OSCILLATION CRITERIA FOR TWO DIMENSIONAL LINEAR NEUTRAL DELAY DIFFERENCE SYSTEMS

ARUN KUMAR TRIPATHY, Sambalpur

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Abstract. In this work, necessary and sufficient conditions for the oscillation of solutions of 2-dimensional linear neutral delay difference systems of the form

\[ \Delta \begin{bmatrix} x(n) + p(n)x(n - m) \\ y(n) + p(n)y(n - m) \end{bmatrix} = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix} \begin{bmatrix} x(n - \alpha) \\ y(n - \beta) \end{bmatrix}, \]

are established, where \( m > 0, \alpha \geq 0, \beta \geq 0 \) are integers and \( a(n), b(n), c(n), d(n), p(n) \) are sequences of real numbers.

Keywords: oscillation; nonoscillation; system of neutral equations; Krasnoselskii’s fixed point theorem

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1. INTRODUCTION

Consider the 2-dimensional difference system

\[ (S_1) \quad \Delta \begin{bmatrix} x(n) + p(n)x(n - m) \\ y(n) + p(n)y(n - m) \end{bmatrix} = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix} \begin{bmatrix} x(n - \alpha) \\ y(n - \beta) \end{bmatrix}, \]

where \( m > 0, \alpha \geq 0, \beta \geq 0 \) are integers and \( a(n), b(n), c(n), d(n), p(n) \) are sequences of real numbers for \( n \in \mathbb{N}(n_0) = \{n_0, n_0 + 1, n_0 + 2, \ldots\}, n_0 \geq 0 \). If \( \alpha = 0, \beta = 0 \) and \( p(n) \equiv 0 \) for all \( n \), then \( (S_1) \) reduces to

\[ (S_2) \quad \begin{bmatrix} x(n + 1) \\ y(n + 1) \end{bmatrix} = \begin{bmatrix} a_1(n) & b_1(n) \\ c_1(n) & d_1(n) \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}. \]

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In [17], Tripathy has studied the oscillatory behaviour of solutions of the system (S₂) along with the oscillatory behaviour of solutions of the system

\[
(S_3) \quad \begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} a_1(n) & b_1(n) \\ c_1(n) & d_1(n) \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} + \begin{bmatrix} f_1(n) \\ f_2(n) \end{bmatrix}.
\]

Indeed, (S₁) and (S₂) are not viewed as the direct discrete analogue of their continuous counterparts, so the work [17] is challenging, being done with the help of the work [11]. In this work, the oscillation and nonoscillation criteria for (S₁) are established unlike to the work [17]. Of course, the study of (S₁) is not so much simple when \( \alpha > 0, \beta > 0 \) and \( p(n) \neq 0 \) for all \( n \).

In [7], [8], [9], Graef and Thandapani, Jiang and Tang, and Li have studied the oscillatory and asymptotic behaviour of all vector solutions of the system of the form

\[
(S_4) \quad \begin{bmatrix} \Delta x(n) \\ \Delta y(n-1) \end{bmatrix} = \begin{bmatrix} 0 & b(n) \\ -c(n) & 0 \end{bmatrix} \begin{bmatrix} f(x(n)) \\ g(y(n)) \end{bmatrix},
\]

where \( f, g \in C(\mathbb{R}, \mathbb{R}) \) such that \( uf(u) > 0 \) and \( ug(u) > 0 \) for \( u \neq 0 \). We may note that (S₄) is a special case of (S₁), if we let \( f(u) = u \) and \( g(u) = u \). It is known that a similar kind of results can be obtained for

\[
(S_5) \quad \Delta \begin{bmatrix} x(n) + p(n)x(n-m) \\ y(n) + p(n)y(n-m) \end{bmatrix} = \begin{bmatrix} a(n) & 0 \\ c(n) & 0 \end{bmatrix} \begin{bmatrix} x(n-\alpha) \\ y(n-\beta) \end{bmatrix}
\]

as long as the works [7], [8] and [9] are concerned.

Consider a particular case of (S₁) as

\[
(S_6) \quad \Delta \begin{bmatrix} x(n) + p(n)x(n-m) \\ y(n) + p(n)y(n-m) \end{bmatrix} = \begin{bmatrix} a(n) & 0 \\ 0 & d(n) \end{bmatrix} \begin{bmatrix} x(n-\alpha) \\ y(n-\beta) \end{bmatrix}
\]

from which we find two first-order neutral delay difference equations

\[
(1.1) \quad \Delta[x(n) + p(n)x(n-m)] - a(n)x(n-\alpha) = 0,
\]

\[
(1.2) \quad \Delta[y(n) + p(n)y(n-m)] - d(n)y(n-\beta) = 0.
\]

A close observation reveals that the oscillation properties of (1.1) and (1.2) are studied by Parhi and Tripathy in their works [12] and [13] and hence the fact that (1.1) and (1.2) are oscillatory implies that (S₆) is oscillatory when \( a(n)d(n) \neq 0 \) for all \( n \). Hence, we do not discuss the oscillation properties of (S₁) when either \( a(n) = 0 = d(n) \) (as in (S₅)) or \( b(n) = 0 = c(n) \) (as in (S₆)) for all \( n \). In this work, our objective is to present the oscillatory behaviour of all vector solutions of (S₁) when \( a(n) \neq 0, b(n) \neq 0, c(n) \neq 0, d(n) \neq 0 \) for all \( n \). Up to our best understanding, the present work is a new finding in the literature. However, there are some works
(see, e.g., [4], [5], [10], [14], [15], [16]) in which the authors have studied oscillation and nonoscillation properties of some kind of neutral and nonneutral systems of equations that are not in the closed forms like \((S_1), (S_2)\) and \((S_3)\). Concerning difference equations and systems of difference equations, we refer to the monographs by Agarwal et al. (see [3], [1]) and by Elyadi (see [6]).

**Definition 1.1.** By a solution of \((S_1)\) we mean a vector \(X(n) = [x(n), y(n)]^\top\) which satisfies \((S_1)\) for \(n \in \mathbb{N}(-\varrho) = \{-\varrho, -\varrho + 1, \ldots, 0, 1, 2, \ldots\}\), where \(\varrho = \max\{m, \alpha, \beta\}\). We say that the solution \(X(n)\) oscillates componentwise or simply oscillates or strongly oscillates, if each component oscillates. Otherwise, the solution \(X(n)\) is called nonoscillatory. Therefore, a solution of \((S_1)\) is nonoscillatory if it has a component which is eventually positive or eventually negative, and strongly nonoscillatory if both components of \(X(n)\) are nonoscillatory. A vector solution \(X(n)\) of \((S_1)\) has the property that it oscillates or converges to zero as \(n \to \infty\), if each component of \(X(n)\) has this property.

**Lemma 1.1 ([13]).** Let \(f(n), g(n)\) and \(p(n)\) be real valued functions of discrete arguments defined for \(n \geq n_0\) such that \(f(n) = g(n) + p(n)g(n - m)\), \(n \geq n_0 + m\), where \(m \geq 0\) is an integer. Suppose that there exist real numbers \(b_1, b_2, b_3, b_4\) such that \(p(n)\) is in one of the following ranges:

1. \(-\infty < b_1 \leq p(n) \leq 0,
2. \(0 \leq p(n) \leq b_2 < 1,
3. \(1 < b_3 \leq p(n) \leq b_4 < \infty\).

If \(g(n) > 0\) for \(n \geq n_0\), \(\liminf_{n \to \infty} g(n) = 0\), and \(\lim_{n \to \infty} f(n) = L\) exists, then \(L = 0\).

**Theorem 1.1 ([2]).** Let \(X\) be a Banach space. Let \(\Omega\) be a bounded closed convex subset of \(X\) and let \(T_1, T_2\) be maps of \(\Omega\) into \(X\) such that \(T_1x + T_2y \in \Omega\) for every pair \(x, y \in \Omega\). If \(T_1\) is a contraction and \(T_2\) is completely continuous, then the equation \(T_1x + T_2x = x\) has a solution in \(\Omega\).

2. Oscillation Criteria

In this section, necessary and sufficient conditions are established for the oscillation of all vector solutions of the system \((S_1)\).

**Theorem 2.1.** Let \(0 < p(n) \leq r < 1\) for large \(n\). Assume that \(a(n) < 0, b(n) > 0, c(n) > 0, d(n) < 0\) are for large \(n\) such that

\[ (A_1) \quad \sum_{n=0}^{\infty} b(n) < \infty, \quad \sum_{n=0}^{\infty} c(n) < \infty. \]
Then every bounded vector solution of \((S_1)\) either strongly oscillates or converges to zero if and only if
\[
(A_2) \quad \sum_{n=0}^{\infty} a(n) = -\infty, \quad \sum_{n=0}^{\infty} d(n) = -\infty.
\]

**Proof.** On the contrary, let \(X(n) = [x(n), y(n)]^T\) be a strongly nonoscillatory bounded vector solution of \((S_1)\) such that \(x(n) > 0, x(n-m) > 0, x(n-\alpha) > 0, x(n-\beta) > 0\) and \(y(n) > 0, y(n-m) > 0, y(n-\alpha) > 0, y(n-\beta) > 0\) for \(n \geq n_0 > q\).

Setting
\[
K(n) = \sum_{i=n}^{\infty} b(i)y(i-\beta), \quad T(n) = \sum_{i=n}^{\infty} c(i)x(i-\alpha);
\]
\[
u(n) = x(n) + p(n)x(n-m), \quad v(n) = y(n) + p(n)y(n-m)
\]
for \((S_1)\), we find that
\[
\Delta[u(n) + K(n)] = a(n)x(n-\alpha) \leq 0, \tag{2.1}
\]
\[
\Delta[v(n) + T(n)] = d(n)y(n-\beta) \leq 0 \tag{2.2}
\]
for \(n \geq n_1 > n_0\). Hence, there exists \(n_2 > n_1\) such that \([u(n) + K(n)]\) and \([v(n) + T(n)]\) are monotonic for \(n \geq n_2\). Since \(u(n) > 0, v(n) > 0\) and \(\lim_{n \to \infty} K(n) < \infty, \lim_{n \to \infty} T(n) < \infty\), then \(\lim_{n \to \infty} u(n)\) exists and \(\lim_{n \to \infty} v(n)\) exists. We claim that \(\liminf_{n \to \infty} x(n) = 0 = \liminf_{n \to \infty} y(n)\). If not, we can find \(n_3 > n_2\) such that \(x(n-\alpha) > \gamma\) and \(y(n-\beta) > \eta\) for \(n \geq n_3\). Therefore, summing (2.1) and (2.2) from \(n_3\) to \(\infty\), we obtain contradictions to the hypothesis \((A_2)\). So, our claim holds. By Lemma 1.1, it follows that \(\lim_{n \to \infty} u(n) = 0 = \lim_{n \to \infty} v(n)\). Ultimately, \(u(n) \geq x(n)\) and \(v(n) \geq y(n)\)
implies that \(\lim_{n \to \infty} x(n) = 0 = \lim_{n \to \infty} y(n)\). The above argument is analogous, if we assume that \(x(n) < 0, x(n-m) < 0, x(n-\alpha) < 0, x(n-\beta) < 0\) and \(y(n) < 0, y(n-m) < 0, y(n-\alpha) < 0, y(n-\beta) < 0\) for \(n \geq n_0 > q\).

Next, we consider the case when \(x(n) > 0, x(n-m) > 0, x(n-\alpha) > 0, x(n-\beta) > 0\) and \(y(n) < 0, y(n-m) < 0, y(n-\alpha) < 0, y(n-\beta) < 0\) for \(n \geq n_0 > q\). Then
\[
\Delta[u(n) + K(n)] = a(n)x(n-\alpha) \leq 0, \tag{2.3}
\]
\[
\Delta[v(n) + T(n)] = d(n)y(n-\beta) \geq 0 \tag{2.4}
\]
and hence \([u(n) + K(n)]\) and \([v(n) + T(n)]\) are monotonic as well as bounded also for \(n \geq n_2\). Consequently, \(\lim_{n \to \infty} [u(n) + K(n)]\) and \(\lim_{n \to \infty} [v(n) + T(n)]\) exist. Using the above argument, it is easy to see that \(\lim_{n \to \infty} X(n) = [0,0]^T\). The case \(x(n) < 0, x(n-m) < 0, x(n-\alpha) < 0, x(n-\beta) < 0\) and \(y(n) > 0, y(n-m) > 0, y(n-\alpha) > 0, y(n-\beta) > 0\) for \(n \geq n_0 > q\) is similar.
Conversely, let us assume that \((A_2)\) fails to hold. Let \(B\) denote the Banach space of all bounded sequences in \(\mathbb{R}^2\) with the supremum norm, i.e., \(B = \{X: \mathbb{N} \rightarrow \mathbb{R}^2: \|X\| = \sup_{n \in \mathbb{N}} |X| < \infty \}\). For a fixed real number \(k > 0\), put

\[
\Omega = \{X \in B: x(n), y(n) \in I, n \in \mathbb{N}\},
\]

where \(I = \left[\frac{1}{3}k(1-r), k\right]\). Indeed, \(\Omega \subset B\) is closed, bounded and convex. Due to \((A_1)\), we can find \(n_1 > 0\) such that

\[
\sum_{n=n_1}^{\infty} |a(n)| < \frac{(1-r)}{6}, \quad \sum_{n=n_1}^{\infty} |b(n)| < \frac{(1-r)}{6},
\]

\[
\sum_{n=n_1}^{\infty} |c(n)| < \frac{(1-r)}{6}, \quad \sum_{n=n_1}^{\infty} |d(n)| < \frac{(1-r)}{6}.
\]

Let us define the maps \(G, H: \Omega \rightarrow B\) such that

\[
(GX)(n) = \begin{bmatrix}
\frac{(2+r)k}{3} - p(n)x(n-m) - \sum_{s=n}^{\infty} a(s)x(s-\alpha) \\
\frac{(2+r)k}{3} - p(n)y(n-m) - \sum_{s=n}^{\infty} d(s)y(s-\beta)
\end{bmatrix}
\]

for \(n \geq n_1\),

\[
(GX)(n) = (GX)(n_1) \quad \text{for } 0 < n < n_1
\]

and

\[
(HX)(n) = \begin{bmatrix}
- \sum_{s=n}^{\infty} b(s)y(s-\beta) \\
- \sum_{s=n}^{\infty} c(s)x(s-\alpha)
\end{bmatrix}
\]

for \(n \geq n_1\),

\[
(HX)(n) = (HX)(n_1) \quad \text{for } 0 < n < n_1.
\]

We rewrite \(G, H\) as

\[
G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.
\]

Let \(X, Y \in \Omega\). Then for \(n \geq n_1\),

\[
(G_1X)(n) + (H_1Y)(n) = \frac{(2+r)k}{3} - p(n)x(n-m)
- \sum_{s=n}^{\infty} a(s)x(s-\alpha) - \sum_{s=n}^{\infty} b(s)y(s-\beta)
\leq \frac{(2+r)k}{3} + \sum_{s=n}^{\infty} |a(s)|x(s-\alpha) + \sum_{s=n}^{\infty} |b(s)|y(s-\beta)
\leq \frac{(2+r)k}{3} + \frac{(1-r)k}{6} + \frac{(1-r)k}{6} = k.
\]
and

\[(G_1X)(n) + (H_1Y)(n)\]
\[= \frac{(2 + r)k}{3} - p(n)x(n - m) - \sum_{s=n}^{\infty} a(s)x(s - \alpha) - \sum_{s=n}^{\infty} b(s)y(s - \beta)\]
\[\geq \frac{(2 + r)k}{3} - p(n)x(n - m) - \sum_{s=n}^{\infty} |a(s)||x(s - \alpha) - \sum_{s=n}^{\infty} |b(s)||y(s - \beta)\]
\[\geq \frac{(2 + r)k}{3} - rk - \frac{(1 - r)k}{6} - \frac{(1 - r)k}{6} = \frac{k(1 - r)}{3} .\]

A similar observation can be made for \((G_2X)(n) + (H_2Y)(n)\), \(n \geq n_1\). Hence, \(GX + HY \in \Omega\). For \(X_1, X_2 \in \Omega\), it is easy to verify that

\[|(G_1X_1)(n) - (G_1X_2)(n)| \leq r|x_1(n - m) - x_2(n - m)| + \sum_{s=n}^{\infty} |a(s)||x_1(s - \alpha) - x_2(s - \alpha)|\]
\[\leq \left[ r + \frac{(1 - r)k}{6} \right] \|x_1 - x_2\| = \frac{(5r + 1)}{6} \|x_1 - x_2\| ,\]

and

\[|(G_2X_1)(n) - (G_2X_2)(n)| \leq \frac{(5r + 1)}{6} \|y_1 - y_2\| \]

for \(n \geq n_1\) implies that

\[\|GX_1 - GX_2\| \leq \frac{(5r + 1)}{6} \|X_1 - X_2\| ,\]

that is, \(G\) is a contraction mapping.

Next, we show that \(H\) is continuous. Let \(X_j = [x_j, y_j]^T \in \Omega\) for any \(j \in \mathbb{N}\). Let \(X_j(n)\) be such that \(x_j(n) \to x(n)\) and \(y_j(n) \to y(n)\) as \(j \to \infty\). If we choose \(X = [x, y]^T\), then \(X_j \in \Omega\) implies that \(X \in \Omega\) and hence \(x(n), y(n) \in I\) for \(n \geq n_1\). Therefore,

\[|(H_1X_j)(n) - (H_1X)(n)| \leq \sum_{s=n}^{\infty} |b(s)||y_j(s - \beta) - y(s - \beta)| \to 0 \quad \text{as} \quad j \to \infty ,\]
\[|(H_2X_j)(n) - (H_2X)(n)| \leq \sum_{s=n}^{\infty} |c(s)||x_j(s - \alpha) - x(s - \alpha)| \to 0 \quad \text{as} \quad j \to \infty ,\]

imply that

\[||(HX_j) - (HX)|| \to 0 \quad \text{as} \quad j \to \infty ,\]

that is, \(H\) is continuous. To complete the proof of the theorem, we need to show that \(H\Omega\) is uniformly Cauchy. Indeed, for \(\varepsilon > \frac{2}{3}k(1 - r) > 0\), we can find \(n_2 > n_1\)
such that for \( n \geq n_2 \)

\[
\sum_{s=n}^{\infty} |b(s)||y(s - \beta)| < \frac{\varepsilon}{2}, \quad \sum_{s=n}^{\infty} |c(s)||x(s - \alpha)| < \frac{\varepsilon}{2}.
\]

Hence for \( n_4 > n_3 > n_2 \), it follows that

\[
|(H_1 X)(n_4) - (H_1 X)(n_3)| = \left| \sum_{s=n_4}^{\infty} b(s)y(s - \beta) - \sum_{s=n_3}^{\infty} b(s)y(s - \beta) \right| \\
\leq \sum_{s=n_4}^{\infty} |b(s)||y(s - \beta)| + \sum_{s=n_3}^{\infty} |b(s)||y(s - \beta)| < \varepsilon
\]

and

\[
|(H_2 X)(n_4) - (H_2 X)(n_3)| = \left| \sum_{s=n_4}^{\infty} c(s)x(s - \alpha) - \sum_{s=n_3}^{\infty} c(s)x(s - \alpha) \right| \\
\leq \sum_{s=n_4}^{\infty} |c(s)||x(s - \alpha)| + \sum_{s=n_3}^{\infty} |c(s)||x(s - \alpha)| < \varepsilon,
\]

that is, \( H\Omega \) is uniformly Cauchy.

Hence by Krasnoselskii’s fixed point theorem, there exists a solution \( X(n) = [x(n), y(n)]^T \) of (S1) in \( \Omega \) such that \((GX)(n) + (HX)(n) = X(n)\) for \( n \geq n_1 \). Keeping in view that

\[
(G_1 X)(n) + (H_1 X)(n) = x(n), \quad (G_2 X)(n) + (H_2 X)(n) = y(n) \quad \text{for } n \geq n_1,
\]

it is easy to verify that \( X(n) = [x(n), y(n)]^T \) is the required vector solution of (S1).

This completes the proof of the theorem. \( \square \)

**Theorem 2.2.** Let \( 1 < t \leq p(n) \leq t_1 \leq \frac{1}{2}t^2 < \infty \) for large \( n \). If \( (A_1) \) holds, then the conclusion of Theorem 2.1 remains intact.

**Proof.** The sufficient part of the proof is the same as in Theorem 2.1. For the necessary part, let \( B \) denote the Banach space of all bounded sequences in \( \mathbb{R}^2 \) with the sup norm, i.e.,

\[
B = \left\{ X: \mathbb{N} \rightarrow \mathbb{R}^2: \|X\| = \sup_{n \in \mathbb{N}} |X| < \infty \right\}.
\]

For a fixed real number \( k > 0 \), put

\[
\Omega_1 = \left\{ X \in B: x(n), y(n) \in I_1, \ n \in \mathbb{N} \right\},
\]

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where $I_1 = [k(t-1)/(8t t_1), k]$. It is easy to see that $\Omega_1 \subset B$ is closed, bounded and convex. Because of (A₁), we can find $n_1 > 0$ such that

$$
\sum_{n=n_1}^{\infty} |a(n)| < \frac{(t-1)}{4t}, \quad \sum_{n=n_1}^{\infty} |b(n)| < \frac{(t-1)}{8t_1},
$$

$$
\sum_{n=n_1}^{\infty} |c(n)| < \frac{(t-1)}{8t_1}, \quad \sum_{n=n_1}^{\infty} |d(n)| < \frac{(t-1)}{4t}.
$$

We define the maps $G, H : \Omega_1 \to B$ as

$$
(G X)(n) = \begin{bmatrix}
\frac{(2t^2 + t - 1)k}{4tp(n + m)} - \frac{x(n + m)}{p(n + m)} & - \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} a(s)x(s - \alpha) \\
\frac{(2t^2 + t - 1)k}{4tp(n + m)} - \frac{y(n + m)}{p(n + m)} & - \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} d(s)y(s - \beta)
\end{bmatrix}
$$

for $n \geq n_1$,

$$
(G X)(n) = (G X)(n_1) \quad \text{for } 0 < n < n_1
$$

and

$$
(H X)(n) = \begin{bmatrix}
- \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} b(s)y(s - \beta) \\
- \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} c(s)x(s - \alpha)
\end{bmatrix}
$$

for $n \geq n_1$,

$$
(H X)(n) = (H X)(n_1) \quad \text{for } 0 < n < n_1.
$$

We rewrite $G, H$ as

$$
G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.
$$

Let $X, Y \in \Omega_1$. Then for $n \geq n_1$,

$$
(G_1 X)(n) + (H_1 Y)(n) = \frac{(2t^2 + t - 1)k}{4tp(n + m)} - \frac{x(n + m)}{p(n + m)} - \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} a(s)x(s - \alpha)
$$

$$
- \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} b(s)y(s - \beta)
$$

$$
\leq \frac{(2t^2 + t - 1)k}{4t^2} + \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} |a(s)||x(s - \alpha)|
$$

$$
+ \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} |b(s)||y(s - \beta)|
$$

$$
\leq \frac{(2t^2 + t - 1)k}{4t^2} + \frac{(t-1)k}{8tt_1} + \frac{(t-1)k}{4t^2}
$$

$$
\leq \frac{(2t^2 + t - 1)k}{4t^2} + \frac{(t-1)k}{8t} + \frac{(t-1)k}{4t^2} = k \frac{4t^2 + 5t - 5}{8t^2} < k
$$
and

\[(G_1X)(n) + (H_1Y)(n) = \frac{(2t^2 + t - 1)k}{4tp(n + m)} - \frac{x(n + m)}{p(n + m)} - \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} a(s)x(s - \alpha) \]

\[- \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} b(s)y(s - \beta) \]

\[\geq \frac{(2t^2 + t - 1)k}{4tt1} - \frac{x(n + m)}{p(n + m)} - \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} |b(s)|y(s - \beta) \]

\[\geq \frac{(2t^2 + t - 1)k}{4tt1} - \frac{k}{t} - \frac{(t - 1)k}{8tt1} \]

\[= k\frac{4t^2 + t - 8t1 - 1}{8tt1} > k\frac{t - 1}{8tt1}.\]

A similar observation can be obtained for \((G_2X)(n) + (H_2Y)(n), n \geq n_1.\) Hence, \(GX + HY \in \Omega_1.\) For \(X_1, X_2 \in \Omega_1,\) it is easy to verify that

\[|(G_1X_1)(n) - (G_1X_2)(n)| \leq \frac{1}{t} |x_1(n + m) - x_2(n + m)| \]

\[+ \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} |a(s)||x_1(s - \alpha) - x_2(s - \alpha)| \]

\[\leq \left[\frac{1}{t} + \frac{(t - 1)k}{4tt1}\right] \|x_1 - x_2\| = \frac{(3 + t)}{4tt1}\|x_1 - x_2\|,\]

and

\[|(G_2X_1)(n) - (G_2X_2)(n)| \leq \frac{(3 + t)}{4tt1}\|y_1 - y_2\| \]

for \(n \geq n_1\) implies that

\[\|GX_1 - GX_2\| \leq \frac{(3 + t)}{4tt1}\|X_1 - X_2\|,\]

that is, \(G\) is a contraction mapping.

Proceeding as in the proof of Theorem 2.1, we can show that \(H\) is continuous and \(H\Omega_1\) is uniformly Cauchy. Hence by Krasnoselskii’s fixed point theorem, there exists a solution \(X(n) = [x(n), y(n)]^\top\) of (S_1) in \(\Omega_1\) such that \((GX)(n) + (HX)(n) = X(n)\) for \(n \geq n_1.\) Therefore, the theorem is proved. \(\Box\)

**Theorem 2.3.** Let \(-1 < r_1 \leq p(n) \leq 0\) for large \(n.\) If \((A_1)\) holds, then the conclusion of Theorem 2.1 remains intact.

**Proof.** Proceeding as in the proof of Theorem 2.1, we can find an \(n_2 > n_1\) such that \([u(n) + K(n)]\) and \([v(n) + T(n)]\) are monotonic for \(n \geq n_2.\) Since \(\lim\limits_{n \to \infty} K(n) < \infty\) and \(\lim\limits_{n \to \infty} T(n) < \infty,\) then \(\lim\limits_{n \to \infty} u(n)\) exists and \(\lim\limits_{n \to \infty} v(n)\) exists. Using the same
bounded sequences in \( R \), where 

\[ \| I \| \text{ is omitted.} \]

The rest of the proof follows from the proof of Theorem 2.1 and hence the details

Let us define the maps \( G, H : \Omega_2 \to B \) such that

\[
(GX)(n) = \begin{cases}
\frac{(1 + r_1)k}{6} - p(n)x(n - m) - \sum_{s=n}^{\infty} a(s)x(s - \alpha) & \text{for } n \geq n_1,
\frac{(1 + r_1)k}{6} - p(n)y(n - m) - \sum_{s=n}^{\infty} d(s)y(s - \beta) & \text{for } n < n_1
\end{cases}
\]

\[
(HX)(n) = \begin{cases}
- \sum_{s=n}^{\infty} b(s)y(s - \beta) & \text{for } n \geq n_1,
- \sum_{s=n}^{\infty} c(s)x(s - \alpha) & \text{for } n < n_1
\end{cases}
\]

We note that

\[
G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.
\]

The rest of the proof follows from the proof of Theorem 2.1 and hence the details are omitted.
Theorem 2.4. Let $-\infty < r_2 \leq p(n) \leq r_3 < -1$ for large $n$. If $(A_1)$ holds, then the conclusion of Theorem 2.1 remains intact.

Proof. The sufficient part of the proof is similar to that of Theorem 2.3. By Lemma 1.1, it follows that $\lim_{n \to \infty} u(n) = \lim_{n \to \infty} v(n)$. Hence,

$$0 = \lim_{n \to \infty} u(n) = \liminf_{n \to \infty} (x(n) + p(n)x(n - m)) \leq \liminf_{n \to \infty} (x(n) + r_3x(n - m))$$

$$\leq \limsup_{n \to \infty} x(n) + \liminf_{n \to \infty} (r_3x(n - m)) = (1 + r_3) \limsup_{n \to \infty} x(n)$$

implies that $\lim_{n \to \infty} x(n) = 0$. Similarly, we can show that $\lim_{n \to \infty} y(n) = 0$.

For the necessary part of the proof, let $B$ denote the Banach space of all bounded sequences in $\mathbb{R}^2$ with the sup norm, i.e., $B = \{X: \mathbb{N} \to \mathbb{R}^2: \|X\| = \sup_{n \in \mathbb{N}}|X| < \infty\}$. For a fixed real number $k > 0$, put

$$\Omega_3 = \{X \in B: x(n), y(n) \in I_3, n \in \mathbb{N}\},$$

where $I_3 = [-kr_3/(M - r_3), Lk]$, and

$$M > \max \left\{-r_2, r_3 + \frac{r_3}{1 + r_3}\right\}, \quad L = \frac{2M - (M + 1)r_3}{(r_3 - M)(1 + r_3)} > 0.$$

Indeed, $\Omega_3 \subset B$ is closed, bounded and convex. Due to $(A_1)$, we can find $n_1 > 0$ such that

$$\sum_{n=n_1}^{\infty} |a(n)| < \frac{-r_3}{(M - r_3)}, \quad \sum_{n=n_1}^{\infty} |b(n)| < \frac{-r_3}{(M - r_3)}$$

$$\sum_{n=n_1}^{\infty} |c(n)| < \frac{-r_3}{(M - r_3)}, \quad \sum_{n=n_1}^{\infty} |d(n)| < \frac{-r_3}{(M - r_3)}.$$

Let us define the maps $G, H: \Omega_3 \to B$ such that

$$(GX)(n) = \begin{bmatrix} \frac{-(2 - r_3)Mk}{(M - r_3)p(n + m)} - \frac{x(n + m)}{p(n + m)} & -\frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} a(s)x(s - \alpha) \\ \frac{-(2 - r_3)Mk}{(M - r_3)p(n + m)} - \frac{y(n + m)}{p(n + m)} & -\frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} d(s)y(s - \beta) \end{bmatrix}$$

for $n \geq n_1$,

$$(GX)(n) = (GX)(n_1) \quad \text{for } 0 < n < n_1$$

and

$$(HX)(n) = \begin{bmatrix} -\frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} b(s)y(s - \beta) \\ -\frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} c(s)x(s - \alpha) \end{bmatrix}$$

for $n \geq n_1$,

$$(HX)(n) = (HX)(n_1) \quad \text{for } 0 < n < n_1.$$
We note that
\[ G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}. \]

Let \( X, Y \in \Omega_3 \). Then for \( n \geq n_1 \),
\[
(G_1X)(n) + (H_1Y)(n) = \frac{-(2 - r_3)Mk}{(M - r_3)p(n + m)} - \frac{x(n + m)}{p(n + m)}
\]
\[
- \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} a(s)x(s - \alpha)
\]
\[
- \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} b(s)y(s - \beta)
\]
\[
\leq \frac{-(2 - r_3)Mk}{(M - r_3)r_3} - \frac{x(n + m)}{p(n + m)} - \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} b(s)y(s - \beta)
\]
\[
\leq \frac{-(2 - r_3)Mk}{(M - r_3)r_3} - \frac{Lk}{r_3} + \frac{Lk}{(M - r_3)}
\]
\[
= -k \left[ \frac{L(M - r_3) + 2M - (M + 1)r_3}{(M - r_3)r_3} \right] = kL
\]

and
\[
(G_1X)(n) + (H_1Y)(n) = \frac{-(2 - r_3)Mk}{(M - r_3)p(n + m)} - \frac{x(n + m)}{p(n + m)}
\]
\[
- \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} a(s)x(s - \alpha)
\]
\[
- \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} b(s)y(s - \beta)
\]
\[
\geq \frac{-(2 - r_3)Mk}{(M - r_3)r_2} - \frac{kr_3}{(M - r_3)r_2} + \frac{1}{r_2} \sum_{s=n+m}^{\infty} |a(s)||x(s - \alpha)
\]
\[
\geq \frac{-(2 - r_3)Mk}{(M - r_3)r_2} - 2kr_3 + kr_3 \geq -\frac{kr_3}{(M - r_3)}
\]

A similar observation can be obtained for \((G_2X)(n) + (H_2Y)(n)\), \( n \geq n_1 \). Hence, \( GX + HY \in \Omega_3 \). For \( X_1, X_2 \in \Omega_3 \), it is easy to verify that
\[
|(G_1X_1)(n) - (G_1X_2)(n)| \leq -\frac{1}{r_3} |x_1(n + m) - x_2(n + m)|
\]
\[
- \frac{1}{p(n + m)} \sum_{s=n+m}^{\infty} |a(s)||x_1(s - \alpha) - x_2(s - \alpha)|
\]
\[
\leq \left[ -\frac{1}{r_3} + \frac{1}{M - r_3} \right] \|x_1 - x_2\|
\]
and
\[(G_2X_1)(n) - (G_2X_2)(n) \leq \left[ -\frac{1}{r_3} + \frac{1}{M - r_3} \right] \|y_1 - y_2\|\]

for \(n \geq n_1\) implies that
\[\|GX_1 - GX_2\| \leq \left[ -\frac{1}{r_3} + \frac{1}{M - r_3} \right] \|X_1 - X_2\|,\]

that is, \(G\) is a contraction mapping.

Proceeding as in the proof of Theorem 2.3, we can show that \(H\) is continuous and \(H \Omega_3\) is uniformly Cauchy. Hence by Krasnosel’skii’s fixed point theorem, there exists a solution \(X(n) = [x(n), y(n)]^\top\) of (S1) in \(\Omega_3\) such that \((GX)(n) + (HX)(n) = X(n)\) for \(n \geq n_1\). Therefore, the theorem is proved. \(\Box\)

**Remark 2.1.** It would be interesting to keep this work up for any solution of the system (S1) (i.e., not necessarily the bounded solution).

**Example 2.1.** Consider a 2-dimensional linear neutral difference system of the form:

\[(S_7)\]

\[
\begin{bmatrix}
  x(n) + e^{-n}x(n-2) \\
  y(n) + e^{-n}y(n-2)
\end{bmatrix}
\begin{bmatrix}
  e^{-(n+2)} \\
  e^{-n}
\end{bmatrix}
\begin{bmatrix}
  -(2 + e^{-n} + 2e^{-(n+1)}) \\
  e^{-(n+2)}
\end{bmatrix}
\begin{bmatrix}
  x(n-4) \\
  y(n-6)
\end{bmatrix}
\]

for \(n > 6\).

Clearly, (A1) and (A2) are satisfied for (S7). By Theorem 2.1, every bounded vector solution \(X(n)\) of (S7) is strongly oscillatory. Indeed, \(X(n) = [(-1)^n, e(-1)^n]^\top\) is one of such solutions of (S7).

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**References**


Author’s address: Arun Kumar Tripathy, Department of Mathematics, Sambalpur University, Sambalpur-768019, India, e-mail: arun_tripathy70@rediffmail.com.