{A}}}, \quad \langle Z_t, Z_s \rangle_{\mathcal{B}_1^f} = Y_{\beta_{t^{-1}s}}$$

for any $t, s \in G$.

Proof. First, we show that the left C -valued inner product and the right D -valued inner product on W are compatible. Let $y, z, w \in Z$ and $t, s, r \in G$. Since Z is a $C_1 - D_1$ -equivalence bimodule, by routine computations, we can see that

$$\begin{aligned} {}_C \langle e_A \cdot z \cdot \beta_t(e_B), e_A \cdot y \cdot \beta_s(e_B) \rangle \diamond [e_A \cdot w \cdot \beta_r(e_B)] \\ = [e_A \cdot z \cdot \beta_t(e_B)] \diamond \langle e_A \cdot y \cdot \beta_s(e_B), e_A \cdot w \cdot \beta_r(e_B) \rangle_D. \end{aligned}$$

Hence, the left C -valued inner product and the right D -valued inner product are compatible. Thus, by Lemmas 4.1–4.3, Z is an $\mathcal{A}_1 - \mathcal{B}_1^f$ -equivalence bundle over G . Next, we show that

$${}_{\mathcal{A}_1} \langle Z_t, Z_s \rangle = Y_{\alpha_{ts}^{\mathcal{A}}}, \quad \langle Z_t, Z_s \rangle_{\mathcal{B}_1^f} = Y_{\beta_{t^{-1}s}}$$

for any $t, s \in G$. Let $t, s \in G$. Since E^B is of Watatani index-finite type, there is a quasi-basis $\{(d_j, d_j^*)\} \subset D \times D$ for E^B . Thus $\sum_j d_j e_B d_j^* = 1$. Since Z is a $C_1 - D_1$ -equivalence bimodule, there is a finite subset $\{z_i\}$ of Z such that $\sum_i {}_{C_1} \langle z_i, z_i \rangle = 1$.

Let $c \in C$. Then

$$\begin{aligned}
& \sum_{i,j} {}_C \langle e_{AC} \cdot \lambda_t(z_i) \cdot \beta_t(d_j e_B), e_A \cdot \lambda_s(z_i) \cdot \beta_s(d_j e_B) \rangle \\
&= \sum_{i,j} C_1 \langle e_{AC} \cdot \lambda_t(z_i) \cdot \beta_t(d_j e_B), \lambda_{ts^{-1}}(e_A \cdot \lambda_s(z_i) \cdot \beta_s(d_j e_B)) \rangle \\
&= \sum_{i,j} C_1 \langle e_{AC} \cdot \lambda_t(z_i) \cdot \beta_t(d_j e_B), \alpha_{ts^{-1}}^A(e_A) \cdot \lambda_t(z_i) \cdot \beta_t(d_j e_B) \rangle \\
&= \sum_{i,j} e_{AC_1} \langle c \cdot \lambda_t(z_i) \cdot \beta_t(d_j e_B d_j^*), \lambda_t(z_i) \rangle \alpha_{ts^{-1}}^A(e_A) \\
&= \sum_i e_{AC_1} \langle \lambda_t(z_i), \lambda_t(z_i) \rangle \alpha_{ts^{-1}}^A(e_A) \\
&= \sum_i e_{AC} \alpha_t^A(C_1 \langle z_i, z_i \rangle) \alpha_{ts^{-1}}^A(e_A) = e_{AC} \alpha_{ts^{-1}}^A(e_A).
\end{aligned}$$

Hence, we obtain that ${}_C \langle Z_t, Z_s \rangle = Y_{\alpha_{ts^{-1}}^A}$ for any $t, s \in G$. Also, since E^A is of Watatani index-finite type, there is a quasi-basis $\{(c_j, c_j^*)\} \subset C \times C$ for E^A . Thus $\sum_j c_j e_A c_j^* = 1$. Since Z is a $C_1 - D_1$ -equivalence bimodule, there is a finite subset $\{w_i\}$ of Z such that $\sum_i \langle w_i, w_i \rangle_{D_1} = 1$. In the same way as above, for any $d \in D_1$,

$$\sum_{i,j} \langle e_{AC_j^*} \cdot w_i \cdot \beta_t(e_B), e_{AC_j^*} \cdot w_i \cdot d \beta_s(e_B) \rangle_D = e_B \beta_{t^{-1}}(d) \beta_{t^{-1}s}(e_B).$$

Hence, we obtain that $\langle Z_t, Z_s \rangle_D = Y_{\beta_{t^{-1}s}}$ for any $t, s \in G$. Therefore we obtain the conclusion. \square

Theorem 4.6. *Let $\mathcal{A} = \{A_t\}_{t \in G}$ and $\mathcal{B} = \{B_t\}_{t \in G}$ be saturated C^* -algebraic bundles over a finite group G . Let e be the unit element in G . Let $C = \bigoplus_{t \in G} A_t$, $D = \bigoplus_{t \in G} B_t$ and $A = A_e$, $B = B_e$. We suppose that C and D are unital and that A and B have the unit elements in C and D , respectively. Let $A \subset C$ and $B \subset D$ be the unital inclusions of unital C^* -algebras induced by \mathcal{A} and \mathcal{B} , respectively. Also, we suppose that $A' \cap C = \mathbf{C}1$. If $A \subset C$ and $B \subset D$ are strongly Morita equivalent, then there are an automorphism f of G and an $\mathcal{A} - \mathcal{B}^f$ -equivalence bundle $\mathcal{Z} = \{Z_t\}_{t \in G}$ satisfying that*

$${}_C \langle Z_t, Z_s \rangle = A_{ts^{-1}}, \quad \langle Z_t, Z_s \rangle_D = B_{f(t^{-1}s)}$$

for any $t, s \in G$, where \mathcal{B}^f is the C^* -algebraic bundle over G induced by \mathcal{B} and f defined by $\mathcal{B}^f = \{B_{f(t)}\}_{t \in G}$.

Proof. This is immediate by Lemma 3.1 and Proposition 4.5. \square

5. APPLICATION

Let A and B be unital C^* -algebras and X a Hilbert $A - B$ -bimodule. Let \tilde{X} be its dual Hilbert $B - A$ -bimodule. For any $x \in X$, \tilde{x} denotes the element in \tilde{X} induced by $x \in X$.

Lemma 5.1. *Let A, B and C be unital C^* -algebras. Let X be a Hilbert $A - B$ -bimodule and Y a Hilbert $B - C$ -bimodule. Then $\widetilde{X \otimes_B Y} \cong \tilde{Y} \otimes_B \tilde{X}$ as Hilbert $C - A$ -bimodules.*

Proof. Let π be the map from $\widetilde{X \otimes_B Y}$ to $\tilde{Y} \otimes_B \tilde{X}$ defined by $\pi(\widetilde{x \otimes y}) = \tilde{y} \otimes \tilde{x}$ for any $x \in X, y \in Y$. Then by routine computations, we can see that π is a Hilbert $C - A$ -bimodule isomorphism of $\widetilde{X \otimes_B Y}$ onto $\tilde{Y} \otimes_B \tilde{X}$. \square

We identify $\widetilde{X \otimes_B Y}$ with $\tilde{Y} \otimes_B \tilde{X}$ by the isomorphism π defined in the proof of Lemma 5.1. Next, we give the definition of an involutive Hilbert $A - A$ -bimodule modifying [8].

Definition 5.2. We say that a Hilbert $A - A$ -bimodule X is *involutive* if there exists a conjugate linear map $x \in X \mapsto x^\natural \in X$ such that

- (1) $(x^\natural)^\natural = x, x \in X,$
- (2) $(a \cdot x \cdot b)^\natural = b^* \cdot x^\natural \cdot a^*, x \in X, a, b \in A,$
- (3) ${}_A \langle x, y^\natural \rangle = \langle x^\natural, y \rangle_A, x, y \in X.$

We call the above conjugate linear map \natural an *involution* on X . If X is full with the both inner products, X is an involutive $A - A$ -equivalence bimodule. For each involutive Hilbert $A - A$ -bimodule, let L_X be the linking C^* -algebra induced by X and C_X the C^* -subalgebra of L_X , which is defined in [8], that is,

$$C_X = \left\{ \begin{bmatrix} a & x \\ \tilde{x}^\natural & a \end{bmatrix} : a \in A, x \in X \right\}.$$

We note that C_X acts on $X \oplus A$ (see Brown, Green and Rieffel [2] and Rieffel [12]). The norm of C_X is defined as the operator norm on $X \oplus A$.

Let A be a unital C^* -algebra and X an involutive Hilbert $A - A$ -bimodule. Let \tilde{X} be its dual Hilbert $A - A$ -bimodule. We define the map \natural on \tilde{X} by $(\tilde{x})^\natural = \widetilde{(x^\natural)}$ for any $\tilde{x} \in \tilde{X}$.

Lemma 5.3. *With the above notation, the above map \natural is an involution on \tilde{X} .*

Proof. This is immediate by direct computations. \square

For each involutive Hilbert $A - A$ -bimodule X , we regard \widetilde{X} as an involutive $A - A$ -bimodule in the same manner as in Lemma 5.3.

Let $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$ and we suppose that \mathbf{Z}_2 consists of the unit element 0 and another element 1. Let X be an involutive Hilbert $A - A$ -bimodule. We construct a C^* -algebraic bundle over \mathbf{Z}_2 induced by X . Let $A_0 = A$ and $A_1 = X$. Let $\mathcal{A}_X = \{A_t\}_{t \in \mathbf{Z}_2}$. We define a product \bullet and an involution \sharp as follows:

- (1) $a \bullet b = ab$, $a, b \in A$,
- (2) $a \bullet x = a \cdot x$, $x \bullet a = x \cdot a$, $a \in A$, $x \in X$,
- (3) $x \bullet y = {}_A \langle x, y^\sharp \rangle = \langle x^\sharp, y \rangle_A$, $x, y \in X$,
- (4) $a^\sharp = a^*$, $a \in A$,
- (5) $x^\sharp = x^\natural$, $x \in X$.

Then $A \oplus X$ is a $*$ -algebra and by routine computations, $A \oplus X$ is isomorphic to C_X as $*$ -algebras. We identify $A \oplus X$ with C_X as $*$ -algebras. We define a norm of $A \oplus X$ as the operator norm on $X \oplus A$. Hence, \mathcal{A}_X is a C^* -algebraic bundle over \mathbf{Z}_2 . Thus, we obtain a correspondence from the involutive Hilbert $A - A$ -bimodules to the C^* -algebraic bundles over \mathbf{Z}_2 . Next, let $\mathcal{A} = \{A_t\}_{t \in \mathbf{Z}_2}$ be a C^* -algebraic bundle over \mathbf{Z}_2 . Then A_1 is an involutive Hilbert $A - A$ -bimodule. Hence, we obtain a correspondence from the C^* -algebraic bundles over \mathbf{Z}_2 to the involutive Hilbert $A - A$ -bimodules. Clearly, the above two correspondences are the inverse correspondences of each other. Furthermore, the inclusion of unital C^* -algebras $A \subset C_X$ induced by X and the inclusion of unital C^* -algebras $A \subset A \oplus X$ induced by the C^* -algebraic bundle \mathcal{A}_X coincide.

Lemma 5.4. *Let X and Y be involutive Hilbert $A - A$ -bimodules and \mathcal{A}_X and \mathcal{A}_Y the C^* -algebraic bundles over \mathbf{Z}_2 induced by X and Y , respectively. Then $\mathcal{A}_X \cong \mathcal{A}_Y$ as C^* -algebraic bundles over \mathbf{Z}_2 if and only if $X \cong Y$ as involutive Hilbert $A - A$ -bimodules.*

Proof. We suppose that $\mathcal{A}_X \cong \mathcal{A}_Y$ as C^* -algebraic bundles over \mathbf{Z}_2 . Then there is a C^* -algebraic bundle isomorphism $\{\pi_t\}_{t \in \mathbf{Z}_2}$ of \mathcal{A}_X onto \mathcal{A}_Y . We identify A with $\pi_0(A)$. Then π_1 is an involutive Hilbert $A - A$ -bimodule isomorphism of X onto Y . Next, we suppose that there is an involutive Hilbert $A - A$ -bimodule isomorphism π of X onto Y . Let $\pi_0 = \text{id}_A$ and $\pi_1 = \pi$. Then $\{\pi_t\}_{t \in \mathbf{Z}_2}$ is a C^* -algebraic bundle isomorphism \mathcal{A}_X onto \mathcal{A}_Y . \square

Lemma 5.5. *Let X be an involutive Hilbert $A - A$ -bimodule and \mathcal{A}_X the C^* -algebraic bundle over \mathbf{Z}_2 induced by X . Then X is full with the both inner products if and only if \mathcal{A}_X is saturated.*

Proof. We suppose that X is full with the both inner products. Then

$$A_1 \bullet A_1^\sharp = {}_A\langle X, X \rangle = A = A_0.$$

Also,

$$\begin{aligned} A_0 \bullet A_1^\sharp &= A \cdot X^\natural = A \cdot X = X = A_1, \\ A_1 \bullet A_0^\sharp &= X \cdot A^* = X \cdot A = X = A_1 \end{aligned}$$

by [3], Proposition 1.7. Clearly $A_0 \bullet A_0 = AA = A = A_0$. Hence \mathcal{A}_X is saturated. Next, we suppose that \mathcal{A}_X is saturated. Then

$${}_A\langle X, X \rangle = A_1 \bullet A_1^\sharp = A_1 = A, \quad \langle X, X \rangle_A = {}_A\langle X^\natural, X^\natural \rangle = {}_A\langle X, X \rangle = A.$$

Thus, X is full with the both inner products. \square

Remark 5.6. Let X be an involutive Hilbert $A - A$ -bimodule. Then by the above proof, we see that X is full with the left A -valued inner product if and only if X is full with the right A -valued inner product.

Lemma 5.7. Let A and B be unital C^* -algebras and M an $A - B$ -equivalence bimodule. Let X be an involutive Hilbert $A - A$ -bimodule. Then $\widetilde{M} \otimes_A X \otimes_A M$ is an involutive Hilbert $B - B$ -bimodule whose involution \natural is defined by

$$(\widetilde{m} \otimes x \otimes n)^\natural = \widetilde{n} \otimes x^\natural \otimes m$$

for any $m, n \in M$, $x \in X$.

Proof. This is immediate by routine computations. \square

Let A, B, X and M be as in Lemma 5.7. Let Y be an involutive Hilbert $B - B$ -bimodule. We suppose that there is an involutive Hilbert $B - B$ -bimodule isomorphism Φ of $\widetilde{M} \otimes_A X \otimes_A M$ onto Y . Let $\widetilde{\Phi}$ be the linear map from $\widetilde{M} \otimes_A \widetilde{X} \otimes_A M$ onto \widetilde{Y} defined by

$$\widetilde{\Phi}(\widetilde{m} \otimes \widetilde{x} \otimes n) = \widetilde{\Phi}((\widetilde{n} \otimes x \otimes m)^\sim) = [\Phi(\widetilde{n} \otimes x \otimes m)]^\sim$$

for any $m, n \in M$, $x \in X$.

Lemma 5.8. With the above notation, $\widetilde{\Phi}$ is an involutive Hilbert $B - B$ -bimodule isomorphism of $\widetilde{M} \otimes_A \widetilde{X} \otimes_A M$ onto \widetilde{Y} .

Proof. This is immediate by routine computations. \square

Again, let A, B, X and M be as in Lemma 5.7. Let Y be an involutive Hilbert $B - B$ -bimodule. We suppose that there is an involutive Hilbert $B - B$ -bimodule isomorphism Φ of $\widetilde{M} \otimes_A X \otimes_A M$ onto Y . We identify A and X with $M \otimes_B \widetilde{M}$ and $A \otimes_A X$ by the isomorphisms defined by

$$m \otimes n \in M \otimes_B \widetilde{M} \mapsto_A \langle m, n \rangle \in A, \quad a \otimes x \in A \otimes_A X \mapsto a \cdot x \in X,$$

respectively. Since M is an $A - B$ -equivalence bimodule, there is a finite subset $\{u_i\}$ of M with $\sum_i \langle u_i, u_i \rangle = 1$. Let $x \in X, m \in M$. Then

$$x \otimes m = 1_A \cdot x \otimes m = \sum_i \langle u_i, u_i \rangle \cdot x \otimes m = \sum_i u_i \otimes \widetilde{u}_i \otimes x \otimes m.$$

Hence, there is the linear map Ψ from $X \otimes_A M$ to $M \otimes_B Y$ defined by

$$\Psi(x \otimes m) = \sum_i u_i \otimes \Phi(\widetilde{u}_i \otimes x \otimes m)$$

for any $x \in X, m \in M$. By the definition of Ψ , we can see that Ψ is a Hilbert $A - B$ -bimodule isomorphism of $X \otimes_A M$ onto $M \otimes_B Y$.

Lemma 5.9. *With the above notation, the Hilbert $A - B$ -bimodule isomorphism Ψ of $X \otimes_A M$ onto $M \otimes_B Y$ is independent of the choice of a finite subset $\{u_i\}$ of M with $\sum_i \langle u_i, u_i \rangle = 1$.*

Proof. Let $\{v_j\}$ be another finite subset of M with $\sum_j \langle v_j, v_j \rangle = 1$. Then for any $x \in X, m \in M$,

$$\begin{aligned} & \sum_i u_i \otimes \Phi(\widetilde{u}_i \otimes x \otimes m) \\ &= \sum_{i,j} \langle v_j, v_j \rangle \cdot u_i \otimes \Phi(\widetilde{u}_i \otimes x \otimes m) = \sum_{i,j} v_j \cdot \langle v_j, u_i \rangle_B \otimes \Phi(\widetilde{u}_i \otimes x \otimes m) \\ &= \sum_{i,j} v_j \otimes \Phi([u_i \cdot \langle u_i, v_j \rangle_B]^\sim \otimes x \otimes m) = \sum_j v_j \otimes \Phi(\widetilde{v}_j \otimes x \otimes m). \end{aligned}$$

Therefore, we obtain the conclusion. \square

Similarly, let $\widetilde{\Psi}$ be the Hilbert $A - B$ -bimodule isomorphism of $\widetilde{X} \otimes_A M$ onto $M \otimes_B \widetilde{Y}$ defined by

$$\widetilde{\Psi}(\widetilde{x} \otimes m) = \sum_i u_i \otimes \widetilde{\Phi}(\widetilde{u}_i \otimes \widetilde{x} \otimes m)$$

for any $x \in X$, $m \in M$. We construct the inverse map of Ψ , which is a Hilbert $A - B$ -bimodule isomorphism of $M \otimes_B Y$ onto $X \otimes_A M$. Let Θ be the linear map from $M \otimes_B Y$ to $X \otimes_A M$ defined by

$$\Theta(m \otimes y) = m \otimes \Phi^{-1}(y)$$

for any $m \in M$, $y \in Y$, where we identify $M \otimes_B \widetilde{M} \otimes_A X \otimes_A M$ with $X \otimes_A M$ as Hilbert $A - B$ -bimodules by the map

$$m \otimes \tilde{n} \otimes x \otimes m_1 \in M \otimes_B \widetilde{M} \otimes_A X \otimes_A M \mapsto {}_A\langle m, n \rangle \cdot x \otimes m_1 \in X \otimes_A M.$$

Lemma 5.10. *With the above notation, Θ is the Hilbert $A - B$ -bimodule isomorphism of $M \otimes_B Y$ onto $X \otimes_A M$ such that $\Theta \circ \Psi = \text{id}_{X \otimes_A M}$ and $\Psi \circ \Theta = \text{id}_{M \otimes_B Y}$.*

Proof. Let $m, m_1 \in M$, $y, y_1 \in Y$. Then

$$\begin{aligned} {}_A\langle \Theta(m \otimes y), \Theta(m_1 \otimes y_1) \rangle &= {}_A\langle m \otimes \Phi^{-1}(y), m_1 \otimes \Phi^{-1}(y_1) \rangle \\ &= {}_A\langle m \cdot {}_B\langle \Phi^{-1}(y), \Phi^{-1}(y_1) \rangle, m_1 \rangle \\ &= {}_A\langle m \cdot {}_B\langle y, y_1 \rangle, m_1 \rangle = {}_A\langle m \otimes y, m_1 \otimes y_1 \rangle. \end{aligned}$$

Hence, Θ preserves the left A -valued inner products. Similarly, we can see that Θ preserves the right B -valued inner products. Furthermore, for any $x \in X$, $m \in M$,

$$\begin{aligned} (\Theta \circ \Psi)(x \otimes m) &= \sum_i \Theta(u_i \otimes \Phi(\tilde{u}_i \otimes x \otimes m)) = \sum_i u_i \otimes \tilde{u}_i \otimes x \otimes m \\ &= \sum_i {}_A\langle u_i, u_i \rangle \cdot x \otimes m = x \otimes m \end{aligned}$$

since we identify $M \otimes \widetilde{M}$ with A as $A - A$ -equivalence bimodules by the map $m \otimes \tilde{n} \in M \otimes_B \widetilde{M} \mapsto {}_A\langle m, n \rangle \in A$. Hence, $\Theta \circ \Psi = \text{id}_{X \otimes_A M}$. Hence, $\Psi \circ \Theta \circ \Psi = \Psi$ on $X \otimes_A M$. Since Ψ is surjective, $\Psi \circ \Theta = \text{id}_{M \otimes_B Y}$. Therefore, by the remark after [4], Definition 1.1.18, Θ is a Hilbert $A - B$ -bimodule isomorphism of $M \otimes_B Y$ onto $X \otimes_A M$ such that $\Theta \circ \Psi = \text{id}_{X \otimes_A M}$ and $\Psi \circ \Theta = \text{id}_{M \otimes_B Y}$. \square

Similarly, we see that the inverse map of $(\tilde{\Psi})^{-1}$ is defined by

$$(\tilde{\Psi})^{-1}(m \otimes \tilde{y}) = m \otimes (\tilde{\Phi})^{-1}(\tilde{y})$$

for any $m \in M$, $y \in Y$, where we identify $M \otimes_B \widetilde{M} \otimes_A \tilde{X} \otimes_A M$ with $\tilde{X} \otimes_A M$ as Hilbert $A - B$ -bimodules by the map

$$m \otimes \tilde{n} \otimes \tilde{x} \otimes m_1 \in M \otimes_B \widetilde{M} \otimes_A \tilde{X} \otimes_A M \mapsto {}_A\langle m, n \rangle \cdot \tilde{x} \otimes m_1 \in \tilde{X} \otimes_A M.$$

We prepare some lemmas in order to show Proposition 5.14.

Lemma 5.11. *Let A and B be unital C^* -algebras. Let X and Y be an involutive Hilbert $A - A$ -bimodule and an involutive Hilbert $B - B$ -bimodule, respectively. Let $\mathcal{A}_X = \{A_t\}_{t \in \mathbf{Z}_2}$ and $\mathcal{A}_Y = \{B_t\}_{t \in \mathbf{Z}_2}$ be C^* -algebraic bundles over \mathbf{Z}_2 induced by X and Y , respectively. We suppose that there is an $\mathcal{A}_X - \mathcal{A}_Y$ -equivalence bundle $\mathcal{M} = \{M_t\}_{t \in \mathbf{Z}_2}$ over \mathbf{Z}_2 such that*

$${}_C \langle M_t, M_s \rangle = A_{ts^{-1}}, \quad \langle M_t, M_s \rangle_D = B_{t^{-1}s}$$

for any $t, s \in \mathbf{Z}_2$, where $C = A \oplus X$ and $D = B \oplus Y$. Then there is an $A - B$ -equivalence bimodule M such that $Y \cong \widetilde{M} \otimes_A X \otimes_A M$ as involutive Hilbert $B - B$ -bimodules.

Proof. By the assumptions, M_0 is an $A - B$ -equivalence bimodule. Let $M = M_0$. Then by Lemma 5.7, $\widetilde{M} \otimes_A X \otimes_A M$ is an involutive Hilbert $B - B$ -bimodule whose involution is defined by $(\widetilde{m} \otimes x \otimes n)^\sharp = \widetilde{n} \otimes x^\sharp \otimes m$ for any $m, n \in M$, $x \in X$. We show that $Y \cong \widetilde{M} \otimes_A X \otimes_A M$ as involutive Hilbert $B - B$ -bimodules. Let Φ be the map from $\widetilde{M} \otimes_A X \otimes_A M$ to Y defined by

$$\Phi(\widetilde{m} \otimes x \otimes n) = \langle m, x \cdot n \rangle_D$$

for any $m, n \in M$, $x \in X$. Since $A_1 = X$ and $M = M_0$, $X \cdot M_0 \subset M_1$. And $\langle M_0, M_1 \rangle_D \in B_1 = Y$. Hence, Φ is a map from $\widetilde{M} \otimes_A X \otimes_A M$ to Y . Clearly, Φ is a linear and $B - B$ -bimodule map. We show that Φ is surjective. Indeed,

$$X \cdot M = A_1 \cdot M_0 = {}_C \langle M_1, M_0 \rangle \cdot M_0 = M_1 \cdot \langle M_0, M_0 \rangle_D = M_1 \cdot B = M_1$$

by [3], Proposition 1.7. Hence, $\langle M, X \cdot M \rangle_D = \langle M, M_1 \rangle_D = Y$. Thus, Φ is surjective. Let $m, n, m_1, n_1 \in M$, $x, x_1 \in X$. Then

$$\begin{aligned} & {}_B \langle \widetilde{m} \otimes x \otimes n, \widetilde{m}_1 \otimes x_1 \otimes n_1 \rangle \\ &= {}_B \langle \widetilde{m} \cdot {}_A \langle x \otimes n, x_1 \otimes n_1 \rangle, \widetilde{m}_1 \rangle = {}_B \langle [{}_A \langle x_1 \otimes n_1, x \otimes n \rangle \cdot m]^\sim, \widetilde{m}_1 \rangle \\ &= \langle {}_A \langle x_1 \otimes n_1, x \otimes n \rangle \cdot m, m_1 \rangle_B = \langle {}_A \langle x_1 \cdot {}_A \langle n_1, n \rangle, x \rangle \cdot m, m_1 \rangle_B \\ &= \langle [({}_x \bullet_C \langle n_1, n \rangle) \bullet x^\sharp] \cdot m, m_1 \rangle_B = \langle [{}_C \langle x_1 \cdot n_1, n \rangle \bullet x^\sharp] \cdot m, m_1 \rangle_B \\ &= \langle {}_C \langle [x_1 \cdot n_1], n \rangle \cdot [x^\sharp \cdot m], m_1 \rangle_B = \langle [x_1 \cdot n_1] \cdot \langle n, x^\sharp \cdot m \rangle_D, m_1 \rangle_B \\ &= \langle x^\sharp \cdot m, n \rangle_D \bullet \langle x_1 \cdot n_1, m_1 \rangle_D = \langle m, x \cdot n \rangle_D \bullet \langle m_1, x_1 \cdot n_1 \rangle_D^\sharp \\ &= {}_B \langle \langle m, x \cdot n \rangle_D, \langle m_1, x_1 \cdot n_1 \rangle_D \rangle = {}_B \langle \Phi(\widetilde{m} \otimes x \otimes n), \Phi(\widetilde{m}_1 \otimes x_1 \otimes n_1) \rangle. \end{aligned}$$

Hence, Φ preserves the left B -valued inner products. Also, similarly we can see that Φ preserves the right B -valued inner products. Furthermore,

$$\Phi(\widetilde{m} \otimes x \otimes n)^\sharp = \langle m, x \cdot n \rangle_Y^\sharp = \langle m, x \cdot n \rangle_D^\sharp = \langle x \cdot n, m \rangle_D = \langle x \cdot n, m \rangle_Y.$$

On the other hand,

$$\Phi((\tilde{m} \otimes x \otimes n)^\natural) = \Phi(\tilde{n} \otimes x^\natural \otimes m) = \langle n, x^\natural \cdot m \rangle_Y = \langle x \cdot n, m \rangle_Y = \Phi(\tilde{m} \otimes x \otimes n)^\natural.$$

Hence, Φ preserves the involutions \natural . Therefore $Y \cong \widetilde{M} \otimes_A X \otimes_A M$ as involutive Hilbert $B - B$ -bimodules. \square

Let A and B be unital C^* -algebras. Let X and Y be an involutive Hilbert $A - A$ -bimodule and an involutive Hilbert $B - B$ -bimodule and let \mathcal{A}_X and \mathcal{A}_Y be the C^* -algebraic bundles over \mathbf{Z}_2 induced by X and Y , respectively. We suppose that there is an $A - B$ -equivalence bimodule M such that

$$Y \cong \widetilde{M} \otimes_A X \otimes_A M$$

as involutive Hilbert $B - B$ -bimodules. We construct an $\mathcal{A}_X - \mathcal{A}_Y$ -equivalence bundle $\mathcal{M} = \{M_t\}_{t \in \mathbf{Z}_2}$ over \mathbf{Z}_2 such that

$$C \langle M_t, M_s \rangle = A_{ts^{-1}}, \langle M_t, M_s \rangle_D = B_{t^{-1}s}$$

for any $t, s \in \mathbf{Z}_2$, where $C = A \oplus X$ and $D = B \oplus Y$.

Let Φ be an involutive Hilbert $B - B$ -bimodule isomorphism of $\widetilde{M} \otimes_A X \otimes_A M$ onto Y . Then by the above discussions, there are the Hilbert $A - B$ -bimodule isomorphisms Ψ of $X \otimes_A M$ onto $M \otimes_B Y$ and $\tilde{\Psi}$ of $\tilde{X} \otimes_A M$ onto $M \otimes_B \tilde{Y}$, respectively. We construct a $C_X - C_Y$ -equivalence bimodule C_M from M . Let C_M be the linear span of the set

$${}^X C_M = \left\{ \begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix} : m_1, m_2 \in M, x \in X \right\}.$$

We define the left C_X -action on C_M by

$$\begin{bmatrix} a & z \\ \tilde{z}^\natural & a \end{bmatrix} \cdot \begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix} = \begin{bmatrix} a \otimes m_1 + z \otimes \tilde{x}^\natural \otimes m_2 & a \otimes x \otimes m_2 + z \otimes m_1 \\ \tilde{z}^\natural \otimes m_1 + a \otimes \tilde{x}^\natural \otimes m_2 & \tilde{z}^\natural \otimes x \otimes m_2 + a \otimes m_1 \end{bmatrix}$$

for any $a \in A$, $m_1, m_2 \in M$, $x, z \in X$, where we regard the tensor product as a left C_X -action on C_M in the formal manner. But we identify $A \otimes_A M$ and $X \otimes_A \tilde{X}$, $\tilde{X} \otimes_A X$ with M and closed two-sided ideals of A by the isomorphism and the monomorphisms defined by

$$\begin{aligned} a \otimes m \in A \otimes_A M &\mapsto a \cdot m \in M, & x \otimes \tilde{z} \in X \otimes_A \tilde{X} &\mapsto_A \langle x, z \rangle \in A, \\ \tilde{x} \otimes z \in \tilde{X} \otimes_A X &\mapsto \langle x, z \rangle_A \in A. \end{aligned}$$

Hence, we obtain that

$$\begin{aligned} & \begin{bmatrix} a & z \\ \tilde{z}^{\natural} & a \end{bmatrix} \cdot \begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^{\natural} \otimes m_2 & m_1 \end{bmatrix} \\ &= \begin{bmatrix} a \cdot m_1 + {}_A\langle z, x^{\natural} \rangle \cdot m_2 & a \cdot x \otimes m_2 + z \otimes m_1 \\ \tilde{z}^{\natural} \otimes m_1 + \widetilde{(a \cdot x)}^{\natural} \otimes m_2 & \langle \tilde{z}^{\natural}, x \rangle_A \cdot m_2 + a \cdot m_1 \end{bmatrix} \in C_M. \end{aligned}$$

We define the right C_Y -action on C_M by

$$\begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^{\natural} \otimes m_2 & m_1 \end{bmatrix} \cdot \begin{bmatrix} b & y \\ \tilde{y}^{\natural} & b \end{bmatrix} = \begin{bmatrix} m_1 \otimes b + x \otimes m_2 \otimes \tilde{y}^{\natural} & m_1 \otimes y + x \otimes m_2 \otimes b \\ \tilde{x}^{\natural} \otimes m_2 \otimes b + m_1 \otimes \tilde{y}^{\natural} & \tilde{x}^{\natural} \otimes m_2 \otimes y + m_1 \otimes b \end{bmatrix}$$

for any $b \in B$, $x \in X$, $y \in Y$, $m_1, m_2 \in M$, where we regard the tensor product as a right C_Y -action on C_M in the formal manner. But we identify $X \otimes_A M$ and $\tilde{X} \otimes_A M$ with $M \otimes_B Y$ and $M \otimes_B \tilde{Y}$ by Ψ and $\tilde{\Psi}$, respectively. Hence, we obtain that

$$\begin{aligned} & \begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^{\natural} \otimes m_2 & m_1 \end{bmatrix} \cdot \begin{bmatrix} b & y \\ \tilde{y}^{\natural} & b \end{bmatrix} \\ &= \begin{bmatrix} m_1 \otimes b + x \otimes (\tilde{\Psi})^{-1}(m_2 \otimes \tilde{y}^{\natural}) & \Psi^{-1}(m_1 \otimes y) + x \otimes m_2 \otimes b \\ \tilde{x}^{\natural} \otimes m_2 \otimes b + (\tilde{\Psi})^{-1}(m_1 \otimes \tilde{y}^{\natural}) & \tilde{x}^{\natural} \otimes \Psi^{-1}(m_2 \otimes y) + m_1 \otimes b \end{bmatrix}. \end{aligned}$$

Furthermore, we identify $M \otimes_B B$ and $Y \otimes_B \tilde{Y}$, $\tilde{Y} \otimes_B Y$ with M and closed two-sided ideals of B by the isomorphism and the monomorphisms defined by

$$\begin{aligned} m \otimes b &\in M \otimes_B B \mapsto m \cdot b \in M, \\ y \otimes \tilde{z} &\in Y \otimes_B \tilde{Y} \mapsto {}_B\langle y, z \rangle \in B, \\ \tilde{y} \otimes z &\in \tilde{Y} \otimes_B Y \mapsto \langle y, z \rangle_B \in B, \end{aligned}$$

respectively. Then $x \otimes (\tilde{\Psi})^{-1}(m_2 \otimes y^{\natural}) = \tilde{x}^{\natural} \otimes \Psi^{-1}(m_2 \otimes y)$ and we see that

$$\begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^{\natural} \otimes m_2 & m_1 \end{bmatrix} \cdot \begin{bmatrix} b & y \\ \tilde{y}^{\natural} & b \end{bmatrix} \in C_M.$$

Indeed, for any $\varepsilon > 0$, there are finite sets $\{n_k\}, \{l_k\} \subset M$ and $\{z_k\} \subset X$ such that

$$\left\| \Phi^{-1}(y) - \sum_k \tilde{n}_k \otimes z_k \otimes l_k \right\| < \varepsilon.$$

Also,

$$\begin{aligned} \left\| (\tilde{\Phi})^{-1}(\tilde{y}^{\natural}) - \left[\left(\sum_k \tilde{n}_k \otimes z_k \otimes l_k \right)^{\natural} \right]^{\sim} \right\| &= \left\| [\Phi^{-1}(y)^{\natural}]^{\sim} - \left[\left(\sum_k \tilde{n}_k \otimes z_k \otimes l_k \right)^{\natural} \right]^{\sim} \right\| \\ &= \left\| \Phi^{-1}(y) - \sum_k \tilde{n}_k \otimes z_k \otimes l_k \right\| < \varepsilon. \end{aligned}$$

Thus

$$\begin{aligned} & \left\| x \otimes (\tilde{\Psi})^{-1}(m_2 \otimes \tilde{y}^\natural) - x \otimes m_2 \otimes \left[\left(\sum_k \tilde{n}_k \otimes z_k \otimes l_k \right)^\natural \right]^\sim \right\| \\ &= \left\| x \otimes m_2 \otimes (\tilde{\Phi})^{-1}(\tilde{y}^\natural) - x \otimes m_2 \otimes \sum_k [(\tilde{n}_k \otimes z_k \otimes l_k)^\natural]^\sim \right\| \leq \|x\| \|m_2\| \varepsilon \end{aligned}$$

and

$$\begin{aligned} & \left\| \tilde{x}^\natural \otimes \Psi^{-1}(m_2 \otimes y) - \tilde{x}^\natural \otimes m_2 \otimes \sum_k \tilde{n}_k \otimes z_k \otimes l_k \right\| \\ &= \left\| \tilde{x}^\natural \otimes m_2 \otimes \Phi^{-1}(y) - \tilde{x}^\natural \otimes m_2 \otimes \sum_k \tilde{n}_k \otimes z_k \otimes l_k \right\| \leq \|x\| \|m_2\| \varepsilon. \end{aligned}$$

Furthermore, we can see that

$$\begin{aligned} & x \otimes m_2 \otimes \left[\left(\sum_k \tilde{n}_k \otimes \tilde{z}_k \otimes l_k \right)^\natural \right]^\sim \\ &= \sum_k x \otimes m_2 \otimes \tilde{n}_k \otimes \tilde{z}_k^\natural \otimes l_k = \sum_k {}_A \langle x \cdot {}_A \langle m_2, n_k \rangle, z_k^\natural \rangle \cdot l_k \\ &= \sum_k \tilde{x}^\natural \otimes m_2 \otimes \tilde{n}_k \otimes z_k \otimes l_k = \tilde{x}^\natural \otimes m_2 \otimes \sum_k \tilde{n}_k \otimes z_k \otimes l_k, \end{aligned}$$

where we identify $A \otimes_A M$ and $X \otimes_A \tilde{X}$, $\tilde{X} \otimes_A X$ with M and closed two-sided ideals of A by the isomorphism and the monomorphisms defined by

$$\begin{aligned} a \otimes m &\in A \otimes_A M \mapsto a \cdot m \in M, \\ x \otimes \tilde{z} &\in X \otimes_A \tilde{X} \mapsto {}_A \langle x, z \rangle \in A, \\ \tilde{x} \otimes z &\in \tilde{X} \otimes_A X \mapsto \langle x, z \rangle_A \in A. \end{aligned}$$

Hence

$$x \otimes m_2 \otimes \left[\left(\sum_k \tilde{n}_k \otimes \tilde{z}_k \otimes l_k \right)^\natural \right]^\sim = \tilde{x}^\natural \otimes m_2 \otimes \sum_k \tilde{n}_k \otimes z_k \otimes l_k.$$

It follows that

$$\|x \otimes (\tilde{\Psi})^{-1}(m_2 \otimes y^\natural) - \tilde{x}^\natural \otimes \Psi^{-1}(m_2 \otimes y)\| \leq 2\|x\| \|m_2\| \varepsilon.$$

Since ε is arbitrary, we can see that $x \otimes (\tilde{\Psi})^{-1}(m_2 \otimes y^\natural) = \tilde{x}^\natural \otimes \Psi^{-1}(m_2 \otimes y)$ and that

$$\begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix} \cdot \begin{bmatrix} b & y \\ \tilde{y}^\natural & b \end{bmatrix} \in C_M.$$

Before we define a left C_X -valued inner product and a right C_Y -valued inner product on C_M , we define a conjugate linear map on C_M ,

$$\begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix} \in C_M \mapsto \begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix}^\sim \in C_M$$

by

$$\begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix}^\sim = \begin{bmatrix} \tilde{m}_1 & (\tilde{x}^\natural \otimes m_2)^\sim \\ (x \otimes m_2)^\sim & \tilde{m}_1 \end{bmatrix}$$

for any $m_1, m_2 \in M$, $x \in X$. Since we identify $\widetilde{X \otimes_A M}$ and $\widetilde{\tilde{X} \otimes_A M}$ with $\widetilde{M \otimes_A \tilde{X}}$ and $\widetilde{M \otimes_A X}$ by Lemma 5.1, respectively, we obtain that

$$\begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix}^\sim = \begin{bmatrix} \tilde{m}_1 & \tilde{m}_2 \otimes x^\natural \\ \tilde{m}_2 \otimes \tilde{x} & \tilde{m}_1 \end{bmatrix}.$$

We define the left C_X -valued inner product on C_M by

$$\begin{aligned} C_X \left\langle \begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix}, \begin{bmatrix} n_1 & z \otimes n_2 \\ \tilde{z}^\natural \otimes n_2 & n_1 \end{bmatrix} \right\rangle \\ = \begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix} \cdot \begin{bmatrix} n_1 & z \otimes n_2 \\ \tilde{z}^\natural \otimes n_2 & n_1 \end{bmatrix}^\sim \\ = \begin{bmatrix} {}_A \langle m_1, n_1 \rangle + {}_A \langle x \cdot {}_A \langle m_2, n_2 \rangle, z \rangle & {}_A \langle m_1, n_2 \rangle \cdot z^\natural + x \cdot {}_A \langle m_2, n_1 \rangle \\ \tilde{x}^\natural \cdot {}_A \langle m_2, n_1 \rangle + {}_A \langle m_1, n_2 \rangle \cdot \tilde{z} & {}_A \langle x \cdot {}_A \langle m_2, n_2 \rangle, z \rangle + {}_A \langle m_1, n_1 \rangle \end{bmatrix} \end{aligned}$$

for any $m_1, m_2, n_1, n_2 \in M$, $x, z \in X$, where we regard the tensor product as a product in C_M in the formal manner and identify in the same way as above. Similarly, we define the right C_Y -valued inner product on C_M by

$$\begin{aligned} \left\langle \begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix}, \begin{bmatrix} n_1 & z \otimes n_2 \\ \tilde{z}^\natural \otimes n_2 & n_1 \end{bmatrix} \right\rangle_{C_Y} \\ = \begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix}^\sim \cdot \begin{bmatrix} n_1 & z \otimes n_2 \\ \tilde{z}^\natural \otimes n_2 & n_1 \end{bmatrix} \\ = \begin{bmatrix} \langle m_1, n_1 \rangle_B + \langle m_2, \langle x, z \rangle_A \cdot n_2 \rangle_B & \tilde{m}_1 \otimes \Psi(z \otimes n_2) + \tilde{m}_2 \otimes \Psi(x^\natural \otimes n_1) \\ \tilde{m}_2 \otimes \tilde{\Psi}(\tilde{x} \otimes n_1) + \tilde{m}_1 \otimes \tilde{\Psi}(\tilde{z}^\natural \otimes n_2) & \langle m_2, \langle x, z \rangle_A \cdot n_2 \rangle_B + \langle m_1, n_1 \rangle_B \end{bmatrix} \end{aligned}$$

for any $m_1, m_2, n_1, n_2 \in M$, $x, z \in X$, where we regard the tensor product as a product in C_M in the formal manner, identifying in the same way as above and by the isomorphisms Ψ and $\tilde{\Psi}$. Here, we have to show that the value of the above inner product on C_M exists in C_M . Indeed, by routine computations,

$$\begin{aligned} \tilde{m}_1 \otimes \Psi(z \otimes n_2) &= \sum_i \tilde{m}_1 \otimes u_i \otimes \Phi(\tilde{u}_i \otimes z \otimes n_2) = \Phi(\tilde{m}_1 \otimes z \otimes n_2) \in Y, \\ \tilde{m}_2 \otimes \Psi(x^\natural \otimes n_1) &= \Phi(\tilde{m}_2 \otimes x^\natural \otimes n_1) \in Y. \end{aligned}$$

Also,

$$\begin{aligned}\tilde{m}_2 \otimes \tilde{\Psi}(\tilde{x} \otimes n_1) &= \sum_i \tilde{m}_2 \otimes u_i \otimes \tilde{\Phi}(\tilde{u}_i \otimes \tilde{x} \otimes n_1) = \Phi(\tilde{n}_1 \otimes x \otimes m_2)^\sim \in \tilde{Y}, \\ \tilde{n}_1 \otimes \tilde{\Psi}(\tilde{z}^\natural \otimes n_2) &= \sum_i \tilde{m}_1 \otimes u_i \otimes \tilde{\Phi}(\tilde{u}_i \otimes \tilde{z}^\natural \otimes n_2) = \sum_i \Phi(\tilde{n}_2 \otimes z^\natural \otimes m_1)^\sim \in \tilde{Y}.\end{aligned}$$

Thus

$$\begin{aligned}[\tilde{m}_2 \otimes \tilde{\Psi}(\tilde{x} \otimes n_1) + \tilde{n}_1 \otimes \tilde{\Psi}(\tilde{z}^\natural \otimes n_2)]^\natural &= \Phi(\tilde{n}_1 \otimes x \otimes m_2)^\natural + \Phi(\tilde{n}_2 \otimes z^\natural \otimes m_1)^\natural \\ &= \Phi(\tilde{m}_2 \otimes x^\natural \otimes n_1) + \Phi(\tilde{m}_1 \otimes z \otimes n_2) \\ &= \tilde{m}_1 \otimes \Psi(z \otimes x) + \tilde{m}_2 \otimes \Psi(x^\natural \otimes n_1).\end{aligned}$$

Hence

$$\left\langle \left[\begin{array}{cc} m_1 & x \otimes m_2 \\ \tilde{x}^\natural \otimes m_2 & m_1 \end{array} \right], \left[\begin{array}{cc} n_1 & z \otimes n_2 \\ \tilde{z}^\natural \otimes n_2 & n_1 \end{array} \right] \right\rangle_{C_Y} \in C_Y.$$

By the above definitions, C_M has the left C_X - and the right C_Y -actions and the left C_X -valued and the right C_Y -valued inner products.

Let C_M^1 be the linear span of the set

$$C_M^Y = \left\{ \left[\begin{array}{cc} m_1 & m_2 \otimes y \\ m_2 \otimes \tilde{y}^\natural & m_1 \end{array} \right] : m_1, m_2 \in M, y \in Y \right\}.$$

In the similar way to the above, we define a left C_X - and a right C_Y -actions on C_M^1 and a left C_X -valued and a right C_Y -valued inner products. But identifying $X \otimes_A M$ and $\tilde{X} \otimes_A M$ with $M \otimes_B Y$ and $M \otimes_B \tilde{Y}$ by Ψ and $\tilde{\Psi}$, respectively, we can see that each of them coincides with the other by routine computations. Hence, we obtain the following lemma:

Lemma 5.12. *With the above notation, C_M is a $C_X - C_Y$ -equivalence bimodule.*

Proof. By the definitions of the left C_X -action and the left C_X -valued inner product on C_M , we can see that Conditions (a)–(d) in [6], Proposition 1.12 hold. By the definitions of the right C_Y -action and the right C_Y -valued inner product on C_M , we can also see that the similar conditions to Conditions (a)–(d) in [6], Proposition 1.12 hold. Furthermore, we can easily see that the associativity of the left C_X -valued inner product and the right C_Y -valued inner product hold. Since M is an $A - B$ -equivalence bimodule, there are finite subsets $\{u_i\}_{i=1}^n$ and $\{v_j\}_{j=1}^m$ of M such that

$$\sum_{i=1}^n {}_A \langle u_i, u_i \rangle = 1, \quad \sum_{j=1}^m \langle v_j, v_j \rangle_B = 1.$$

Let $U_i = \begin{bmatrix} u_i & 0 \\ 0 & u_i \end{bmatrix}$ for any i and let $V_j = \begin{bmatrix} v_j & 0 \\ 0 & v_j \end{bmatrix}$ for any j . Then $\{U_i\}$ and $\{V_j\}$ are finite subsets of C_M and

$$\sum_{i=1}^n {}_{C_X} \langle U_i, U_i \rangle = \sum_{i=1}^n \begin{bmatrix} u_i & 0 \\ 0 & u_i \end{bmatrix} \begin{bmatrix} \tilde{u}_i & 0 \\ 0 & \tilde{u}_i \end{bmatrix} = \sum_{i=1}^n \begin{bmatrix} {}_A \langle u_i, u_i \rangle & 0 \\ 0 & {}_A \langle u_i, u_i \rangle \end{bmatrix} = 1_{C_X}.$$

Similarly, $\sum_{j=1}^m \langle V_j, V_j \rangle_{C_Y} = 1_{C_Y}$. Thus, since the associativity of the left C_X -valued inner product and the right C_Y -valued inner product on C_M holds, we can see that $\{U_i\}$ and $\{V_j\}$ are a right C_Y -basis and a left C_X -basis of C_M , respectively. Hence by [6], Proposition 1.12, C_M is a $C_X - C_Y$ -equivalence bimodule. \square

Lemma 5.13. *Let A and B be unital C^* -algebras. Let X and Y be an involutive Hilbert $A - A$ -bimodule and an involutive Hilbert $B - B$ -bimodule, respectively. Let $\mathcal{A}_X = \{A_t\}_{t \in \mathbf{Z}_2}$ and $\mathcal{A}_Y = \{B_t\}_{t \in \mathbf{Z}_2}$ be C^* -algebraic bundles over \mathbf{Z}_2 induced by X and Y , respectively. We suppose that there is an $A - B$ -equivalence bimodule M such that*

$$Y \cong \widetilde{M} \otimes_A X \otimes_A M$$

as involutive Hilbert $B - B$ -bimodules. Then there is an $\mathcal{A}_X - \mathcal{A}_Y$ -equivalence bundle $\mathcal{M} = \{M_t\}_{t \in \mathbf{Z}_2}$ over \mathbf{Z}_2 such that

$${}_C \langle M_t, M_s \rangle = A_{ts^{-1}}, \quad \langle M_t, M_s \rangle_D = B_{t^{-1}s}$$

for any $t, s \in \mathbf{Z}_2$, where $C = A \oplus X$ and $D = B \oplus Y$.

Proof. Let C_M be the $C_X - C_Y$ -equivalence bimodule induced by M , which is defined in the above. We identify $M \oplus (X \otimes_A M)$ with C_M as vector spaces over \mathbf{C} by the isomorphism defined by

$$m_1 \oplus (x \otimes m_2) \in M \oplus (X \otimes_A M) \mapsto \begin{bmatrix} m_1 & x \otimes m_2 \\ \tilde{x}^\natural \otimes m_2 & m_1 \end{bmatrix} \in C_M.$$

Since we identify $C = A \oplus X$ and $D = B \oplus Y$ with C_X and C_Y , respectively, $M \oplus (X \otimes_A M)$ is a $C - D$ -equivalence bimodule by above identifications and Lemma 5.12. Let $M_0 = M$ and $M_1 = X \otimes_A M$. We note that $X \otimes_A M$ is identified with $M \otimes_B Y$ by the Hilbert $A - B$ -bimodule isomorphism Ψ . Let $\mathcal{M} = \{M_t\}_{t \in \mathbf{Z}_2}$. Then by routine computations, \mathcal{M} is an $\mathcal{A}_X - \mathcal{A}_Y$ -equivalence bundle over \mathbf{Z}_2 such that

$${}_C \langle M_t, M_s \rangle = A_{ts^{-1}}, \quad \langle M_t, M_s \rangle_D = B_{t^{-1}s}$$

for any $t, s \in \mathbf{Z}_2$. \square

Proposition 5.14. *Let A and B be unital C^* -algebras. Let X and Y be an involutive Hilbert A – A -bimodule and an involutive Hilbert B – B -bimodule, respectively. Let $\mathcal{A}_X = \{A_t\}_{t \in \mathbf{Z}_2}$ and $\mathcal{A}_Y = \{B_t\}_{t \in \mathbf{Z}_2}$ be the C^* -algebraic bundles over \mathbf{Z}_2 induced by X and Y , respectively. Then the following conditions are equivalent:*

- (1) *There is an \mathcal{A}_X – \mathcal{A}_Y -equivalence bundle $\mathcal{M} = \{M_t\}_{t \in \mathbf{Z}_2}$ over \mathbf{Z}_2 such that*

$${}_C \langle M_t, M_s \rangle = A_{ts^{-1}}, \quad \langle M_t, M_s \rangle_D = B_{t^{-1}s}$$

for any $t, s \in \mathbf{Z}_2$, where $C = A \oplus X$ and $D = B \oplus Y$.

- (2) *There is an A – B -equivalence bimodule M such that*

$$Y \cong \widetilde{M} \otimes_A X \otimes_A M$$

as involutive Hilbert B – B -bimodules.

Proof. This is immediate by Lemmas 5.11 and 5.13. □

Theorem 5.15. *Let A and B be unital C^* -algebras. Let X and Y be an involutive Hilbert A – A -bimodule and an involutive Hilbert B – B -bimodule, respectively. Let $A \subset C_X$ and $B \subset C_Y$ be the unital inclusions of unital C^* -algebras induced by X and Y , respectively. Then the following hold:*

- (1) *If there is an A – B -equivalence bimodule M such that*

$$\widetilde{M} \otimes_A X \otimes_A M \cong Y$$

as involutive Hilbert B – B -bimodules, then the unital inclusions $A \subset C_X$ and $B \subset C_Y$ are strongly Morita equivalent.

- (2) *We suppose that X and Y are full with the both inner products and that $A' \cap C_X = \mathbf{C}1$. If the unital inclusions $A \subset C_X$ and $B \subset C_Y$ are strongly Morita equivalent, then there is an A – B -equivalence bimodule M such that*

$$\widetilde{M} \otimes_A X \otimes_A M \cong Y$$

as involutive Hilbert B – B -bimodules.

Proof. Let $\mathcal{A}_X = \{A_t\}_{t \in \mathbf{Z}_2}$ and $\mathcal{A}_Y = \{B_t\}_{t \in \mathbf{Z}_2}$ be the C^* -algebraic bundles over \mathbf{Z}_2 induced by X and Y , respectively. We prove (1). We suppose that there is an A – B -equivalence bimodule M such that

$$\widetilde{M} \otimes_A X \otimes_A M \cong Y$$

as involutive Hilbert $B - B$ -bimodules. Then by Proposition 5.14, there is an $\mathcal{A}_X - \mathcal{A}_Y$ -equivalence bundle $\mathcal{M} = \{M_t\}_{t \in \mathbf{Z}_2}$ over \mathbf{Z}_2 such that

$${}_C \langle M_t, M_s \rangle = A_{ts^{-1}}, \quad \langle M_t, M_s \rangle_D = B_{t^{-1}s}$$

for any $t, s \in \mathbf{Z}_2$, where $C = A \oplus X$ and $D = B \oplus Y$. Hence, by Proposition 2.1, the unital inclusions of unital C^* -algebras $A \subset C$ and $B \subset D$ are strongly Morita equivalent. Since we identify $A \subset C$ and $B \subset D$ with $A \subset C_X$ and $B \subset C_Y$, respectively, $A \subset C_X$ and $B \subset C_Y$ are strongly Morita equivalent. Next, we prove (2). We suppose that X and Y are full with the both inner products and that $A' \cap C_X = \mathbf{C}1$. Also, we suppose that $A \subset C_X$ and $B \subset C_Y$ are strongly Morita equivalent. Then \mathcal{A}_X and \mathcal{A}_Y are saturated by Lemma 5.5. Since the identity map $\text{id}_{\mathbf{Z}_2}$ is the only automorphism of \mathbf{Z}_2 , by Theorem 4.6 there is an $\mathcal{A}_X - \mathcal{A}_Y$ -equivalence bundle $\mathcal{M} = \{M_t\}_{t \in \mathbf{Z}_2}$ such that

$${}_C \langle M_t, M_s \rangle = A_{ts^{-1}}, \quad \langle M_t, M_s \rangle_D = B_{t^{-1}s}$$

for any $t, s \in \mathbf{Z}$. Hence, from Proposition 5.14, there is an $A - B$ -equivalence bimodule M such that

$$Y \cong \widetilde{M} \otimes_A X \otimes_A M$$

as involutive Hilbert $B - B$ -bimodules. □

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