GENERALIZED ATOMIC SUBSPACES
FOR OPERATORS IN HILBERT SPACES

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Abstract. We introduce the notion of a \( g \)-atomic subspace for a bounded linear operator and construct several useful resolutions of the identity operator on a Hilbert space using the theory of \( g \)-fusion frames. Also, we shall describe the concept of frame operator for a pair of \( g \)-fusion Bessel sequences and some of their properties.

Keywords: frame; atomic subspace; \( g \)-fusion frame; \( K \)-\( g \)-fusion frame

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1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer in 1952 to study some fundamental problems in non-harmonic Fourier series (see [7]). Later on, after some decades, frame theory was popularized by Daubechies, Grossman, Meyer (see [5]). At present, frame theory has been widely used in signal and image processing, filter bank theory, coding and communications, system modeling and so on. Several generalizations of frames, namely \( K \)-frames, \( g \)-frames, fusion frames etc. have been introduced in recent times.

\( K \)-frames were introduced by Gavruta (see [8]) to study the atomic system with respect to a bounded linear operator. Using frame theory techiques, the author also studied the atomic decompositions for operators on reproducing kernel Hilbert spaces, see [9]. Sun in [15] introduced a \( g \)-frame and a \( g \)-Riesz basis in complex Hilbert spaces and discussed several properties of them. Huang in [12] began to study \( K \)-\( g \)-frame by combining \( K \)-frame and \( g \)-frame. Casazza (see [3]) was first to introduce the notion of fusion frames or frames of subspaces and gave various ways to obtain a resolution of the identity operator from a fusion frame. The concept of

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an atomic subspace with respect to a bounded linear operator were introduced by Bhandari and Mukherjee in [2]. Construction of $K$-$g$-fusion frames and their dual were presented by Sadri and Rahimi (see [1]) to generalize the theory of $K$-frame, fusion frame and $g$-frame. Ghosh and Samanta in [11] studied the stability of dual $g$-fusion frames in Hilbert spaces.

In this paper, we present some useful results about resolution of the identity operator on a Hilbert space using the theory of $g$-fusion frames. We give the notion of $g$-atomic subspace with respect to a bounded linear operator. The frame operator for a pair of $g$-fusion Bessel sequences are discussed and some properties are going to be established.

The paper is organized as follows: in Section 2, we briefly recall the basic definitions and results. Various ways of obtaining resolution of the identity operator on a Hilbert space in $g$-fusion frame are studied in Section 3. $g$-atomic subspaces are introduced and discussed in Section 4. In Section 5, frame operators for a pair of $g$-fusion Bessel sequences are given and various properties are established.

Throughout this paper, $H$ is considered to be a separable Hilbert space with associated inner product $\langle \cdot, \cdot \rangle$ and $\{H_j\}_{j \in J}$ are the collection of Hilbert spaces, where $J$ is a subset of integers $\mathbb{Z}$. $I_H$ is the identity operator on $H$. $\mathcal{B}(H_1, H_2)$ is a collection of all bounded linear operators from $H_1$ to $H_2$. In particular, $\mathcal{B}(H)$ denotes the space of all bounded linear operators on $H$. For $T \in \mathcal{B}(H)$, we denote $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for null space and range of $T$, respectively. Also, $P_V \in \mathcal{B}(H)$ is the orthonormal projection onto a closed subspace $V \subset H$. Define the space

$$l^2(\{H_j\}_{j \in J}) = \left\{ \{f_j\}_{j \in J} : f_j \in H_j, \sum_{j \in J} \|f_j\|^2 < \infty \right\}$$

with inner product given by

$$\langle \{f_j\}_{j \in J}, \{g_j\}_{j \in J} \rangle = \sum_{j \in J} \langle f_j, g_j \rangle_{H_j}.$$  

Clearly $l^2(\{H_j\}_{j \in J})$ is a Hilbert space with the pointwise operations (see [1]).

2. Preliminaries

**Theorem 2.1** ([6], Douglas' factorization theorem). Let $U, V \in \mathcal{B}(H)$. Then the following conditions are equivalent:

1. $\mathcal{R}(U) \subseteq \mathcal{R}(V)$.
2. $UU^* \leq \lambda^2VV^*$ for some $\lambda > 0$.
3. $U = VW$ for some bounded linear operator $W$ on $H$.  

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**Theorem 2.2** ([13]). The set $S(H)$ of all self-adjoint operators on $H$ is a partially ordered set with respect to the partial order $\leq$ which is defined as for $T, S \in S(H)$

$$T \leq S \iff \langle T f, f \rangle \leq \langle S f, f \rangle \quad \forall f \in H.$$ 

**Theorem 2.3** ([10]). Let $V \subset H$ be a closed subspace and $T \in B(H)$. Then $P_V T^* = P_V T^* P_{TV}$. If $T$ is a unitary operator (i.e. $T^* T = I_H$), then $P_{TV} T = T P_V$.

**Definition 2.4** ([4]). A sequence $\{f_j\}_{j \in J}$ of elements in $H$ is a frame for $H$ if there exist constants $A, B > 0$ such that

$$A \|f\|^2 \leq \sum_{j \in J} |\langle f, f_j \rangle|^2 \leq B \|f\|^2 \quad \forall f \in H.$$ 

The constants $A$ and $B$ are called frame bounds.

**Definition 2.5** ([3]). Let $\{W_j\}_{j \in J}$ be a collection of closed subspaces of $H$ and $\{v_j\}_{j \in J}$ be a collection of positive weights. A family of weighted closed subspaces $\{(W_j, v_j): j \in J\}$ is called a fusion frame for $H$ if there exist constants $0 < A \leq B < \infty$ such that

$$A \|f\|^2 \leq \sum_{j \in J} v_j^2 \|P_{W_j}(f)\|^2 \leq B \|f\|^2 \quad \forall f \in H.$$ 

The constants $A, B$ are called fusion frame bounds. If $A = B$, then the fusion frame is called a tight fusion frame, if $A = B = 1$, then it is called a Parseval fusion frame.

**Definition 2.6** ([2]). Let $\{W_j\}_{j \in J}$ be a family of closed subspaces of $H$ and $\{v_j\}_{j \in J}$ be a family of positive weights and $K \in B(H)$. Then $\{(W_j, v_j): j \in J\}$ is said to be an atomic subspace of $H$ with respect to $K$ if the following conditions hold:

(I) $\sum_{j \in J} v_j f_j$ is convergent for all $\{f_j\}_{j \in J} \in \left( \sum_{j \in J} \oplus W_j \right)_{l^2}$.

(II) For every $f \in H$ there exists $\{f_j\}_{j \in J} \in \left( \sum_{j \in J} \oplus W_j \right)_{l^2}$ such that

$$K(f) = \sum_{j \in J} v_j f_j \quad \text{and} \quad \|\{f_j\}\| \left( \sum_{j \in J} \oplus W_j \right)_{l^2} \leq C \|f\|_H$$

for some $C > 0$, where

$$\left( \sum_{j \in J} \oplus W_j \right)_{l^2} = \left\{ \{f_j\}_{j \in J}: f_j \in W_j, \sum_{j \in J} \|f_j\|^2 < \infty \right\}$$

with inner product given by $\langle \{f_j\}_{j \in J}, \{g_j\}_{j \in J} \rangle = \sum_{j \in J} \langle f_j, g_j \rangle_H$. 

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Definition 2.7 ([15]). A sequence \( \{\Lambda_j \in B(H, H_j) : j \in J\} \) is called a generalized frame or \( g \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) if there are two positive constants \( A \) and \( B \) such that
\[
A \|f\|^2 \leq \sum_{j \in J} \|\Lambda_j f\|^2 \leq B \|f\|^2 \quad \forall f \in H.
\]
The constants \( A \) and \( B \) are called the lower and upper frame bounds, respectively.

Definition 2.8 ([14], [1]). Let \( \{W_j\}_{j \in J} \) be a collection of closed subspaces of \( H \) and \( \{v_j\}_{j \in J} \) be a collection of positive weights and let \( \Lambda_j \in B(H, H_j) \) for each \( j \in J \). Then the family \( \Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J} \) is called a generalized fusion frame or a \( g \)-fusion frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) if there exist constants \( 0 < A \leq B < \infty \) such that
\[
(2.1) \quad A \|f\|^2 \leq \sum_{j \in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \leq B \|f\|^2 \quad \forall f \in H.
\]
The constants \( A \) and \( B \) are called the lower and upper bounds of \( g \)-fusion frame, respectively. If \( A = B \), then \( \Lambda \) is called tight \( g \)-fusion frame and if \( A = B = 1 \), then we say \( \Lambda \) is a Parseval \( g \)-fusion frame. If \( \Lambda \) satisfies only the condition
\[
\sum_{j \in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \leq B \|f\|^2 \quad \forall f \in H,
\]
then it is called a \( g \)-fusion Bessel sequence with bound \( B \) in \( H \).

Definition 2.9 ([1]). Let \( \Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J} \) be a \( g \)-fusion Bessel sequence in \( H \) with a bound \( B \). The synthesis operator \( T_\Lambda \) of \( \Lambda \) is defined as
\[
T_\Lambda : l^2(\{H_j\}_{j \in J}) \to H, \quad T_\Lambda(\{f_j\}_{j \in J}) = \sum_{j \in J} v_j P_{W_j} \Lambda_j^* f_j \quad \forall \{f_j\}_{j \in J} \in l^2(\{H_j\}_{j \in J})
\]
and the analysis operator is given by
\[
T_\Lambda^* : H \to l^2(\{H_j\}_{j \in J}), \quad T_\Lambda^*(f) = \{v_j \Lambda_j P_{W_j}(f)\}_{j \in J} \quad \forall f \in H.
\]
The \( g \)-fusion frame operator \( S_\Lambda : H \to H \) is defined as
\[
S_\Lambda(f) = T_\Lambda T_\Lambda^*(f) = \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}(f)
\]
and it can be easily verified that
\[
\langle S_\Lambda(f), f \rangle = \sum_{j \in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \quad \forall f \in H.
\]

4 Online first
Furthermore, if $\Lambda$ is a $g$-fusion frame with bounds $A$ and $B$, then from (2.1),

$$\langle Af, f \rangle \leq \langle S_\Lambda(f), f \rangle \leq \langle Bf, f \rangle \quad \forall f \in H.$$ 

The operator $S_\Lambda$ is bounded, self-adjoint, positive and invertible. Now, according to Theorem 2.2, we can write $AI_H \leq S_\Lambda \leq BI_H$ and this gives

$$B^{-1}I_H \leq S^{-1}_\Lambda \leq A^{-1}I_H.$$ 

**Definition 2.10 ([1]).** Let $\{W_j\}_{j \in J}$ be a collection of closed subspaces of $H$ and $\{v_j\}_{j \in J}$ be a collection of positive weights and let $\Lambda_j \in B(H, H_j)$ for each $j \in J$ and $K \in B(H)$. Then the family $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is called a $K$-$g$-fusion frame for $H$ if there exist constants $0 < A \leq B < \infty$ such that

$$(2.2) \quad A\|K^*f\|^2 \leq \sum_{j \in J} v_j^2\|\Lambda_j P_{W_j}(f)\|^2 \leq B\|f\|^2 \quad \forall f \in H.$$ 

**Theorem 2.11 ([1]).** Let $\Lambda$ be a $g$-fusion Bessel sequence in $H$. Then $\Lambda$ is a $K$-$g$-fusion frame for $H$ if and only if there exists $A > 0$ such that $S_\Lambda \geq AKK^*$. 

**Definition 2.12 ([3]).** A family of bounded operators $\{T_j\}_{j \in J}$ on $H$ is called a resolution of identity operator on $H$ if for all $f \in H$ we have $f = \sum_{j \in J} T_j(f)$, provided the series converges unconditionally for all $f \in H$.

### 3. Resolution of the Identity Operator in $g$-Fusion Frame

In this section, we present several useful results of resolution of the identity operator on a Hilbert space using the theory of $g$-fusion frames.

**Theorem 3.1.** Let $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ be a $g$-fusion frame for $H$ with frame bounds $C$, $D$ and $S_\Lambda$ be its associated $g$-fusion frame operator. Then the family $\{v_j^2 P_{W_j} \Lambda_j^* T_j\}_{j \in J}$ is the resolution of the identity operator on $H$, where $T_j = \Lambda_j P_{W_j} S^{-1}_\Lambda$, $j \in J$. Furthermore, for all $f \in H$ we have

$$\frac{C}{D^2} \|f\|^2 \leq \sum_{j \in J} v_j^2 \|T_j(f)\|^2 \leq \frac{D}{C^2} \|f\|^2.$$
Proof. For any $f \in H$ we have the reconstruction formula for $g$-fusion frame:

$$f = S_{\Lambda}S_{\Lambda}^{-1}(f) = \sum_{j \in J} v^2_j P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} S_{\Lambda}^{-1}(f) = \sum_{j \in J} v^2_j P_{W_j} \Lambda_j^* T_j(f).$$

Thus, $\{v^2_j P_{W_j} \Lambda_j^* T_j\}_{j \in J}$ is a resolution of the identity operator on $H$. Since $\Lambda$ is a $g$-fusion frame with bounds $C$ and $D$, for each $f \in H$ we have

$$\sum_{j \in J} v^2_j \|T_j(f)\|^2 = \sum_{j \in J} v^2_j \|\Lambda_j P_{W_j} S_{\Lambda}^{-1}(f)\|^2 \leq D \|S_{\Lambda}^{-1}(f)\|^2 \leq D \|S_{\Lambda}^{-1}\|^2 \|f\|^2 \leq \frac{D}{C^2} \|f\|^2 \ (\text{since } D^{-1} I_H \leq S_{\Lambda}^{-1} \leq C^{-1} I_H).$$

On the other hand,

$$\sum_{j \in J} v^2_j \|T_j(f)\|^2 = \sum_{j \in J} v^2_j \|\Lambda_j P_{W_j} S_{\Lambda}^{-1}(f)\|^2 \geq C \|S_{\Lambda}^{-1}(f)\|^2 \geq \frac{C}{D^2} \|f\|^2.$$

Therefore

$$\frac{C}{D^2} \|f\|^2 \leq \sum_{j \in J} v^2_j \|T_j(f)\|^2 \leq \frac{D}{C^2} \|f\|^2 \ \forall \ f \in H.$$

Theorem 3.2. Let $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ be a $g$-fusion frame for $H$ with frame bounds $C$, $D$ and let $T_j: H \rightarrow H_j$ be a bounded operator such that $\{v^2_j P_{W_j} \Lambda_j^* T_j\}_{j \in J}$ is a resolution of the identity operator on $H$. Then

$$\frac{1}{D} \left\| \sum_{j \in J} v^2_j P_{W_j} \Lambda_j^* T_j(f) \right\|^2 \leq \sum_{j \in J} v^2_j \|T_j(f)\|^2 \ \forall \ f \in H.$$

Proof. Assume $I \subset J$ with $|I| < \infty$. If our inequality holds for all finite subsets, then it would hold for all subsets. Let $f \in H$ and set $g = \sum_{j \in I} v^2_j P_{W_j} \Lambda_j^* T_j(f)$. Then

$$\|g\|^4 = \langle g, g \rangle^2 = \left\langle g, \sum_{j \in I} v^2_j P_{W_j} \Lambda_j^* T_j(f) \right\rangle^2 = \left( \sum_{j \in I} v_j \langle \Lambda_j P_{W_j}(g), v_j T_j(f) \rangle \right)^2 \leq \left( \sum_{j \in I} v_j \|\Lambda_j P_{W_j}(g)\| \|v_j T_j(f)\| \right)^2 \leq \sum_{j \in I} v^2_j \|\Lambda_j P_{W_j}(g)\|^2 \sum_{j \in I} \|v_j T_j(f)\|^2 \leq D \|g\|^2 \sum_{j \in I} \|v_j T_j(f)\|^2 \ (\text{since } \Lambda \text{ is a } g\text{-fusion frame})$$

$$\Rightarrow \frac{1}{D} \|g\|^2 \leq \sum_{j \in I} \|v_j T_j(f)\|^2$$

$$\Rightarrow \frac{1}{D} \left\| \sum_{j \in I} v^2_j P_{W_j} \Lambda_j^* T_j(f) \right\|^2 \leq \sum_{j \in I} v^2_j \|T_j(f)\|^2 \ \forall \ f \in H.$$
Since the inequality holds for any finite subset $I \subset J$, we have
\[ \frac{1}{D} \left\| \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* T_j(f) \right\|_2^2 \leq \sum_{j \in J} v_j^2 \| T_j(f) \|_2^2 \quad \forall f \in H. \]
This completes the proof. \( \square \)

**Theorem 3.3.** Let $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ be a $g$-fusion frame for $H$ with frame bounds $C$, $D$ and let $T_j : H \to H_j$ be a bounded operator such that $\{v_j^2 P_{W_j} \Lambda_j^* T_j\}_{j \in J}$ is a resolution of the identity operator on $H$. If $T_j^* \Lambda_j P_{W_j} = T_j$, then
\[ \frac{1}{D} \| f \|_2^2 \leq \sum_{j \in J} v_j^2 \| T_j(f) \|_2^2 \leq DE \| f \|_2^2 \quad \forall f \in H, \]
where $E = \sup_j \| T_j \|_2^2 < \infty$.

**Proof.** Since $\{v_j^2 P_{W_j} \Lambda_j^* T_j\}_{j \in J}$ is a resolution of the identity on $H$,
\[ f = \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* T_j(f), \quad f \in H. \]
Now, for each $f \in H$, using Theorem 3.2, we get
\[
\begin{align*}
\frac{1}{D} \| f \|_2^2 &= \frac{1}{D} \left\| \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* T_j(f) \right\|_2^2 \\
&\leq \sum_{j \in J} v_j^2 \| T_j(f) \|_2^2 \\
&= \sum_{j \in J} v_j^2 \| T_j^* \Lambda_j P_{W_j}(f) \|_2^2 \quad \text{(since $T_j^* \Lambda_j P_{W_j} = T_j$)} \\
&\leq \sum_{j \in J} v_j^2 \| T_j \|_2^2 \| \Lambda_j P_{W_j}(f) \|_2^2 \\
&\leq E \sum_{j \in J} v_j^2 \| \Lambda_j P_{W_j}(f) \|_2^2 \quad \text{(using $E = \sup_j \| T_j \|_2^2$)} \\
&\leq DE \| f \|_2^2 \quad \text{(since $\Lambda$ is a $g$-fusion frame).}
\end{align*}
\]
This completes the proof. \( \square \)

**Theorem 3.4.** Let $\{W_j\}_{j \in J}$ be a family of closed subspaces of $H$ and $\{v_j\}_{j \in J}$ be a family of bounded weights and let $\Lambda_j \in \mathcal{B}(H, H_j)$, $j \in J$. Then $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is a $g$-fusion frame for $H$ if the following conditions hold:

1. For all $f \in H$ there exists $A > 0$ such that
\[ \sum_{j \in J} \| \Lambda_j P_{W_j}(f) \|_2^2 \leq \frac{1}{A} \| f \|_2^2. \]

2. $\{v_j P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}\}_{j \in J}$ is a resolution of the identity operator on $H$. 

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Proof. Since \( \{v_j P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}\}_{j \in J} \) is a resolution of the identity operator on \( H \), for \( f \in H \) we have
\[
f = \sum_{j \in J} v_j P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}(f).
\]

By Cauchy-Schwarz inequality, we have
\[
\|f\|^4 = \langle f, f \rangle^2 = \left( \sum_{j \in J} v_j \langle \Lambda_j P_{W_j}(f), \Lambda_j P_{W_j}(f) \rangle \right)^2 = \left( \sum_{j \in J} v_j \| \Lambda_j P_{W_j}(f) \|^2 \right)^2 \
\leq \sum_{j \in J} \| \Lambda_j P_{W_j}(f) \|^2 \sum_{j \in J} v_j^2 \| \Lambda_j P_{W_j}(f) \|^2 \
\leq \frac{1}{A} \|f\|^2 \sum_{j \in J} v_j^2 \| \Lambda_j P_{W_j}(f) \|^2 \quad \text{(using given condition (I))} \
\Rightarrow A \|f\|^2 \leq \sum_{j \in J} v_j^2 \| \Lambda_j P_{W_j}(f) \|^2.
\]

On the other hand,
\[
\sum_{j \in J} v_j^2 \| \Lambda_j P_{W_j}(f) \|^2 \leq B \sum_{j \in J} \| \Lambda_j P_{W_j}(f) \|^2 \quad \text{(where } B = \sup_{j \in J} \{v_j^2\})
\]
\[
\leq \frac{B}{A} \|f\|^2 \quad \text{(using given condition (I))}
\]

and hence, \( \Lambda \) is a \( g \)-fusion frame.

\[\square\]

4. \( g \)-ATOMIC SUBSPACE

In this section, we define a generalized atomic subspace or a \( g \)-atomic subspace of a Hilbert space with respect to a bounded linear operator.

**Definition 4.1.** Let \( K \in \mathcal{B}(H) \) and \( \{W_j\}_{j \in J} \) be a collection of closed subspaces of \( H \), let \( \{v_j\}_{j \in J} \) be a collection of positive weights and \( \Lambda_j \in \mathcal{B}(H, H_j) \) for each \( j \in J \). Then the family \( \Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J} \) is said to be a generalized atomic subspace or \( g \)-atomic subspace of \( H \) with respect to \( K \) if the following statements hold:

(I) \( \Lambda \) is a \( g \)-fusion Bessel sequence in \( H \).

(II) For every \( f \in H \) there exists \( \{f_j\}_{j \in J} \in l^2(\{H_j\}_{j \in J}) \) such that
\[
K(f) = \sum_{j \in J} v_j P_{W_j} \Lambda_j^* f_j \quad \text{and} \quad \|\{f_j\}_{j \in J}\|_{l^2(\{H_j\}_{j \in J})} \leq C \|f\|_H
\]
for some \( C > 0 \).
Theorem 4.2. Let $K \in \mathcal{B}(H)$ and $\{W_j\}_{j \in J}$ be a collection of closed subspaces of $H$, let $\{v_j\}_{j \in J}$ be a collection of positive weights and $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in J$. Then the following statements are equivalent:

(I) $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is a $g$-atomic subspace of $H$ with respect to $K$.

(II) $\Lambda$ is a $K$-$g$-fusion frame for $H$.

Proof. (I) $\Rightarrow$ (II): Suppose $\Lambda$ is a $g$-atomic subspace of $H$ with respect to $K$. Then $\Lambda$ is a $g$-fusion Bessel sequence, so there exists $B > 0$ such that

$$\sum_{j \in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \leq B \|f\|^2 \quad \forall f \in H.$$ 

Now, for any $f \in H$ we have

$$\|K^* f\| = \sup_{\|g\|=1} |\langle K^* f, g \rangle| = \sup_{\|g\|=1} |\langle f, Kg \rangle|,$$

by Definition 4.1, for $g \in H$ there exists $\{f_j\}_{j \in J} \in l^2(\{H_j\}_{j \in J})$ such that

$$K(g) = \sum_{j \in J} v_j P_{W_j} \Lambda_j^* f_j \quad \text{and} \quad \|\{f_j\}_{j \in J}\|_{l^2(\{H_j\}_{j \in J})} \leq C \|g\|_H$$

for some $C > 0$. Thus

$$\|K^* f\| = \sup_{\|g\|=1} \left| \sum_{j \in J} v_j P_{W_j} \Lambda_j^* f_j \right| = \sup_{\|g\|=1} \left| \sum_{j \in J} v_j \langle \Lambda_j P_{W_j}(f), f_j \rangle \right|$$

$$\leq \sup_{\|g\|=1} \left( \sum_{j \in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \right)^{1/2} \left( \sum_{j \in J} \|f_j\|^2 \right)^{1/2}$$

$$\leq C \sup_{\|g\|=1} \left( \sum_{j \in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \right)^{1/2} \|g\|$$

$$\Rightarrow \frac{1}{C^2} \|K^* f\|^2 \leq \sum_{j \in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2.$$

Therefore $\Lambda$ is a $K$-$g$-fusion frame for $H$ with bounds $1/C^2$ and $B$.

(II) $\Rightarrow$ (I): Suppose that $\Lambda$ is a $K$-$g$-fusion frame with the corresponding synthesis operator $T_\Lambda$. Then obviously $\Lambda$ is a $g$-fusion Bessel sequence in $H$. Now, for each $f \in H$,

$$A \|K^* f\|^2 \leq \sum_{j \in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 = \|T_\Lambda^* f\|^2.$$
gives $AKK^* \leq T\Lambda T^*_\Lambda$ and by Theorem 2.1, exists $L \in \mathcal{B}(H, l^2(\{H_j\}_{j \in J}))$ such that $K = T\Lambda L$. Define $L(f) = \{f_j\}_{j \in J}$ for every $f \in H$. Then for each $f \in H$ we have

$$K(f) = T\Lambda L(f) = T\Lambda(\{f_j\}_{j \in J}) = \sum_{j \in J} v_j P_{W_j} A^*_j f_j$$

and

$$\|\{f_j\}_{j \in J} \|_{l^2(\{H_j\}_{j \in J})} = \| L(f) \|_{l^2(\{H_j\}_{j \in J})} \leq C \| f \|,$$

where $C = \| L \|$. Hence, $\Lambda$ is a $g$-atomic subspace of $H$ with respect to $K$. \hfill \Box

**Theorem 4.3.** Let $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ be a $g$-fusion frame for $H$. Then $\Lambda$ is a $g$-atomic subspace of $H$ with respect to its $g$-fusion frame operator $S_\Lambda$.

**Proof.** Since $\Lambda$ is a $g$-fusion frame in $H$, there exist $A, B > 0$ such that

$$A \| f \|^2 \leq \sum_{j \in J} v_j^2 \| \Lambda_j P_{W_j}(f) \|^2 \leq B \| f \|^2 \quad \forall f \in H.$$

Since $\mathcal{R}(T\Lambda) = H = \mathcal{R}(S_\Lambda)$, by Theorem 2.1, there exists $\alpha > 0$ such that $\alpha S_\Lambda S^*_\Lambda \leq T\Lambda T^*_\Lambda$ and therefore for each $f \in H$ we have

$$\alpha \| S^*_\Lambda f \|^2 \leq \| T^*_\Lambda f \|^2 = \sum_{j \in J} v_j^2 \| \Lambda_j P_{W_j}(f) \|^2 \leq B \| f \|^2.$$

Thus, $\Lambda$ is a $S_\Lambda$-$g$-fusion frame and hence by Theorem 4.2, $\Lambda$ is a $g$-atomic subspace of $H$ with respect to $S_\Lambda$. \hfill \Box

**Theorem 4.4.** Let $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ and $\Gamma = \{(W_j, \Gamma_j, v_j)\}_{j \in J}$ be two $g$-atomic subspaces of $H$ with respect to $K \in \mathcal{B}(H)$ with the corresponding synthesis operators $T\Lambda$ and $T\Gamma$, respectively. If $T\Lambda T^*_\Gamma = \theta_H$ ($\Theta_H$ is a null operator on $H$) and $U, V \in \mathcal{B}(H)$ such that $U + V$ is invertible operator on $H$ with $K(U+V) = (U+V)K$, then

$$\{(U + V) W_j, (\Lambda_j + \Gamma_j) P_{W_j}(U + V)^*, v_j\}_{j \in J}$$

is a $g$-atomic subspace of $H$ with respect to $K$.

**Proof.** Since $\Lambda$ and $\Gamma$ are $g$-atomic subspaces with respect to $K$, by Theorem 4.2, they are $K$-$g$-fusion frames for $H$. So, for each $f \in H$ there exist positive constants $(A_1, B_1)$ and $(A_2, B_2)$ such that

$$A_1 \| K^* f \|^2 \leq \sum_{j \in J} v_j^2 \| \Lambda_j P_{W_j}(f) \|^2 \leq B_1 \| f \|^2$$

and

$$A_2 \| K^* f \|^2 \leq \sum_{j \in J} v_j^2 \| \Gamma_j P_{W_j}(f) \|^2 \leq B_2 \| f \|^2.$$
Since $T_\Lambda T_\Gamma^* = \theta_H$, for any $f \in H$ we have

\[(4.1) \quad T_\Lambda \{v_j \Gamma_j P_{W_j}(f)\}_{j \in J} = \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* \Gamma_j P_{W_j}(f) = 0.\]

Also, $U + V$ is invertible, so

\[(4.2) \quad \|K^* f\|^2 = \|((U + V)^{-1})^* (U + V)^* K^* f\|^2 \leq \|(U + V)^{-1}\|^2 \| (U + V)^* K^* f\|^2.\]

Now, for any $f \in H$ we have

\[
\sum_{j \in J} v_j^2 \| (\Lambda_j + \Gamma_j) P_{W_j}(U + V)^* P_{(U + V)W_j}(f) \|^2 \\
= \sum_{j \in J} v_j^2 \| (\Lambda_j + \Gamma_j) P_{W_j}(U + V)^* (f)\|^2 \quad \text{(using Theorem 2.3)} \\
= \sum_{j \in J} v_j^2 \langle (\Lambda_j + \Gamma_j) P_{W_j}(T^*) f, (\Lambda_j + \Gamma_j) P_{W_j}(T^*) f \rangle \quad \text{(taking $T = U + V$)} \\
= \sum_{j \in J} v_j^2 \| \Lambda_j P_{W_j}(T^*) f \|^2 \quad \text{(using (4.1))} \\
\leq B_1 \|T^* f\|^2 + B_2 \|T^* f\|^2 \quad \text{(since $\Lambda, \Gamma$ are $K$-g-fusion frames)} \\
= (B_1 + B_2) \|(U + V)^* f\|^2 \quad \text{(since $T = U + V$)} \\
\leq (B_1 + B_2) \|U + V\|^2 \|f\|^2 \quad \text{(as $U + V$ is bounded).}
\]

On the other hand,

\[
\sum_{j \in J} v_j^2 \| (\Lambda_j + \Gamma_j) P_{W_j}(U + V)^* P_{(U + V)W_j}(f) \|^2 \\
= \sum_{j \in J} v_j^2 \| \Lambda_j P_{W_j}(U + V)^* f \|^2 + \sum_{j \in J} v_j^2 \| \Gamma_j P_{W_j}(U + V)^* f \|^2 \\
\geq \sum_{j \in J} v_j^2 \| \Lambda_j P_{W_j}(U + V)^* f \|^2 \\
\geq A_1 \|K^* (U + V)^* f\|^2 \quad \text{(since $\Lambda$ is $K$-g-fusion frame)} \\
= A_1 \| (U + V)^* K^* f\|^2 \quad \text{(using $K(U + V) = (U + V)K$)} \\
\geq A_1 \| (U + V)^{-1}\|^{-2} \|K^* f\|^2 \quad \text{(using (4.2)).}
\]

Therefore \{((U + V)W_j, (\Lambda_j + \Gamma_j) P_{W_j}(U + V)^*, v_j)\}_{j \in J} is a $K$-g-fusion frame and by Theorem 4.2, it is a $g$-atomic subspace of $H$ with respect to $K$. \qed
Corollary 4.5. Let Λ = \{(W_j, Λ_j, v_j)\}_{j \in J} and Γ = \{(W_j, Γ_j, v_j)\}_{j \in J} be two g-atomic subspaces of H with respect to K ∈ B(H) with the corresponding synthesis operators T_Λ and T_Γ. If T_Λ T_Γ^* = θ_H and U ∈ B(H) is an invertible operator with KU = UK, then \{(UW_j, (Λ_j + Γ_j)P_{W_j}U^*, v_j)\}_{j \in J} is a g-atomic subspace of H with respect to K.

Proof. The proof of this Corollary directly follows from Theorem 4.4 by putting \(V = \theta_H\).

Theorem 4.6. Let Λ = \{(W_j, Λ_j, v_j)\}_{j \in J} is a g-atomic subspace for K ∈ B(H) and S_Λ be the frame operator of Λ. If U ∈ B(H) is a positive and invertible operator on H, then \(Λ' = \{(\sum (I + U)W_j, Λ_jP_{W_j}(I + U)^*, v_j)\}_{j \in J}\) is a g-atomic subspace of H with respect to K. Moreover, for any natural number \(n\), \(Λ'' = \{(\sum (I + U^n)W_j, Λ_jP_{W_j}(I + U^n)^*, v_j)\}_{j \in J}\) is a g-atomic subspace of H with respect to K.

Proof. Since Λ is a g-atomic subspace with respect to K, by Theorem 4.2, it is a K-g-fusion frame for H. Then according to Theorem 2.11, there exists \(A > 0\) such that \(S_Λ ≥ AKK^*\). Now, for each \(f ∈ H\) we have

\[
\sum_{j \in J} v_j^2 ∥Λ_jP_{W_j}(I + U)^*P_{(t_H + U)}W_j(f)∥^2 \\
= \sum_{j \in J} v_j^2 ∥Λ_jP_{W_j}(I + U)^*(f)∥^2 \quad \text{(using Theorem 2.3)}
\]

\[
≤ B∥(I + U)^*(f)∥^2 \quad \text{(since Λ is a K-g-fusion frame)}
\]

\[
≤ B∥I + U∥^2∥f∥^2 \quad \text{(since (I + U) ∈ B(H)).}
\]

Thus, Λ' is a g-fusion Bessel sequence in H. Also, for each \(f ∈ H\) we have

\[
\sum_{j \in J} v_j^2 P_{(t_H + U)}W_j(Λ_jP_{W_j}(I + U)^*)Λ_jP_{W_j}(I + U)^*P_{(t_H + U)}W_j(f)
\]

\[
= \sum_{j \in J} v_j^2 P_{(t_H + U)}W_j(I + U)P_{W_j}Λ_j^*Λ_jP_{W_j}(I + U)^*P_{(t_H + U)}W_j(f)
\]

\[
= \sum_{j \in J} v_j^2 (P_{W_j}(I + U)^*P_{(t_H + U)}W_j)Λ_j^*Λ_jP_{W_j}(I + U)^*P_{(t_H + U)}W_j(f)
\]

\[
= \sum_{j \in J} v_j^2 (P_{W_j}(I + U)^*)Λ_j^*Λ_jP_{W_j}(I + U)^*(f) \quad \text{(using Theorem 2.3)}
\]

\[
= \sum_{j \in J} v_j^2 (I + U)P_{W_j}Λ_j^*Λ_jP_{W_j}(I + U)^*(f)
\]

\[
= (I + U)\sum_{j \in J} v_j^2 P_{W_j}Λ_j^*Λ_jP_{W_j}(I + U)^*(f) = (I + U)S_Λ(I + U)^*(f).
\]
This shows that the frame operator of $\Lambda'$ is $(I_H + U)S_{\Lambda}(I_H + U)^*$. Now,

$$(I_H + U)S_{\Lambda}(I_H + U)^* \geq S_{\Lambda} \geq AKK^* \quad \text{(since } U, S_{\Lambda} \text{ are positive}).$$

Then by Theorem 2.11, we can conclude that $\Lambda'$ is a $K$-$g$-fusion frame and therefore by Theorem 4.2, $\Lambda'$ is a $g$-atomic subspace of $H$ with respect to $K$. According to the preceding procedure, for any natural number $n$, the frame operator of $\Lambda''$ is $(I_H + U^n)S_{\Lambda}(I_H + U^n)^*$ and similarly, it can be shown that $\Lambda''$ is a $g$-atomic subspace of $H$ with respect to $K$.

\[\square\]

5. FRAME OPERATOR FOR A PAIR OF $g$-FUSION BESSEL SEQUENCES

In this section, we shall discuss the frame operator for a pair of $g$-fusion Bessel sequences and establish some properties relative to frame operator. At the end of this section, we shall construct a new $g$-fusion frame for the Hilbert space $H \oplus X$, using the $g$-fusion frames of the Hilbert spaces $H$ and $X$.

**Definition 5.1.** Let $\Lambda = \{(W_j, \Lambda_j, w_j)\}_{j \in J}$ and $\Gamma = \{(V_j, \Gamma_j, v_j)\}_{j \in J}$ be two $g$-fusion Bessel sequences in $H$ with bounds $D_1$ and $D_2$. Then the operator $S_{\Gamma\Lambda}: H \to H$, defined by

$$S_{\Gamma\Lambda}(f) = \sum_{j \in J} v_j w_j P_{V_j} \Gamma_j^* \Lambda_j P_{W_j}(f) \quad \forall f \in H,$$

is called the frame operator for the pair of $g$-fusion Bessel sequences $\Lambda$ and $\Gamma$.

**Theorem 5.2.** The frame operator $S_{\Gamma\Lambda}$ for the pair of $g$-fusion Bessel sequences $\Lambda$ and $\Gamma$ is bounded and $S_{\Gamma\Lambda}^* = S_{\Lambda\Gamma}$.

**Proof.** For each $f, g \in H$ we have

$$\langle S_{\Gamma\Lambda}(f), g \rangle = \left\langle \sum_{j \in J} v_j w_j P_{V_j} \Gamma_j^* \Lambda_j P_{W_j}(f), g \right\rangle = \sum_{j \in J} v_j w_j \langle \Lambda_j P_{W_j}(f), \Gamma_j P_{V_j}(g) \rangle.$$

By the Cauchy-Schwarz inequality, we obtain

$$|\langle S_{\Gamma\Lambda}(f), g \rangle| \leq \left(\sum_{j \in J} v_j^2 \|\Gamma_j P_{V_j}(g)\|^2\right)^{1/2} \left(\sum_{j \in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2\right)^{1/2} \leq \sqrt{D_2} \|g\| \sqrt{D_1} \|f\|.$$
This shows that $S_{\Gamma \Lambda}$ is a bounded operator with $\|S_{\Gamma \Lambda}\| \leq \sqrt{D_1 D_2}$. Now,

\[(5.3)\quad \|S_{\Gamma \Lambda} f\| = \sup_{\|g\|=1} |\langle S_{\Gamma \Lambda} (f), g \rangle| \leq \sup_{\|g\|=1} \sqrt{D_2} \|g\| \left( \sum_{j \in J} w_j^2 \|\Lambda_j P_{W_j} (f)\|^2 \right)^{1/2} \] (using (5.2))

and similarly, it can be shown that

\[(5.4)\quad \|S_{\Gamma \Lambda}^* g\| \leq \sqrt{D_1} \left( \sum_{j \in J} v_j^2 \|\Gamma_j P_{V_j} (g)\|^2 \right)^{1/2}.\]

Also, for each $f, g \in H$ we have

\[\langle S_{\Gamma \Lambda} (f), g \rangle = \left\langle \sum_{j \in J} v_j w_j P_{V_j} \Gamma_j^* \Lambda_j P_{W_j} (f), g \right\rangle = \sum_{j \in J} v_j w_j \langle f, P_{V_j} \Gamma_j^* \Lambda_j P_{W_j} (g) \rangle \]

and hence $S_{\Gamma \Lambda}^* = S_{\Lambda \Gamma}$.

**Theorem 5.3.** Let $S_{\Gamma \Lambda}$ be the frame operator for a pair of $g$-fusion Bessel sequences $\Lambda$ and $\Gamma$ with bounds $D_1$ and $D_2$, respectively. Then the following statements are equivalent:

(I) $S_{\Gamma \Lambda}$ is bounded below.

(II) There exists $K \in B(H)$ such that $\{T_j\}_{j \in J}$ is a resolution of the identity operator on $H$, where $T_j = v_j w_j K P_{V_j} \Gamma_j^* \Lambda_j P_{W_j}$, $j \in J$.

If one of the given conditions holds, then $\Lambda$ is a $g$-fusion frame.

**Proof.** (I) $\Rightarrow$ (II): Suppose that $S_{\Gamma \Lambda}$ is bounded below. Then for each $f \in H$ there exists $A > 0$ such that

\[\|f\|^2 \leq A \|S_{\Gamma \Lambda} f\|^2 \Rightarrow \langle I_H f, f \rangle \leq A \langle S_{\Gamma \Lambda}^* S_{\Gamma \Lambda} f, f \rangle \Rightarrow I_H^* I_H \leq A S_{\Gamma \Lambda}^* S_{\Gamma \Lambda}.\]

So, by Theorem 2.1, there exists $K \in B(H)$ such that $K S_{\Gamma \Lambda} = I_H$. Therefore for each $f \in H$ we have

\[f = K S_{\Gamma \Lambda} (f) = K \sum_{j \in J} v_j w_j P_{V_j} \Gamma_j^* \Lambda_j P_{W_j} (f) = \sum_{j \in J} v_j w_j K P_{V_j} \Gamma_j^* \Lambda_j P_{W_j} (f) = \sum_{j \in J} T_j (f)\]

and hence $\{T_j\}_{j \in J}$ is a resolution of the identity operator on $H$, where $T_j = v_j w_j K P_{V_j} \Gamma_j^* \Lambda_j P_{W_j}$.
(II) \implies (I): Since \( \{T_j\}_{j \in J} \) is a resolution of the identity operator on \( H \), for any \( f \in H \) we have
\[
f = \sum_{j \in J} T_j(f) = \sum_{j \in J} v_j w_j K \Gamma_j^* \Lambda_j P_{W_j}(f) = K \sum_{j \in J} v_j w_j \Gamma_j^* \Lambda_j P_{W_j}(f) = KS_{\Gamma \Lambda}(f).
\]
Thus, \( I_H = KS_{\Gamma \Lambda} \). So, by Theorem 2.1, there exists \( \alpha > 0 \) such that \( I_H I_H^* \leq \alpha S_{\Gamma \Lambda} S_{\Gamma \Lambda}^* \) and hence \( S_{\Gamma \Lambda} \) is bounded below.

Last part: First we suppose that \( S_{\Gamma \Lambda} \) is bounded below. Then for all \( f \in H \) there exists \( M > 0 \) such that \( \|S_{\Gamma \Lambda} f\| \geq M \|f\| \) and this implies that
\[
M^2 \|f\|^2 \leq \|S_{\Gamma \Lambda} f\|^2 \leq D_2 \sum_{j \in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \quad \text{(using (5.3))}
\]
\[
\Rightarrow M^2 \|f\|^2 \leq \sum_{j \in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2.
\]
Hence, \( \Lambda \) is a \( g \)-fusion frame for \( H \) with bounds \( M^2/D_2 \) and \( D_1 \).

Next, we suppose that the given condition (II) holds. Then for any \( f \in H \) we have
\[
f = \sum_{j \in J} v_j w_j K \Gamma_j^* \Lambda_j P_{W_j}(f), \quad K \in \mathcal{B}(H).
\]
By Cauchy-Schwarz inequality, for each \( f \in H \) we have
\[
\|f\|^2 = \langle f, f \rangle = \left( \sum_{j \in J} v_j w_j K \Gamma_j^* \Lambda_j P_{W_j}(f), f \right) = \sum_{j \in J} v_j w_j \langle \Lambda_j P_{W_j}(f), \Gamma_j P_{V_j}(K^* f) \rangle
\]
\[
\leq \left( \sum_{j \in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \right)^{1/2} \left( \sum_{j \in J} v_j^2 \|\Gamma_j P_{V_j}(K^* f)\|^2 \right)^{1/2}
\]
\[
\leq \sqrt{D_2} \|K^* f\| \left( \sum_{j \in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \right)^{1/2}
\]
\[
\leq \sqrt{D_2} \|K\| \|f\| \left( \sum_{j \in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \right)^{1/2}
\]
\[
\Rightarrow \frac{1}{D_2 \|K\|^2} \|f\|^2 \leq \sum_{j \in J} w_j^2 \|\Lambda_j P_{W_j}(f)\|^2.
\]
Therefore, in this case \( \Lambda \) is also a \( g \)-fusion frame for \( H \). \hfill \Box

**Theorem 5.4.** Let \( S_{\Gamma \Lambda} \) be the frame operator for a pair of \( g \)-fusion Bessel sequences \( \Lambda \) and \( \Gamma \) with bounds \( D_1 \) and \( D_2 \), respectively. Suppose \( \lambda_1 < 1, \lambda_2 > -1 \) such that for each \( f \in H \), \( \|f - S_{\Gamma \Lambda} f\| \leq \lambda_1 \|f\| + \lambda_2 \|S_{\Gamma \Lambda} f\| \). Then \( \Lambda \) is a \( g \)-fusion frame for \( H \).
Thus, \( \Lambda \) is a \( g \)-fusion frame for \( H \) with bounds \((1 - \lambda_1)^2(1 + \lambda_2)^{-2}D_2^{-1} \) and \( D_1 \).

**Theorem 5.5.** Let \( S_{\Gamma\Lambda} \) be the frame operator for a pair of \( g \)-fusion Bessel sequences \( \Lambda \) and \( \Gamma \) of bounds \( D_1 \) and \( D_2 \), respectively. Assume \( \lambda \in [0,1) \) such that

\[
\|f - S_{\Gamma\Lambda}f\| \leq \lambda\|f\| \quad \forall f \in H.
\]

Then \( \Lambda \) and \( \Gamma \) are \( g \)-fusion frames for \( H \).

**Proof.** By putting \( \lambda_1 = \lambda \) and \( \lambda_2 = 0 \) in (5.5), we get

\[
\frac{(1 - \lambda)^2}{D_2} \|f\|^2 \leq \sum_{j \in J} w_j^2 \|\Lambda_j P_{V_j} (f)\|^2
\]

and therefore \( \Lambda \) is a \( g \)-fusion frame. Now, for each \( f \in H \) we have

\[
\|f - S_{\Gamma\Lambda}^* f\| = \|(I_H - S_{\Gamma\Lambda})^* f\| \leq \|(I_H - S_{\Gamma\Lambda})\||f\| \leq \lambda\|f\|
\]

\[
\Rightarrow (1 - \lambda)\|f\| \leq \|S_{\Gamma\Lambda}^* f\| \leq \sqrt{D_1} \left( \sum_{j \in J} v_j^2 \|\Gamma_j P_{V_j} (f)\|^2 \right)^{1/2} \quad \text{(using (5.4))}
\]

\[
\Rightarrow \sum_{j \in J} v_j^2 \|\Gamma_j P_{V_j} (f)\|^2 \geq \frac{(1 - \lambda)^2}{D_1} \|f\|^2 \quad \forall f \in H.
\]

Hence, \( \Gamma \) is a \( g \)-fusion frame with bounds \((1 - \lambda)^2/D_1 \) and \( D_2 \).

**Definition 5.6.** Let \( H \) and \( X \) be two Hilbert spaces. Define

\[
H \oplus X = \{(f, g) : f \in H, g \in X\}.
\]

Then \( H \oplus X \) forms a Hilbert space with respect to point-wise operations and inner product defined by

\[
\langle (f, g), (f', g') \rangle = \langle f, f' \rangle_H + \langle g, g' \rangle_X \quad \forall f, f' \in H \text{ and } \forall g, g' \in X.
\]

Now, if \( U \in B(H, Z), V \in B(X, Y) \), then for all \( f \in H, g \in X \) we define

\[
U \oplus V \in B(H \oplus X, Z \oplus Y) \quad \text{by} \quad (U \oplus V) (f, g) = (U f, V g),
\]
and \((U \oplus V)^* = U^* \oplus V^*\), where \(Z, Y\) are Hilbert spaces and also we define 
\[P_{M \oplus N}(f, g) = (P_M f, P_N g),\]
where \(P_M, P_N\) and \(P_{M \oplus N}\) are orthonormal projections onto the closed subspaces \(M \subset H, N \subset X\) and \(M \oplus N \subset H \oplus X\), respectively.

From here we assume that for each \(j \in J\), \(W_j \oplus V_j\) are the closed subspaces of \(H \oplus X\) and \(\Gamma_j \in \mathcal{B}(X, X_j)\), where \(\{X_j\}_{j \in J}\) is the collection of Hilbert spaces and \(\Lambda_j \oplus \Gamma_j \in \mathcal{B}(H \oplus X, H_j \oplus X_j)\).

**Theorem 5.7.** Let \(\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}\) be a \(g\)-fusion frame for \(H\) with bounds \(A, B\) and \(\Gamma = \{(V_j, \Gamma_j, v_j)\}_{j \in J}\) be a \(g\)-fusion frame for \(X\) with bounds \(C, D\). Then \(\Lambda \oplus \Gamma = \{(W_j \oplus V_j, \Lambda_j \oplus \Gamma_j, v_j)\}_{j \in J}\) is a \(g\)-fusion frame for \(H \oplus X\) with bounds \(\min\{A, C\}, \max\{B, D\}\). Furthermore, if \(S_\Lambda, S_\Gamma\) and \(S_{\Lambda \oplus \Gamma}\) are \(g\)-fusion frame operators for \(\Lambda, \Gamma\) and \(\Lambda \oplus \Gamma\), respectively, then we have \(S_{\Lambda \oplus \Gamma} = S_\Lambda \oplus S_\Gamma\).

**Proof.** Let \((f, g) \in H \oplus X\) be an arbitrary element. Then
\[
\sum_{j \in J} v_j^2 \| (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} (f, g) \|^2
\]
\[
= \sum_{j \in J} v_j^2 \langle (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} (f, g), (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} (f, g) \rangle
\]
\[
= \sum_{j \in J} v_j^2 \langle \Lambda_j \oplus \Gamma_j (P_{W_j} (f), P_{V_j} (g)), \Lambda_j \oplus \Gamma_j (P_{W_j} (f), P_{V_j} (g)) \rangle
\]
\[
= \sum_{j \in J} v_j^2 \langle (\Lambda_j P_{W_j} (f), \Gamma_j P_{V_j} (g)), (\Lambda_j P_{W_j} (f), \Gamma_j P_{V_j} (g)) \rangle
\]
\[
= \sum_{j \in J} v_j^2 \langle (\Lambda_j P_{W_j} (f), \Lambda_j P_{W_j} (f))_H + (\Gamma_j P_{V_j} (g), \Gamma_j P_{V_j} (g))_X \rangle
\]
\[
= \sum_{j \in J} v_j^2 (\| \Lambda_j P_{W_j} (f) \|^2_H + \| \Gamma_j P_{V_j} (g) \|^2_X)
\]
\[
= \sum_{j \in J} v_j^2 \| \Lambda_j P_{W_j} (f) \|^2_H + \sum_{j \in J} v_j^2 \| \Gamma_j P_{V_j} (g) \|^2_X
\]
\[
\leq B \| f \|^2_H + D \| g \|^2_X \quad \text{(since } \Lambda, \Gamma \text{ are } g\text{-fusion frames)}
\]
\[
\leq \max\{B, D\} (\| f \|^2_H + \| g \|^2_X) = \max\{B, D\} \| (f, g) \|^2.
\]

Similarly, it can be shown that
\[
\min\{A, C\} \| (f, g) \|^2 \leq \sum_{j \in J} v_j^2 \langle (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} (f, g), (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} (f, g) \rangle.
\]

Therefore, for all \((f, g) \in H \oplus X\) we have
\[
A_1 \| (f, g) \|^2 \leq \sum_{j \in J} v_j^2 \| (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} (f, g) \|^2 \leq B_1 \| (f, g) \|^2
\]
and hence $\Lambda \oplus \Gamma$ is a $g$-fusion frame for $H \oplus X$ with bounds $A_1 = \min\{A, C\}$ and $B_1 = \max\{B, D\}$. Furthermore, for $(f, g) \in H \oplus X$ we have

$$S_{\Lambda \oplus \Gamma}(f, g) = \sum_{j \in J} v_j^2 P_{W_j \oplus V_j} (\Lambda_j \oplus \Gamma_j)^* (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j}(f, g)$$

$$= \sum_{j \in J} v_j^2 P_{W_j \oplus V_j} (\Lambda_j \oplus \Gamma_j)^* (\Lambda_j \oplus \Gamma_j) (P_{W_j}(f), P_{V_j}(g))$$

$$= \sum_{j \in J} v_j^2 P_{W_j \oplus V_j} (\Lambda_j \oplus \Gamma_j)^* (\Lambda_j P_{W_j}(f), \Gamma_j P_{V_j}(g))$$

$$= \sum_{j \in J} v_j^2 P_{W_j \oplus V_j} (\Lambda_j^* \Lambda_j P_{W_j}(f), \Gamma_j^\dagger \Gamma_j P_{V_j}(g))$$

$$= \sum_{j \in J} v_j^2 (P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}(f), P_{V_j} \Gamma_j^\dagger \Gamma_j P_{V_j}(g))$$

$$= \left( \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}(f), \sum_{j \in J} v_j^2 P_{V_j} \Gamma_j^\dagger \Gamma_j P_{V_j}(g) \right)$$

$$= (S_\Lambda(f), S_\Gamma(g))$$

$$= (S_{\Lambda \oplus \Gamma} \oplus \Gamma)(f, g) \quad \forall (f, g) \in H \oplus X.$$

Hence, $S_{\Lambda \oplus \Gamma} = S_\Lambda \oplus S_\Gamma$. This completes the proof. \hfill \Box

**Theorem 5.8.** Let $\Lambda \oplus \Gamma = \{(W_j \oplus V_j, \Lambda_j \oplus \Gamma_j, v_j)\}_{j \in J}$ be a $g$-fusion frame for $H \oplus X$ with frame operator $S_{\Lambda \oplus \Gamma}$. Then

$$\Delta' = \{(S_{\Lambda \oplus \Gamma}^{-1/2}(W_j \oplus V_j), (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1/2}, v_j)\}_{j \in J}$$

is a Parseval $g$-fusion frame for $H \oplus X$.

**Proof.** Since $S_{\Lambda \oplus \Gamma}$ is a positive operator, there exists a unique positive square root $S_{\Lambda \oplus \Gamma}^{1/2}$ (or $S_{\Lambda \oplus \Gamma}^{-1/2}$) and they commute with $S_{\Lambda \oplus \Gamma}$ and $S_{\Lambda \oplus \Gamma}^{-1}$. Therefore, each $(f, g) \in H \oplus X$ can be written as

$$(f, g) = S_{\Lambda \oplus \Gamma}^{-1/2} S_{\Lambda \oplus \Gamma} S_{\Lambda \oplus \Gamma}^{-1/2}(f, g)$$

$$= \sum_{j \in J} v_j^2 S_{\Lambda \oplus \Gamma}^{-1/2} P_{W_j \oplus V_j} (\Lambda_j \oplus \Gamma_j)^* (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1/2}(f, g).$$
Now, for each \((f, g) \in H \oplus X\) we have

\[
\| (f, g) \|^2 = \langle (f, g), (f, g) \rangle = \left\langle \sum_{j \in J} v_j^2 S_{\Lambda \oplus \Gamma}^{-1/2} P_{W_j \oplus V_j} (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1/2} (f, g), (f, g) \right\rangle \\
= \sum_{j \in J} v_j^2 \| (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1/2} (f, g) \|^2 \\
\leq \sum_{j \in J} v_j^2 \| (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1/2} P_{S_{\Lambda \oplus \Gamma}^{-1/2} (W_j \oplus V_j)} (f, g) \|^2
\]

(by Theorem 2.3).

This shows that \(\Delta'\) is a Parseval \(g\)-fusion frame for \(H \oplus X\). \qed

**Theorem 5.9.** Let \(\Lambda \oplus \Gamma = \{(W_j \oplus V_j, \Lambda_j \oplus \Gamma_j, v_j)\}_{j \in J}\) be a \(g\)-fusion frame for \(H \oplus X\) with bounds \(A_1, B_1\) and \(S_{\Lambda \oplus \Gamma}\) be the corresponding frame operator. Then

\[
\Delta = \{(S_{\Lambda \oplus \Gamma}^{-1}(W_j \oplus V_j), (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1}, v_j)\}_{j \in J}
\]

is a \(g\)-fusion frame for \(H \oplus X\) with frame operator \(S_{\Lambda \oplus \Gamma}^{-1}\).

**Proof.** For any \((f, g) \in H \oplus X\) we have

\[
(f, g) = S_{\Lambda \oplus \Gamma} S_{\Lambda \oplus \Gamma}^{-1}(f, g) = \sum_{j \in J} v_j^2 P_{W_j \oplus V_j} (\Lambda_j \oplus \Gamma_j)^* (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1}(f, g).
\]

By Theorem 2.3, for any \((f, g) \in H \oplus X\) we have

\[
\sum_{j \in J} v_j^2 \| (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1} P_{S_{\Lambda \oplus \Gamma}^{-1}(W_j \oplus V_j)} (f, g) \|^2
\]

\[
\leq B_1 \| S_{\Lambda \oplus \Gamma}^{-1} \|^2 \| (f, g) \|^2 \quad \text{(since } \Lambda \oplus \Gamma \text{ is } g\)-fusion frame).
On the other hand, using (5.6), we get
\[
\|(f, g)\|^4 = |\langle (f, g), (f, g) \rangle|^2 \\
= \left| \left\langle \sum_{j \in J} v_j^2 P_{W_j \oplus V_j} (\Lambda_j \oplus \Gamma_j)^* (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1} (f, g), (f, g) \right\rangle \right|^2 \\
= \left| \sum_{j \in J} v_j^2 \langle (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1} (f, g), (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} (f, g) \rangle \right|^2 \\
\leq \sum_{j \in J} v_j^2 \|(\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1} (f, g)\|^2 \sum_{j \in J} v_j^2 \|(\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} (f, g)\|^2 \\
\leq \sum_{j \in J} v_j^2 \|(\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1} (f, g)\|^2 B_1 \|(f, g)\|^2 \\
\quad \text{(as } \Lambda \oplus \Gamma \text{ is } g\text{-fusion frame)} \\
= B_1 \|(f, g)\|^2 \sum_{j \in J} v_j^2 \|(\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1} P_{S_{\Lambda \oplus \Gamma}^{-1} (W_j \oplus V_j)} (f, g)\|^2 \\
\quad \text{(from (5.7)).}
\]

Therefore
\[
B_1^{-1} \|(f, g)\|^2 \leq \sum_{j \in J} v_j^2 \|(\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1} P_{S_{\Lambda \oplus \Gamma}^{-1} (W_j \oplus V_j)} (f, g)\|^2.
\]

Hence, \( \Delta \) is a \( g \)-fusion frame for \( H \oplus X \). Let \( S_\Delta \) be the \( g \)-fusion frame operator for \( \Delta \) and take \( \Delta_j = \Lambda_j \oplus \Gamma_j \). Now, for each \( (f, g) \in H \oplus X, S_\Delta (f, g) \)
\[
= \sum_{j \in J} v_j^2 P_{S_{\Lambda \oplus \Gamma}^{-1} (W_j \oplus V_j)} (\Delta_j P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1} (f, g))’ (\Delta_j P_{W_j \oplus V_j} S_{\Lambda \oplus \Gamma}^{-1} (f, g))’ P_{S_{\Lambda \oplus \Gamma}^{-1} (W_j \oplus V_j)} (f, g) \\
= \sum_{j \in J} v_j^2 (P_{W_j \oplus V_j} S^{-1}_{\Lambda \oplus \Gamma} P_{S_{\Lambda \oplus \Gamma}^{-1} (W_j \oplus V_j)})’ (P_{W_j \oplus V_j} S^{-1}_{\Lambda \oplus \Gamma} P_{S_{\Lambda \oplus \Gamma}^{-1} (W_j \oplus V_j)}) (f, g) \\
= \sum_{j \in J} v_j^2 (P_{W_j \oplus V_j} S^{-1}_{\Lambda \oplus \Gamma})’ (P_{W_j \oplus V_j} S^{-1}_{\Lambda \oplus \Gamma}) (f, g) \\
\quad \text{(using Theorem 2.3)} \\
= \sum_{j \in J} v_j^2 S^{-1}_{\Lambda \oplus \Gamma} P_{W_j \oplus V_j} (\Lambda_j \oplus \Gamma_j)’ (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} S^{-1}_{\Lambda \oplus \Gamma} (f, g) \\
= S^{-1}_{\Lambda \oplus \Gamma} \left( \sum_{j \in J} v_j^2 (P_{W_j \oplus V_j} (\Lambda_j \oplus \Gamma_j)’ (\Lambda_j \oplus \Gamma_j) P_{W_j \oplus V_j} (S^{-1}_{\Lambda \oplus \Gamma} (f, g))) \right) \\
= S^{-1}_{\Lambda \oplus \Gamma} S_{\Lambda \oplus \Gamma} (S^{-1}_{\Lambda \oplus \Gamma} (f, g))’ \quad \text{(by definition of } S_{\Lambda \oplus \Gamma}) \\
= S^{-1}_{\Lambda \oplus \Gamma} (f, g).
\]

Thus, \( S_\Delta = S^{-1}_{\Lambda \oplus \Gamma} \). This completes the proof. \( \square \)
Note 5.10. Form Theorem 5.9 we can conclude that if $\Lambda \oplus \Gamma$ is a $g$-fusion frame for $H \oplus K$, then $\Delta$ is also a $g$-fusion frame for $H \oplus K$. The $g$-fusion frame $\Delta$ is a called the canonical dual $g$-fusion frame of $\Lambda \oplus \Gamma$.

References


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