Abstract. Let $\mu_A$ be the singular measure on the Heisenberg group $H^n$ supported on the graph of the quadratic function $\varphi(y) = y^t Ay$, where $A$ is a $2n \times 2n$ real symmetric matrix. If $\det(2A \pm J) \neq 0$, we prove that the operator of convolution by $\mu_A$ on the right is bounded from $L^{(2n+2)/(2n+1)}(H^n)$ to $L^{2n+2}(H^n)$. We also study the type set of the measures $d\nu_\gamma(y, s) = \eta(y)|y|^{-\gamma}d\mu_A(y, s)$, for $0 \leq \gamma < 2n$, where $\eta$ is a cut-off function around the origin on $\mathbb{R}^{2n}$. Moreover, for $\gamma = 0$ we characterize the type set of $\nu_0$.

Keywords: Heisenberg group; singular Borel measure; $L^p$-improving property

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1. Introduction

Let $I_n$ be the $n \times n$ identity matrix and $J$ be the $2n \times 2n$ skew-symmetric matrix given by

\begin{equation}
J = \begin{pmatrix}
0 & I_n \\
-I_n & 0
\end{pmatrix}.
\end{equation}

The Heisenberg group is $H^n = \mathbb{R}^{2n} \times \mathbb{R}$ endowed with the group law (non-commutative)

$$(x, t) \cdot (y, s) = (x + y, t + s + \langle x, y \rangle),$$

where $\langle x, y \rangle$ is the standard symplectic form on $\mathbb{R}^{2n}$, i.e. $\langle x, y \rangle = x^t J y$ with neutral element $(0, 0)$ and with inverse $(x, t)^{-1} = (-x, -t)$. The topology in $H^n$ is induced by $\mathbb{R}^{2n+1}$, so the borelian sets of $H^n$ are identified with those of $\mathbb{R}^{2n+1}$. The Haar measure in $H^n$ is the Lebesgue measure of $\mathbb{R}^{2n+1}$, thus $L^p(H^n) \equiv L^p(\mathbb{R}^{2n+1})$. Given DOI: 10.21136/MB.2021.0014-20
a borelian function $f: \mathbb{H}^n \to \mathbb{C}$ and a Borel measure $\mu$ on $\mathbb{H}^n$, define the convolution by $\mu$ on the right by

$$(f * \mu)(x, t) = \int_{\mathbb{H}^n} f((x, t) \cdot (y, s)^{-1}) \, d\mu(y, s),$$

provided the integral exists.

A Borel measure $\mu$ on the Heisenberg group $\mathbb{H}^n$ is said to be $L^p$-improving if the operator $T_\mu: f \mapsto f * \mu$ is bounded from $L^p(\mathbb{H}^n)$ into $L^q(\mathbb{H}^n)$ for some $1 \leq p < q < \infty$. A remarkable fact is that singular measures can be $L^p$-improving. If in (2) we replace the Heisenberg group $\mathbb{H}^n$ by $\mathbb{R}^n$ with the ordinary convolution in $\mathbb{R}^n$ and considering there $\mu = \eta\sigma_M$, where $\sigma_M$ is the surface measure on a given manifold $M$ (in $\mathbb{R}^n$) and $\eta$ is a smooth cut-off function, then the $L^p$-improving properties of a measure of this type are closely related to the existence of a certain amount of curvature of the manifold $M$ (see [5], [6], [7]). A similar result holds on general Lie groups (see Theorem 1.1, page 362 in [9]).

A more delicate problem consists in determining the exact range of pairs $(p, q)$ for which $L^p * \mu \subseteq L^q$ embeds continuously. Given a manifold $M$ (in $\mathbb{H}^n$), define the type set $E_{\eta\sigma_M}$ by

$$E_{\eta\sigma_M} = \left\{ \left( \frac{1}{p}, \frac{1}{q} \right) \in [0, 1] \times [0, 1]: \|T_{\eta\sigma_M}\|_{p,q} < \infty \right\}.$$ 

A very interesting survey of results concerning the type sets for convolution operators with singular measures in $\mathbb{R}^n$ can be found in [8].

In the $\mathbb{H}^n$ setting, Secco in [10] and [11] obtained $L^p$-improving properties of measures supported on curves in $\mathbb{H}^1$, under certain assumptions. In [9], Ricci and Stein showed that the type set of the measure given by (3) for the case $\varphi \equiv 0$, $\gamma = 0$ and $n = 1$ is the triangle with vertices $(0, 0)$, $(1, 1)$ and $(\frac{3}{4}, \frac{1}{4})$. In [3] and [4], the author jointly with Godoy generalized the work of Ricci and Stein for the case $\varphi(w) = w^t A w = \sum_{j=1}^n \alpha_j |w_j|^2$, where $A$ is a $2n \times 2n$ real diagonal matrix such that $a_{ii} = a_{(i+1)(i+1)}$ for $i = 2j - 1$ with $j = 1, 2, \ldots, n$, $\alpha_j = a_{(2j-1)(2j-1)}$, $w_j \in \mathbb{R}^2$, $0 \leq \gamma < 2n$ and $n \in \mathbb{N}$. There we also gave some examples of surfaces with degenerate curvature at the origin.

Let $\varphi: \mathbb{R}^{2n} \to \mathbb{R}$ be the function defined by $\varphi(y) = y^t A y$, where $A$ is a $2n \times 2n$ real symmetric matrix. It is well known that if $A$ is an arbitrary matrix, then there exists a symmetric matrix $\tilde{A}$ such that $y^t A y = y^t \tilde{A} y$ for all $y$. We consider two borelian measures on $\mathbb{H}^n$ supported on the graph of $\varphi$, $\mu_A$ and $\nu_\gamma$, $0 \leq \gamma < 2n$, given by

$$\mu_A(E) = \int_{\mathbb{R}^{2n}} \chi_E(y, \varphi(y)) \, dy$$
and

\[ \nu_\gamma(E) = \int_{\mathbb{R}^{2n}} \chi_E(y, \varphi(y))\eta(y)|y|^{-\gamma} \, dy, \]

where \( \eta : \mathbb{R}^{2n} \to [0,1] \) is a smooth cut-off function such that \( \eta(y) = 1 \) if \( |y| \leq 1 \), \( \eta(y) = 0 \) if \( |y| \geq 2 \), and \( E \) is a borelian set of \( \mathbb{H}^n \). Let \( T_{\mu_A} f = f \ast \mu_A \) and \( T_{\nu_\gamma} f = f \ast \nu_\gamma \) be the operators of convolution by \( \mu_A \) and \( \nu_\gamma \) on the right, respectively.

We are interested in studying the \( L^p \)-improving properties of the operator \( T_{\mu_A} \) and in the characterization of the type set \( E_{\nu_\gamma} \). We point out that our measure \( \mu_A \) is not the surface measure on the graph \( \text{gr}(\varphi) \) of \( \varphi \), however the measures \( \mu_A \) and \( \sigma_{\text{gr}(\varphi)} \) are equivalent, see Proposition 2 below, so \( E_{\eta\mu_A} = E_{\eta\sigma_{\text{gr}(\varphi)}} \).

The following restrictions for the type sets \( E_{\nu_\gamma} \), \( 0 \leq \gamma < 2n \), were proved in [3] and [4] for the case \( \varphi(w_1, \ldots, w_n) = \sum_{j=1}^{n} \alpha_j |w_j|^2 \) with \( w_j \in \mathbb{R}^2 \). It is easy to see that such an argument works as well for our function \( \varphi(y) = y^T A y \). Thus, if \( (1/p,1/q) \in E_{\nu_\gamma} \), \( 0 \leq \gamma < 2n \), then

\[ p \leq q, \quad \frac{1}{q} \geq \frac{2n+1}{p} - 2n, \quad \frac{1}{q} \geq \frac{1}{(2n+1)p}. \]

Another necessary condition for the pair \( (1/p,1/q) \) to be in \( E_{\nu_\gamma} \) is the following:

\[ \frac{1}{q} \geq \frac{1}{p} - \frac{2n - \gamma}{2n + 2}. \]

This last condition is relevant only for the case \( 0 < \gamma < 2n \). Let \( D \) be the point of intersection, in the \( (1/p,1/q) \) plane, of the lines \( 1/q = (2n+1)/p - 2n \), \( 1/q = 1/p - (2n - \gamma)/(2n + 2) \), and let \( D' \) be its symmetric image with respect to the symmetry axis \( 1/q = 1 - 1/p \). So

\[ D = \left( \frac{4n^2 + 2n + \gamma}{2n(2n+2)}, \frac{2n + (2n+1)\gamma}{2n(2n+2)} \right) = \left( \frac{1}{pD}, \frac{1}{qD} \right) \quad \text{and} \quad D' = \left( 1 - \frac{1}{qD}, 1 - \frac{1}{pD} \right). \]

Since \( 0 \leq \gamma < 2n \), it is clear that \( \|T_{\nu_\gamma} f\|_p \leq c\|f\|_p \) for all Borel functions \( f \in L^p(\mathbb{H}^n) \) and all \( 1 \leq p \leq \infty \), so \( (1/p,1/p) \in E_{\nu_\gamma} \). Thus, for \( 0 < \gamma < 2n \) the set \( E_{\nu_\gamma} \) is contained in the closed trapezoid with vertices \((0,0),(1,1),D,D'\), and the set \( E_{\nu_0} \) is contained in the closed triangle with vertices \((0,0),(1,1)\) and \((2n+1)/(2n+2),1/(2n+2))\).

In Section 3, our main result appears. There we prove that the operator \( T_{\mu_A} \) is bounded from \( L^{(2n+2)/(2n+1)}(\mathbb{H}^n) \) to \( L^{2n+2}(\mathbb{H}^n) \), see Theorem 3 below. This result allows us to characterize the type set \( E_{\nu_0} \) as well as the interior of \( E_{\nu_\gamma} \) for \( 0 < \gamma < 2n \).
More precisely, we show that $E_{\nu_0}$ is the closed triangle with vertices $(0, 0), (1, 1)$ and $((2n + 1)/(2n + 2), 1/(2n + 2))$ and the interior of $E_{\nu_0}$ coincides with the interior of the closed trapezoid with vertices $(0, 0), (1, 1), D$ and $D'$, see Theorem 4 and Theorem 6 below.

Throughout this paper, $c$ will denote a positive real constant not necessarily the same at each occurrence. The symbol $A \lesssim B$ stands for the inequality $A \leq cB$ for a constant $c$. We use the following convention for the Fourier transform in $\mathbb{R}^n$ $\hat{f}(\xi) = \int f(x) e^{-i\xi \cdot x} \, dx$. The Fourier transform $\hat{u}$ of a distribution $u$ on $\mathbb{R}^n$ is the distribution defined by $(\hat{u}, \varphi) = (u, \hat{\varphi})$ for all rapidly decreasing functions $\varphi$ on $\mathbb{R}^n$.

2. Preliminaries

In the sequel $J$ will denote the $2n \times 2n$ skew-symmetric matrix defined in (1). It is easy to check that

(a) $J^2 = -I$,
(b) $J^t = -J$,
(c) $x^t J x = 0$ for all $x \in \mathbb{R}^{2n}$,
(d) $x^t J y = -y^t J x$ for all $x, y \in \mathbb{R}^{2n}$.

Lemma 1. Let $A$ be a $2n \times 2n$ real diagonal matrix. Then

$$\det(A \pm J) = (a_{11}a_{(n+1)(n+1)} + 1) \cdot (a_{22}a_{(n+2)(n+2)} + 1) \ldots (a_{nn}a_{(2n)(2n)} + 1),$$

where the $a_{ii}$’s are the diagonal entries of $A$.

Proof. Since $\det(A + J) = \det((A + J)^t) = \det(A - J)$, it is sufficient to prove the statement of the lemma for $\det(A + J)$. Applying induction on $n$, the lemma follows.

Proposition 2. Let $A$ be a $2n \times 2n$ real symmetric matrix. Then the graph of the function $\varphi(y) = y^t Ay$ generates all the group $\mathbb{H}^n$. Moreover, the measure $\nu_0 = \eta \mu_A$ is equivalent to the measure $\eta \sigma$, where $\eta$ is a cut-off function and $\sigma$ is the surface measure on the graph of $\varphi$.

Proof. The first statement will follow if we prove that $(x, 0)$ and $(0, t)$ belong to the set $G_{\text{gr}(\varphi)}$ generated by the graph $\text{gr}(\varphi)$ of $\varphi$, since $(x, t) = (x, 0) \cdot (0, t)$. It is clear that $(x, \varphi(x)) \in G_{\text{gr}(\varphi)}$, so $(-t^{1/2}x, \varphi(t^{1/2}x)) = (-t^{1/2}x, \varphi(-t^{1/2}x)) \in G_{\text{gr}(\varphi)}$ for all $x \in \mathbb{R}^{2n}$ and all $t > 0$. From that it follows that $(0, t \varphi(x)) \in G_{\text{gr}(\varphi)}$ for all $t > 0$ and all $x$. If $A$ is a non-null matrix, then $(0, -t) = (0, t)^{-1} \in G_{\text{gr}(\varphi)}$ and $(x, 0) = (x, \varphi(x)) \cdot (0, -\varphi(x)) \in G_{\text{gr}(\varphi)}$. If $A$ is the null matrix, it is sufficient to
prove that \((0, t) \in G_{\text{gr}(\varphi)}\) for all \(t\). Indeed, for \(x\) and \(y\) such that \(\langle x, y \rangle \neq 0\) we have \((0, t) = (x, 0) \cdot (ty/\langle x, y \rangle, 0) \cdot (-x - ty/\langle x, y \rangle, 0) \in G_{\text{gr}(\varphi)}\). So \(G_{\text{gr}(\varphi)} = \mathbb{H}^n\).

For the second part of the proposition, we have that the surface measure on the graph of \(\varphi\) is given by

\[
\sigma(E) = \int_{\varphi^{-1}(E)} \sqrt{\det \begin{bmatrix} \partial_{x_i} \varphi, \partial_{x_j} \varphi \end{bmatrix}} \, dx,
\]

where \(\varphi(x) = (x, \varphi(x))\) and \(E\) is a borelian set of \(\mathbb{R}^{2n+1}\) (see pages 43–45 in [1]). A computation gives

\[
\det \begin{bmatrix} \partial_{x_i} \varphi, \partial_{x_j} \varphi \end{bmatrix} = 1 + 2n \sum_{j=1}^{2n} (\partial_{x_j} \varphi(x))^2 \quad \forall x.
\]

So

\[
\int_{\mathbb{R}^{2n}} \chi_E(\varphi(x)) \eta(x) \, dx \leq \int_{\varphi^{-1}(E)} \sqrt{\det \begin{bmatrix} \partial_{x_i} \varphi, \partial_{x_j} \varphi \end{bmatrix}} \eta(x) \, dx \leq \int_{\mathbb{R}^{2n}} \chi_E(\varphi(x)) \eta(x) \, dx.
\]

Then \(\nu_0\) is equivalent to \(\eta \sigma\). ∎

The \(\lambda\)-twisted convolution is defined by

\[
(f \times_\lambda g)(x) = \int_{\mathbb{R}^{2n}} f(x - y)g(y)e^{-i\lambda x^t J y} \, dy.
\]

Given a \(2n \times 2n\) real symmetric matrix \(A\), we put

\[
e_A(x) = e^{ix^t Ax}.
\]

It is easy to check, using the properties (b) and (c) of the matrix \(J\), that

\[
(f \times_\lambda e_A)(x) = e_A(x)(e_A(\cdot)f(\cdot))^\wedge(\lambda(2A + J)x),
\]

where \(\hat{f}(\xi) = \int_{\mathbb{R}^{2n}} f(x)e^{-ix^t \xi} \, dx\) is the Fourier transform of \(f\). Thus, for each \(f \in L^1(\mathbb{R}^{2n}) \cap L^2(\mathbb{R}^{2n})\) we have

\[
\|f \times_\lambda e_A\|_{L^2(\mathbb{R}^{2n})} = (2\pi)^n |\lambda|^{-n} |\det(2A \pm J)|^{-1/2} \|f\|_{L^2(\mathbb{R}^{2n})}
\]

if \(\det(2A \pm J) \neq 0\).
3. Main Result

To prove the $L^{(2n+2)/(2n+1)}(\mathbb{H}^n) - L^{2n+2}(\mathbb{H}^n)$ boundedness of the operator $T_{\mu_A}$ we embed our operator in an analytic family $\{T_z\}$ of operators on the strip $-n \leq \Re(z) \leq 1$, and then we apply the complex interpolation theorem.

**Theorem 3.** If $\det(2A \pm J) \neq 0$, then the operator $T_{\mu_A}$ is bounded from $L^{(2n+2)/(2n+1)}(\mathbb{H}^n)$ to $L^{2n+2}(\mathbb{H}^n)$.

**Proof.** To prove the statement of the theorem we consider the family $\{|s|z-1\}$ of functions initially defined when $\Re(z) > 0$ and $s \in \mathbb{R} \setminus \{0\}$. This family of functions can be extended in the $z$ variable to an analytic family of distributions on $\mathbb{C} \setminus \{-2k: k \in \mathbb{N} \cup \{0\}\}$. By abuse of notation, we denote this extension by $|s|z-1$. The family $\{|s|z-1\}$ has simple poles in $z = -2k$ for $k \in \mathbb{N} \cup \{0\}$. Since the meromorphic continuation of the function $\Gamma(\frac{1}{2}z)$ (we keep the notation for his continuation) has simple poles at the same points (i.e. $z = -2k$), the family $\{I_z\}$ of distributions defined by

$$I_z(s) = \frac{2^{-z/2}}{\Gamma(\frac{1}{2}z)}|s|^{z-1}$$

results in an entire family of distributions (see pages 55–56 in [2]).

From this construction and by taking the ratios of the corresponding residues at $z = 0$, we have $I_0 = \delta$, where $\delta$ is the Dirac distribution at the origin on $\mathbb{R}$ (see equation (3), page 57 in [2]), also $\widehat{I}_{z} = cI_{1-z}$ for a real constant $c$ independent of $z$ (see equation ($12'\prime$), page 173 in [2]).

For $z \in \mathbb{C}$, we also define $U_z$ as the distribution on $\mathbb{H}^n$ given by the tensor product

$$U_z = \delta_{R^{2n}} \otimes I_z,$$

where $\delta_{R^{2n}}$ is the Dirac distribution at the origin on $\mathbb{R}^{2n}$ and $I_z$ is given by (7). Let $\{T_z\}$ be the analytic family of operators on the strip $-n \leq \Re(z) \leq 1$, given by

$$T_z f = f * \mu_A * U_z.$$

It is clear that $T_0 = T_{\mu_A}$. For $\Re(z) = 1$ we have

$$\|T_z f\|_\infty = \|f * \mu_A * U_z\|_\infty \leq \|f\|_1 \|\mu_A * U_z\|_\infty.$$

Since $\mu_A * U_{1+ib}(x,t) = I_{1+ib}(t - \varphi(x)) = (2^{-(1+ib)/2}/\Gamma(\frac{1}{2}(1+ib)))|t - \varphi(x)|^b$, it follows that

$$\|T_{1+ib}\|_{1,\infty} \leq \left|\frac{2^{-(1+ib)/2}}{\Gamma(\frac{1}{2}(1+ib))}\right| \forall b \in \mathbb{R}.$$
For $\Re(z) = -n$ we will prove that the operator $T_z$ is bounded on $L^2(\mathbb{H}^n)$. This is equivalent to showing that
\[
\int_{\mathbb{R}^{2n}} |(T_z f)^\lambda(x)|^2 \, dx \leq c \int_{\mathbb{R}^{2n}} |f^\lambda(x)|^2 \, dx,
\]
where $h^\lambda(x) := \int_{\mathbb{R}} h(x,t)e^{-\lambda t} \, dt$. A computation gives
\[
(T_{-n+ib})^\lambda(x) = \tilde{T}_{-n+ib}(\lambda) \int_{\mathbb{R}^{2n}} f^\lambda(x - y)e^{\lambda_A(y)}e^{-\lambda x^t J y} \, dy = \tilde{T}_{-n+ib}(\lambda)(f^\lambda \times e_{\lambda_A})(x).
\]
From the identity in (6) and since $\tilde{T}_{i} = c_{I_{1,-z}}$, we get
\[
\| (T_{-n+ib})^\lambda \|_{L^2(\mathbb{R}^{2n})} = \left| \frac{c}{2^{n}} \right| \left| \det(2 \pm J) \right|^{1/2} \| f^\lambda \|_{L^2(\mathbb{R}^{2n})}
\]
for each $b \in \mathbb{R}$. So $T_{-n+ib}$ is bounded on $L^2(\mathbb{H}^n)$ if $\det(2A \pm J) \neq 0$. Finally, it is easy to see, with the aid of the Stirling formula (see e.g. [12]), that the family $\{T_z\}$ satisfies, on the strip $-n \leq \Re(z) \leq 1$, the hypothesis of the complex interpolation theorem (see [13], page 205) and so $T_0 = T_{\mu_A}$ is bounded from $L^{(2n+2)/(2n+1)}(\mathbb{H}^n)$ into $L^{2n+2}(\mathbb{H}^n)$.

**Theorem 4.** Let $\nu_0$ be the measure defined by (3) with $\gamma = 0$. If $\det(2A \pm J) \neq 0$, then the type set $E_{\nu_0}$ is the closed triangle with vertices $(0,0)$, $(1,1)$ and $((2n+1)/(2n+2))$.

**Proof.** Since the inequality $T_{\nu_0} f \leq T_{\mu_A} f$ holds for each borelian function $f \geq 0$, the theorem follows from the restrictions that appear in (4), Theorem 3 and the Riesz convexity theorem.

**Corollary 5.** If $\det(2A \pm J) \neq 0$, then the operator $T_{\mu_A}$ is bounded from $L^p(\mathbb{H}^n)$ into $L^p(\mathbb{H}^n)$ if and only if $p = (2n+2)/(2n+1)$ and $q = 2n+2$.

**Proof.** The “if” part of the corollary is Theorem 3. To see the reciprocal we introduce the action of the dilation group $\mathbb{R}^{>0}$ on $\mathbb{H}^n$, i.e. $\delta \cdot (x,t) = (\delta x, \delta^2 t)$, $\delta > 0$. For a function $f$ defined on $\mathbb{H}^n$ we put $f_\delta(x,t) = f(\delta \cdot (x,t))$. It is easy to check that
\[
(T_{\mu_A} f)_\delta = \delta^{2n} T_{\mu_A}(f_\delta).
\]
If $\|T_{\mu_A} f\|_q \leq c_{p,q} \|f\|_p$, then
\[
\delta^{-(2n+2)/q} \|T_{\mu_A} f\|_q = \|T_{\mu_A} f(\delta)\|_q = \delta^{2n} \|T_{\mu_A}(f_\delta)\|_q \leq \delta^{2n} c \|f_\delta\|_p = \delta^{2n-(2n+2)/p} c \|f\|_p
\]
for all $\delta > 0$. So $1/q = 1/p - 2n/(2n+2)$. Since $T_{\nu_0} f \leq T_{\mu_A} f$ for $f \geq 0$, from Theorem 4 it follows that $p = (2n+2)/(2n+1)$ and $q = 2n+2$. 

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Theorem 6. Let $\nu_\gamma$ be the measure defined by equation (3) with $0 < \gamma < 2n$. If $\det(2A \pm J) \neq 0$, then the type set $E_{\nu_\gamma}$ is contained in the closed trapezoid with vertices $(0,0)$, $(1,1)$, $D$ and $D'$, where

$$D = \left( \frac{4n^2 + 2n + \gamma}{2n(2n + 2)}, \frac{2n + (2n + 1)\gamma}{2n(2n + 2)} \right) = \left( \frac{1}{p_D}, \frac{1}{q_D} \right) \quad \text{and} \quad D' = \left( 1 - \frac{1}{q_D}, 1 - \frac{1}{p_D} \right)$$

and with the only possible exception of the closed segment joining the two points $D$ and $D'$.

Proof. For each $k \in \mathbb{N} \cup \{0\}$ we define the sets $A_k \subset \mathbb{R}^{2n}$ by

$$A_k = \{y \in \mathbb{R}^{2n} : 2^{-k} < |y| \leq 2^{-k+1} \}.$$

Let $\nu_{\gamma,k}$ be the fractional Borel measure given by

$$\nu_{\gamma,k}(E) = \int_{A_k} \chi_E(y, \varphi(y))\eta(y)|y|^{-\gamma} dy$$

and let $T_{\nu_{\gamma,k}}$ be its corresponding convolution operator, i.e. $T_{\nu_{\gamma,k}}f = f * \nu_{\gamma,k}$. Now, it is clear that $\nu_\gamma = \sum_k \nu_{\gamma,k}$ and $\|T_{\nu_\gamma}\|_{p,q} \leq \sum_k \|T_{\nu_{\gamma,k}}\|_{p,q}$. For $f \geq 0$ we have that

$$\int_{\mathbb{R}^{2n}} f(y, s) d\nu_{\gamma,k}(y, s) \leq 2^{k\gamma} \int_{\mathbb{R}^{2n}} f(y, \varphi(y))\eta(y) dy.$$

Thus $\|T_{\nu_{\gamma,k}}\|_{p,q} \leq c 2^{k\gamma} \|T_{\nu_\gamma}\|_{p,q}$, from Theorem 4 it follows that

$$\|T_{\nu_{\gamma,k}}\|_{(2n+2)/(2n+1), 2n+2} \leq c 2^{k\gamma}.$$

It is easy to check that $\|T_{\nu_{\gamma,k}}\|_{1,1} \leq |\nu_{\gamma,k}(\mathbb{R}^{2n+1})| \sim \int_{A_k} |y|^{-\gamma} dy = c 2^{-k(2n-\gamma)}$. For $0 < \theta < 1$ we define

$$\left( \frac{1}{p_\theta}, \frac{1}{q_\theta} \right) = \left( \frac{2n + 1}{2n + 2}, \frac{1}{2n + 2} \right)(1 - \theta) + (1,1)\theta.$$

By the Riesz convexity theorem we have

$$\|T_{\nu_{\gamma,k}}\|_{p_\theta, q_\theta} \leq c 2^{k\gamma(1-\theta) - k(2n-\gamma)\theta}.$$

Choosing $\theta$ such that $k\gamma(1 - \theta) - k(2n - \gamma)\theta = 0$ yields $\sup_{k \in \mathbb{N}}\|T_{\nu_{\gamma,k}}\|_{p_\theta, q_\theta} \leq c < \infty$. A simple computation gives $\theta = (2n - \gamma)/(2n)$, then $(1/p_\theta, 1/q_\theta) = (1/p_D, 1/q_D)$, so
\[ \| T_{\nu, k} \|_{p, q} \leq c, \text{ where } c \text{ is independent of } k. \] Interpolating once again, but now between the points \((1/p, 1/q)\) and \((1, 1)\) we obtain for each \(0 < \tau < 1\) fixed
\[ \| T_{\nu, k} \|_{p, \tau} \leq c 2^{-k(2n-\gamma)\tau}. \]
Since \(\| T_{\nu, k} \|_{p, q} \leq \sum_k \| T_{\nu, k} \|_{p, q}\) and \(0 < \gamma < 2n\), it follows that
\[ \| T_{\nu, k} \|_{p, \tau} \leq c \sum_{k \in \mathbb{N}} 2^{-k(2n-\gamma)\tau} < \infty. \]
By duality we also have
\[ \| T_{\nu, k} \|_{q, \tau} / (q_p - 1) \leq c \tau < \infty. \]
Finally, the theorem follows from the Riesz convexity theorem, and the restrictions that appear in (4) and (5).

We conclude this note with the following remarks.

Remark 7. Let \(\nu_0\) be the measure of compact support defined by (3), but now with \(\det(2A \pm J) = 0\). In this case, by Theorem 1.1 in [9] and Proposition 2, we can be sure that the type set \(E_{\nu_0}\) has a nonempty interior.

Remark 8. Lemma 1 provides us with examples of diagonal matrices \(A\) such that \(\det(2A \pm J) = 0\). By the above remark we know that the interior of the type set of measure \(\nu_0 = \eta \mu_A\) is nonempty. If \(n \geq 2\) and \(A\) also satisfies that \(\varphi(y) = y^T A y = \sum_{j=1}^n \alpha_j |y_j|^2\) \((\alpha_j \in \mathbb{R} \text{ and } y_j \in \mathbb{R}^2)\), then the type set of \(\nu_0\) is the closed triangle with vertices \((0, 0)\), \((1, 1)\) and \(((2n+1)/(2n+2), 1/(2n+2))\). This result is independent of the value of \(\det(2A \pm J)\) (see Theorem 1, page 102 in [3]).

These final comments illustrate the limits of the techniques used in this note as well as of those developed in the works [3] and [4].

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