INVESTIGATING GENERALIZED QUATERNIONS WITH DUAL-GENERALIZED COMPLEX NUMBERS

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Abstract. We aim to introduce generalized quaternions with dual-generalized complex number coefficients for all real values $\alpha$, $\beta$ and $\rho$. Furthermore, the algebraic structures, properties and matrix forms are expressed as generalized quaternions and dual-generalized complex numbers. Finally, based on their matrix representations, the multiplication of these quaternions is restated and numerical examples are given.

Keywords: generalized quaternion; dual-generalized complex number; matrix representation

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1. Introduction

The discovery of the quaternions, which is an associative and non-commutative Clifford algebra over the real numbers, is one of the outstanding contributions of Hamilton (see [20], [22], [21]). As an extension of the quaternions, the octonions are a non-associative and non-commutative algebra. Later on, Cayley and Dickson discussed these algebras, sometimes called the Cayley-Dickson algebras. The extension originates from $\mathbb{R}$ (real numbers 1-D) to $\mathbb{C}$ (complex numbers 2-D) and continues as: from $\mathbb{C}$ to $\mathbb{H}$ (quaternions 4-D), from $\mathbb{H}$ to $\mathbb{O}$ (octonions 8-D), from $\mathbb{O}$ to $\mathbb{S}$ (sedenions 16-D) and from $\mathbb{S}$ to $\mathbb{T}$ (trigintaduonions 32-D) and has been generalized to algebras over fields and rings. This process is known as the Cayley-Dickson doubling process or the Cayley-Dickson process. Hence, one can see the following Cayley-Dickson doubling subalgebras chain:

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O} \subset \mathbb{S} \subset \mathbb{T} \subset \ldots$$

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Since the Cayley-Dickson process is inductive, it is possible to construct \( n \)-ions by applying this process in an arbitrarily repeated pattern.

A quaternion can be written as \( q = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \), where \( a_0, a_1, a_2, a_3 \in \mathbb{R} \) and \( e_1, e_2, e_3 \) are quaternionic units. Real quaternions are commonly used in theoretical and applied mathematics, computer animation and robotics. Cockle (see [11], [10]) discovered split quaternions (co-quaternions or para-quaternions). Moreover, the set of generalized quaternions, denoted by \( Q_{\alpha \beta} \), is examined in [13], [18], [24], [27], [30], [33]. The algebra of generalized quaternions as a non-commutative system includes various well-known 4-dimensional algebras as special cases. For these quaternions, the conditions of the units are given by:

\[
\begin{align*}
   e_1^2 &= -\alpha, \quad e_2^2 = -\beta, \quad e_3^2 = -\alpha\beta, \\
   e_1e_2 &= -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = \beta e_1, \quad e_3e_1 = -e_1e_3 = \alpha e_2,
\end{align*}
\]

where \( e_1, e_2, e_3 \notin \mathbb{R} \) and \( \alpha, \beta \in \mathbb{R} \). For \( \alpha = \beta = 1 \) real quaternions, \( \alpha = 1, \beta = -1 \) split quaternions, \( \alpha = 1, \beta = 0 \) semi quaternions, \( \alpha = -1, \beta = 0 \) split semi quaternions and for \( \alpha = \beta = 0 \) quasi quaternions are obtained.

When it comes to numbers and their relationship to one another, scholars have long been interested in the subject matter. One of the most significant contributions of number theory is the revelation of generalized complex numbers. The generalized complex numbers have the form:

\[
\mathbb{C}_p := \{ z = x_1 + x_2 J : x_1, x_2 \in \mathbb{R}, J^2 = p \in \mathbb{R}, J \notin \mathbb{R} \}.
\]

This is a commutative unitary ring and a vector space over \( \mathbb{R} \), see [4], [5], [6], [23], [25], [37], [39]. Complex numbers \( \mathbb{C} \) (ordinary numbers) in [38], hyperbolic numbers \( \mathbb{P} \) (double, binary, split complex, perplex numbers) in [9], [16], [34] and dual numbers \( \mathbb{D} \) in [29], [35] are obtained for \( p = -1, p = 0, \) and \( p = 1 \), respectively. Furthermore, the construction of the number systems by writing the coefficients as elements of the sets \( \mathbb{C}, \mathbb{P} \) and \( \mathbb{D} \) is another fascinating area for researchers. Hence it is no surprise that hyperbolic-complex numbers are examined in [2], [10], [25]. Furthermore, \( n \)-dimensional hyperbolic-complex and bicomplex numbers are investigated in [32], [31], [36], [17], respectively. Dual-complex numbers are examined in [7], [8], [26], [28]. Dual-hyperbolic numbers and their algebraic properties are discussed in [26]. Besides, the functions and various matrix representations of dual-hyperbolic numbers and complex-hyperbolic numbers are presented in [1]. Hyper-dual numbers are studied in [12], [14], [15]. Additionally, dual-generalized complex (DG\( \mathbb{C} \)) numbers have been constructed by doubling dual numbers over generalized complex numbers using the Cayley-Dickson process. This extension is examined in [19] and denoted by:

\[
\mathbb{DC}_p := \{ a = z_1 + z_2 \varepsilon : z_1 = x_1 + x_2 J, z_2 = x_3 + x_4 J \in \mathbb{C}_p \},
\]

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where $J^2 = p \in \mathbb{R}$, $\varepsilon^2 = 0$, $\varepsilon \neq 0$, $J\varepsilon = \varepsilon J$, and $J, \varepsilon \notin \mathbb{R}$. $\mathbb{DC}_p$\(^1\) generalizes with dual-complex numbers for $p = -1$ (see [7], [8], [28]), dual-hyperbolic numbers for $p = 1$ (see [1]), and hyper-dual numbers for $p = 0$ (see [12], [14], [15]).

The theoretical perspectives and literature review mentioned are the motivating factors of this study and as a result lead to the following:

**Problem.** Is it possible to combine the concepts of generalized quaternions and $\mathbb{DG}_{\mathbb{C}}$ numbers? If the answer is affirmative, what algebraic properties are satisfied?

In this regard, the present paper is organized as follows. In Section 2, generalized quaternions with $\mathbb{DG}_{\mathbb{C}}$ number coefficients are introduced for all real values $\alpha$, $\beta$, and $p$. Moreover, the algebraic notions are investigated as numbers and as quaternions. Then, several matrix representations are given. Finally, the multiplication of these quaternions is presented using different methods and numerical examples are given.

2. **Generalized Quaternions with Dual-Generalized Complex Number**

The structure of this section is as follows: After a brief definition of new generalized quaternions, the algebraic properties and structures are discussed.

**Definition 2.1.** The set of generalized quaternions with $\mathbb{DG}_{\mathbb{C}}$ number coefficients is denoted by $\tilde{Q}_{\alpha\beta}$ and defined as:

$$\tilde{Q}_{\alpha\beta} := \{ \tilde{q} = a_0 + a_1e_1 + a_2e_2 + a_3e_3 : a_0, a_1, a_2, a_3 \in \mathbb{DC}_p \},$$

where $e_1, e_2, e_3$ are quaternionic units as given in equation (1.1) and $\alpha, \beta \in \mathbb{R}$.

It should be noted that the $\mathbb{DG}_{\mathbb{C}}$ units $J, \varepsilon$ and $J\varepsilon$ commute with the three quaternionic units $e_k$; that is $e_kJ = Je_k$, $e_k\varepsilon = \varepsilon e_k$ and $e_kJ\varepsilon = J\varepsilon e_k$ for $1 \leq k \leq 3$. It is evident that $e_1$ is distinct from the usual complex unit for $p = -1, \alpha = 1$, distinct from the usual hyperbolic unit for $p = 1, \alpha = -1$, and distinct from the usual dual unit for $p = -1, \alpha = 0$. This condition also holds for the other quaternionic units.

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\(^1\) $\mathbb{DC}_p$ is a commutative unitary ring and a vector space over $\mathbb{R}$. For $a_1 = z_{11} + z_{12}\varepsilon$, $a_2 = z_{21} + z_{22}\varepsilon \in \mathbb{DC}_p$ and $\lambda \in \mathbb{R}$, the operations are given as follows (see [19]):

- **equality:** $a_1 = a_2 \iff z_{11} + z_{12}\varepsilon = z_{21} + z_{22}\varepsilon \iff z_{11} = z_{21}, z_{12} = z_{22},$
- **addition:** $a_1 + a_2 = (z_{11} + z_{12}\varepsilon) + (z_{21} + z_{22}\varepsilon) = (z_{11} + z_{21}) + (z_{12} + z_{22})\varepsilon,$
- **scalar multiplication:** $\lambda a_1 = \lambda(z_{11} + z_{12}\varepsilon) = (\lambda z_{11}) + (\lambda z_{12})\varepsilon,$
- **multiplication:** $a_1a_2 = (z_{11} + z_{12}\varepsilon)(z_{21} + z_{22}\varepsilon) = (z_{11}z_{21}) + (z_{11}z_{22} + z_{12}z_{21})\varepsilon.$
Additionally, special cases of these quaternions are given by:

- If $\alpha = \beta = 1$, then $\tilde{Q}_{\alpha\beta}$ is the set of real quaternions.
- If $\alpha = 1$, $\beta = -1$, then $\tilde{Q}_{\alpha\beta}$ is the set of split/para/co quaternions.
- If $\alpha = 1$, $\beta = 0$, then $\tilde{Q}_{\alpha\beta}$ is the set of semi quaternions.
- If $\alpha = -1$, $\beta = 0$, then $\tilde{Q}_{\alpha\beta}$ is the set of split semi quaternions.
- If $\alpha = \beta = 0$, then $\tilde{Q}_{\alpha\beta}$ is the set $\frac{1}{4}$/quasi quaternions.

In all cases with DGC number coefficients.

It is also possible to study more specific quaternions with DGC number coefficients depending on the choice of the real values $\alpha$ and $\beta$.

Let $\tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$, $\tilde{p} = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_3 \in \tilde{Q}_{\alpha\beta}$. Algebraic structures are now defined on $\tilde{Q}_{\alpha\beta}$ considering a generalized quaternion form. In general, a quaternion $\tilde{q}$ has 2 parts, a scalar $S_{\tilde{q}} = a_0$, and a vector $V_{\tilde{q}} = a_1 e_1 + a_2 e_2 + a_3 e_3$. So that $\tilde{q} = S_{\tilde{q}} + V_{\tilde{q}}$. Equality and addition (and hence subtraction) are component-wise defined as follows:

$$\tilde{p} = \tilde{q} \iff a_0 = b_0, \ a_1 = b_1, \ a_2 = b_2, \ a_3 = b_3 \iff S_{\tilde{p}} = S_{\tilde{q}}, \ V_{\tilde{p}} = V_{\tilde{q}}$$

and

$$\tilde{Q}_{\alpha\beta} \times \tilde{Q}_{\alpha\beta} \to \tilde{Q}_{\alpha\beta},$$

$$(\tilde{q}, \tilde{p}) \mapsto \tilde{q} + \tilde{p} = (a_0 + b_0) + (a_1 + b_1)e_1 + (a_2 + b_2)e_2 + (a_3 + b_3)e_3$$

$$= (S_{\tilde{p}} + S_{\tilde{q}}) + (V_{\tilde{p}} + V_{\tilde{q}}).$$

An element $\tilde{q}$ called the conjugate of $\tilde{q}$ is defined by:

$$\tilde{Q}_{\alpha\beta} \to \tilde{Q}_{\alpha\beta},$$

$$\tilde{q} \mapsto \tilde{q}^\dagger = a_0 - a_1 e_1 - a_2 e_2 - a_3 e_3 = S_{\tilde{q}} - V_{\tilde{q}}.$$

The scalar multiplication refers to the product of $\tilde{q}$ by $c \in \mathbb{R}$ and is

$$\mathbb{R} \times \tilde{Q}_{\alpha\beta} \to \tilde{Q}_{\alpha\beta},$$

$$(c, \tilde{q}) \mapsto c\tilde{q} = ca_0 + ca_1 e_1 + ca_2 e_2 + ca_3 e_3 = cS_{\tilde{q}} + cV_{\tilde{q}}.$$

Moreover, multiplication of $\tilde{q}$ and $\tilde{p}$ is calculated as:

$$\tilde{Q}_{\alpha\beta} \times \tilde{Q}_{\alpha\beta} \to \tilde{Q}_{\alpha\beta},$$

$$(\tilde{q}, \tilde{p}) \mapsto \tilde{q}\tilde{p} = (a_0b_0 - \alpha a_1 b_1 - \beta a_2 b_2 - \alpha\beta a_3 b_3)$$

$$+ (a_0b_1 + a_1 b_0 + \beta a_2 b_3 - \alpha a_3 b_2)e_1$$

$$+ (a_0b_2 - \alpha a_1 b_3 + a_2 b_0 + \alpha a_3 b_1)e_2$$

$$+ (a_0b_3 + a_1 b_2 - a_2 b_1 + a_3 b_0)e_3.$$

The multiplication is non-commutative but associative and distributive over addition.
The scalar product and the vector product in \( \langle \cdot, \cdot \rangle \) where \( \langle \cdot, \cdot \rangle = a \cdot \hat{q} = a \cdot (\hat{q} + \hat{p}) = a \cdot q + a \cdot \hat{p}, (a + b) \cdot \hat{q} = a \cdot \hat{q} + b \cdot \hat{q}, (ab) \cdot \hat{q} = a \cdot (b \cdot \hat{q}) \) and 1 is the multiplicative identity of \( \mathbb{Q}_{\alpha \beta} \). Hence \( \mathbb{Q}_{\alpha \beta} \) is module over \( \mathbb{D}\mathbb{C}_p \) with base \( \{1, e_1, e_2, e_3\} \) and dimension 4. Similarly, obtaining module properties considering the operation \( \cdot : \mathbb{C}_p \times \mathbb{Q}_{\alpha \beta} \rightarrow \mathbb{Q}_{\alpha \beta} \) is a simple calculation. Thus the proof is completed. \( \square \)

**Definition 2.2.** Let \( \hat{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3, \hat{p} = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_3 \in \mathbb{Q}_{\alpha \beta} \). The scalar product and the vector product in \( \mathbb{Q}_{\alpha \beta} \) are defined as follows, respectively:

\[
\langle \hat{q}, \hat{p} \rangle_{g} = S_{\hat{q}} S_{\hat{p}} + \langle V_{\hat{q}}, V_{\hat{p}} \rangle_{g} = a_0 b_0 + \alpha a_1 b_1 + \beta a_2 b_2 + \alpha \beta a_3 b_3 = S_{\hat{q} \hat{p}},
\]

\[
\hat{q} \times_{g} \hat{p} = S_{\hat{q}} V_{\hat{p}} + S_{\hat{p}} V_{\hat{q}} - V_{\hat{q}} \times_{g} V_{\hat{p}},
\]

where \( \langle \cdot, \cdot \rangle_{g} \) is a generalized scalar product and \( \times_{g} \) is a generalized vector product\(^2\).

**Lemma 2.1.** For all \( \hat{q}, \hat{p} \in \mathbb{Q}_{\alpha \beta}, \hat{q} \hat{p} = \langle \hat{q}, \hat{p} \rangle_{g} + \hat{q} \times_{g} \hat{p} \).

**Proof.** It is clear that

\[
\hat{q} \hat{p} = (a_0 b_0 + \alpha a_1 b_1 + \beta a_2 b_2 + \alpha \beta a_3 b_3) + (-a_0 b_1 + a_1 b_0 - \beta a_2 b_3 + \beta a_3 b_2) e_1 + (-a_0 b_2 + a_1 b_3 + a_2 b_0 - \alpha a_3 b_1) e_2 + (-a_0 b_3 - a_1 b_2 + a_2 b_1 + a_3 b_0) e_3 = S_{\hat{q}} S_{\hat{p}} + \langle V_{\hat{q}}, V_{\hat{p}} \rangle_{g} + S_{\hat{p}} V_{\hat{q}} - V_{\hat{q}} \times_{g} V_{\hat{p}}.
\]

\( \square \)

**Definition 2.3.** For any \( \hat{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \mathbb{Q}_{\alpha \beta} \), the norm operation of \( \hat{q} \) is defined by:

\[
N : \mathbb{Q}_{\alpha \beta} \times \mathbb{Q}_{\alpha \beta} \rightarrow \mathbb{D}\mathbb{C}_p,
\]

\[
\hat{q} \mapsto N_{\hat{q}} = \hat{q} \hat{q} = \hat{q} \hat{q} = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha \beta a_3^2.
\]

\(^2\)For a more general description of the generalized inner and cross product, see [24].
Thus, it is clear that properties analogous to the properties of alternative real Cayley-Dickson algebras:

\[ \overline{Q}_{\alpha\beta} \to \overline{Q}_{\alpha\beta}, \]
\[ \tilde{q} \mapsto (\tilde{q})^{-1} = \frac{\tilde{q}}{N_{\tilde{q}}}, \]

where \( N_{\tilde{q}} \) is non-null\(^3\) number.

For the elements of \( \overline{Q}_{\alpha\beta} \), let us present the following properties, which are properties analogous to the properties of alternative real Cayley-Dickson algebras:

**Proposition 2.1.** Let \( \tilde{q}, \tilde{p} \in \overline{Q}_{\alpha\beta} \) and \( c_1, c_2 \in \mathbb{R} \). Then the basic properties of conjugation and the norm can be given as follows:

(i) \( \overline{\tilde{q}} = \tilde{q} \),
(ii) \( c_1\tilde{p} + c_2\tilde{q} = c_1\tilde{p} + c_2\tilde{q} \),
(iii) \( \overline{\overline{\tilde{q}}} = \tilde{q} \),
(iv) \( N_{c_1\tilde{q}} = c_1^2 N_{\tilde{q}} \),
(v) \( N_{\tilde{q}\tilde{p}} = N_{\tilde{q}} N_{\tilde{p}} \).

**Proof.** Let \( \tilde{q} = a_0 + a_1e_1 + a_2e_2 + a_3e_3, \tilde{p} = b_0 + b_1e_1 + b_2e_2 + b_3e_3 \in \overline{Q}_{\alpha\beta} \). Considering equation (2.1), items (i) and (ii) are obvious.

(iii) Taking the conjugate of equation (2.2), we get:

\[
\overline{\overline{\tilde{q}}} = (a_0b_0 - \alpha a_1b_1 - \beta a_2b_2 - \alpha \beta a_3b_3) - (a_0b_1 + a_1b_0 + \beta a_2b_3 - \beta a_3b_2)e_1 \\
- (a_0b_2 - \alpha a_1b_3 + a_2b_0 + \alpha a_3b_1)e_2 - (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)e_3.
\]

Thus, it is clear that \( \overline{\overline{\tilde{q}}} = \tilde{q} \).

(iv) Using item (ii) and equation (2.5), we have: \( N_{c_1\tilde{q}} = (c_1\tilde{q})(c_1\tilde{q}) = c_1^2 N_{\tilde{q}} \).

(v) Having item (iii) and equation (2.5), we find:

\[
N_{\tilde{q}\tilde{p}} = (\tilde{q}\tilde{p})(\overline{\overline{\tilde{q}}}\tilde{p}) = \tilde{q}\tilde{p}\tilde{p} = N_{\tilde{q}} N_{\tilde{p}}.
\]

\[ \square \]

**Proposition 2.2.** For any \( \tilde{q}, \tilde{p} \) and \( \tilde{r} \in \overline{Q}_{\alpha\beta} \), the inner product possesses the following properties:

(i) \( \langle \tilde{r}, \tilde{q}\tilde{p} \rangle_g = N_{\tilde{r}} \langle \tilde{q}, \tilde{p} \rangle_g \),
(ii) \( \langle \tilde{q}\tilde{r}, \tilde{p} \rangle_g = N_{\tilde{r}} \langle \tilde{q}, \tilde{p} \rangle_g \),
(iii) \( \langle \tilde{r}, \tilde{p} \rangle_g = \langle \tilde{q}, \tilde{p} \rangle_g \),
(iv) \( \langle \tilde{q}\tilde{r}, \tilde{p} \rangle_g = \langle \tilde{q}, \tilde{p} \rangle_g \).

\(^3\) Null numbers are characterized by having zero norm in \( \mathbb{D}C_p \).
**Proof.** Let \( \tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3, \tilde{p} = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_3 \) and \( \tilde{r} = c_0 + c_1 e_1 + c_2 e_2 + c_3 e_3 \in \tilde{\mathbb{Q}}_{\alpha \beta} \). Using equations (2.3) and (2.5), the following proofs can be given:

(i) \( \langle \tilde{r} \bar{q}, \tilde{r} \bar{p} \rangle_g = S_{\tilde{r} \bar{q}} S_{\tilde{r} \bar{p}} + \langle V_{\tilde{r} \bar{q}}, V_{\tilde{r} \bar{p}} \rangle_g \)

\[
= (c_0 a_0 - \alpha c_1 a_1 - \beta c_2 a_2 - \alpha \beta c_3 a_3)(c_0 b_0 - \alpha c_1 b_1 - \beta c_2 b_2 - \alpha \beta c_3 b_3) + \alpha(c_0 a_1 + c_1 a_0 + \beta c_2 a_3 - \beta c_3 a_2)(c_0 b_1 + c_1 b_0 + \beta c_2 b_3 - \beta c_3 b_2) + \beta(c_0 a_2 - \alpha c_1 a_3 + c_2 a_0 + \alpha c_3 a_1)(c_0 b_2 - \alpha c_1 b_3 + c_2 b_0 + \alpha c_3 b_1) + \alpha \beta(c_0 a_3 + c_1 a_2 - c_2 a_1 + c_3 a_0)(c_0 b_3 + c_1 b_2 - c_2 b_1 + c_3 b_0)
\]

\[
= (c_0^2 + \alpha c_1^2 + \beta c_2^2 + \alpha \beta c_3^2)(a_0 b_0 + \alpha a_1 b_1 + \beta a_2 b_2 + \alpha \beta a_3 b_3)
\]

\[
= N_{\tilde{r}}(\tilde{q}, \tilde{p})_g.
\]

(iv) \( \langle \tilde{q} \bar{r}, \tilde{p} \rangle_g = S_{\tilde{q} \bar{r}} S_{\tilde{p}} + \langle V_{\tilde{q} \bar{r}}, V_{\tilde{p}} \rangle_g \)

\[
= (a_0 c_0 - \alpha a_1 c_1 - \beta a_2 c_2 - \alpha \beta a_3 c_3)b_0 + \alpha(a_0 c_1 + a_1 c_0 + \beta a_2 c_3 - \beta a_3 c_2)b_1 + \beta(a_0 c_2 - \alpha a_1 c_3 + a_2 c_0 + \alpha a_3 c_1)b_2 + \alpha \beta(a_0 c_3 + a_1 c_2 - a_2 c_1 + a_3 c_0)b_3
\]

and

\[
\langle \tilde{q}, \tilde{p} \rangle_g = S_{\tilde{q}} S_{\tilde{p}}, q + \langle V_{\tilde{q}}, V_{\tilde{p}} \rangle_g \]

\[
= a_0(b_0 c_0 + \alpha b_1 c_1 + \beta b_2 c_2 + \alpha \beta b_3 c_3) + \alpha(-b_0 c_1 + b_1 c_0 - \beta b_2 c_3 + \beta b_3 c_2)a_1 + \beta(-b_0 c_2 + \alpha b_1 c_3 + b_2 c_0 - \alpha b_3 c_1)a_2 + \alpha \beta(-b_0 c_3 - b_1 c_2 + b_2 c_1 + b_3 c_0)a_3.
\]

Hence we have \( \langle \tilde{q} \bar{r}, \tilde{p} \rangle_g = \langle \tilde{q}, \tilde{p} \rangle_g \). The other items can be proved similarly. \( \Box \)

**Example 2.1.** Let us consider the following elements of \( \tilde{\mathbb{Q}}_{21} \) as generalized quaternions with \( DGC \) number coefficients:

\[
\tilde{q} = (1 - J + 2 \varepsilon - J \varepsilon) + (-1 + 2 \varepsilon + J \varepsilon) e_1 + (J - \varepsilon - J \varepsilon) e_2 + (2 \varepsilon - J \varepsilon) e_3,
\]

\[
\tilde{p} = (1 + J + \varepsilon + J \varepsilon) + (-1 + 2 J + 3 \varepsilon - J \varepsilon) e_1 + (1 - J \varepsilon) e_2 + (1 - J + \varepsilon) e_3,
\]

\[
\tilde{r} = 1 + (1 + J \varepsilon) e_2.
\]

For \( p = 1 \), we have generalized quaternions with dual-hyperbolic number coefficients. Using Definition 2.2, we obtain:

\[
\langle \tilde{q}, \tilde{p} \rangle_g = 2 - 3 J - \varepsilon + 2 J \varepsilon,
\]

and

\[
\tilde{q} \times_g \tilde{p} = (3 - 5 J + 4 \varepsilon - J \varepsilon) e_1 + (-2 + 4 J + 4 \varepsilon - 11 J \varepsilon) e_2 + (1 + J - 7 \varepsilon + 5 J \varepsilon) e_3.
\]
Besides, using Proposition 2.2 item (ii), we verify:

\[ \langle \tilde{q}^r, \tilde{p}^r \rangle_g = 4 - 6J - 8\varepsilon + 8J\varepsilon = N_r \langle \tilde{q}, \tilde{p} \rangle_g, \]

where \( N_r = 2 + 2J\varepsilon \). Moreover, similar calculations can be conducted for the above generalized quaternions with hyper-dual number coefficients \((p = 0)\) and dual-complex number coefficients \((p = -1)\) as a particular case.

The same results can be obtained if these quaternions are rewritten as \(DGC\) numbers with generalized quaternion coefficients. One can see the details easily through the following remark:

**Remark 2.1.** For \( \tilde{q} \in \tilde{Q}_{\alpha\beta} \), the following equation can be written:

\[ \tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 = q_0 + q_1 J + q_2 \varepsilon + q_3 J\varepsilon, \]

where \( a_j = x_{j1} + x_{j2} J + x_{j3} \varepsilon + x_{j4} J\varepsilon \in \mathbb{D}_p \) and \( q_{i-1} = x_{0i} + x_{1i} e_1 + x_{2i} e_2 + x_{3i} e_3 \in Q_{\alpha\beta}, 0 \leq j \leq 3, 1 \leq i \leq 4 \). Thus, it should be noted that there is no difference between a generalized quaternion with \(DGC\) number coefficients and a \(DGC\) number with generalized quaternion coefficients.

For \( \tilde{q} = q_0 + q_1 J + q_2 \varepsilon + q_3 J\varepsilon \) and \( \tilde{p} = p_0 + p_1 J + p_2 \varepsilon + p_3 J\varepsilon \), the algebraic operations considering \(DGC\) number form are given as follows, respectively:

- **equality:** \( \tilde{p} = \tilde{q} \Leftrightarrow p_0 = q_0, p_1 = q_1, p_2 = q_2, p_3 = q_3 \),
- **addition (and hence subtraction):** \( \tilde{p} + \tilde{q} = (p_0 + q_0) + (p_1 + q_1)J \)
  \[ + (p_2 + q_2)\varepsilon + (p_3 + q_3)J\varepsilon, \]
- **scalar multiplication:** \( c\tilde{q} = cq_0 + cq_1 J + cq_2 \varepsilon + cq_3 J\varepsilon, c \in \mathbb{R} \),
- **multiplication:** \( \tilde{p}\tilde{q} = (p_0 q_0 + pp_1 q_1) + (p_0 q_1 + p_1 q_0)J \)
  \[ + (p_0 q_2 + pp_1 q_3 + p_2 q_0 + pp_3 q_1)\varepsilon \]
  \[ + (p_0 q_3 + p_1 q_2 + p_2 q_1 + p_3 q_0)J\varepsilon, \]
- **generalized complex conjugate:** \( \tilde{q}^{\dagger 1} = q_0 - q_1 J + q_2 \varepsilon - q_3 J\varepsilon, \)
- **dual conjugate:** \( \tilde{q}^{\dagger 2} = q_0 + q_1 J - q_2 \varepsilon - q_3 J\varepsilon, \)
- **coupled conjugate:** \( \tilde{q}^{\dagger 3} = q_0 - q_1 J - q_2 \varepsilon + q_3 J\varepsilon. \)

Hence, \( \tilde{Q}_{\alpha\beta} \) is a 4-dimensional module over \( Q_{\alpha\beta} \) with base \( \{1, J, \varepsilon, J\varepsilon\} \) and thus a 16-dimensional vector space over \( \mathbb{R} \) with base

\[ \{1, J, \varepsilon, J\varepsilon, e_1, Je_1, \varepsilon e_1, J\varepsilon e_1, e_2, Je_2, \varepsilon e_2, J\varepsilon e_2, e_3, Je_3, \varepsilon e_3, J\varepsilon e_3\}. \]
For $1 \leq i \leq 3$, the following norm operations for $\tilde{q}$ are defined:

generalized complex module: $N^i_{\tilde{q}} = \tilde{q}\tilde{q}^i$,

dual module: $N^i_{\tilde{q}} = \tilde{q}\tilde{q}^i$,

coupled module: $N^i_{\tilde{q}} = \tilde{q}\tilde{q}^i$.

Additionally, the inverse of a non-null $\tilde{q}$ is defined by:

$$(\tilde{q})^{-1} = \frac{\tilde{q}^i}{N^i_{\tilde{q}}}.$$  

For the elements of $\tilde{Q}_{\alpha\beta}$, the following properties, which are properties analogues to the properties of alternative real Cayley-Dickson algebras, are given as:

**Proposition 2.3.** Let $\tilde{q}, \tilde{p} \in \tilde{Q}_{\alpha\beta}$ and $c_1, c_2 \in \mathbb{R}$. Then, for $1 \leq i \leq 3$, the properties of conjugation and the norm can be given as follows:

(i) $(\tilde{q}^i)^{1i} = \tilde{q}$,

(ii) $(c_1\tilde{q} \pm c_2\tilde{p})^{1i} = c_1\tilde{q}^i \pm c_2\tilde{p}^i$,

(iii) $(\tilde{q}\tilde{p})^{1i} \neq \tilde{p}^{\dagger i} \tilde{q}^i$ in general,

(iv) $\tilde{q} + \tilde{q}^{1i} = 2(q_0 + q_2\varepsilon)$,

(v) $\tilde{q} + \tilde{q}^{2i} = 2(q_0 + q_1J)$,

(vi) $\tilde{q} + \tilde{q}^{3i} = 2(q_0 + q_3J\varepsilon)$,

(vii) $\tilde{q} - \varepsilon\tilde{q}^{1i} = q_0 + q_1J$,

(viii) $\varepsilon\tilde{q} + \tilde{q}^{1i} = q_2 + q_3J$,

(ix) $N^1_{c_1\tilde{q}} = c_1^2 N^1_{\tilde{q}}$,

(x) $N^i_{\tilde{q}\tilde{p}} \neq N^i_{\tilde{q}} N^i_{\tilde{p}}$ in general.

**Proof.** Let us consider $\tilde{q} = (1 + e_1) + J$ and $\tilde{p} = (1 + e_2) + J\varepsilon$ for items (iii) and (x).

(iii) One can easily see the following:

(2.7) \[ \tilde{q}\tilde{p} = (1 + e_1 + e_2 + e_3) + (1 + e_2)J + p\varepsilon + (1 + e_1)J\varepsilon, \]

(2.8) \[ (\tilde{q}\tilde{p})^{1i} = (1 + e_1 + e_2 + e_3) + (1 + e_2)J - p\varepsilon - (1 + e_1)J\varepsilon, \]

and

\[ \tilde{p}^{\dagger i} \tilde{q}^{1i} = (1 + e_1 + e_2 - e_3) + (1 + e_2)J - p\varepsilon - (1 + e_1)J\varepsilon. \]

So $(\tilde{q}\tilde{p})^{1i} \neq \tilde{p}^{\dagger i} \tilde{q}^{1i}$. Since the generalized quaternions are non-commutative, we also get $(\tilde{q}\tilde{p})^{1i} \neq \tilde{p}^{\dagger i} \tilde{q}^{1i}$ for $i = 1, 3$.

(x) By substituting equations (2.7) and (2.8) into $N^i_{\tilde{q}\tilde{p}} = (\tilde{q}\tilde{p})(\tilde{q}\tilde{p})^{1i}$, we have:

\[ N^i_{\tilde{q}\tilde{p}} = (1 + e_1 + e_2 + e_3)^2 + p(1 + e_2)^2 + 2(1 - \beta + e_1 + 2e_2 + e_3)J + 2p\varepsilon + 2(e_3 - \alpha e_2)J\varepsilon. \]
and
\[ N^\dagger_q N^\dagger_p = ((1 + e_1) + J)((1 + e_1) + J)((1 + e_2) + J\varepsilon)((1 + e_2) - J\varepsilon) \]
\[ = ((1 + e_1)^2 + p)(1 + e_2)^2 + 2(1 + e_1)(1 + e_2)^2 J. \]

Hence \( N^\dagger_q \neq N^\dagger_p \). Since the generalized quaternions are non-commutative, \( N^\dagger_{qp} \neq N^\dagger_p \cdot N^\dagger_q \) for \( i = 1, 3 \).

The proof of the other items is a simple calculation so we can omit it. \( \square \)

Through the analogy between a generalized quaternion with \( \mathcal{DG}C \) numbers and a \( \mathcal{DG}C \) number with generalized quaternions, the following remark can be given:

**Remark 2.2.** Let \( \tilde{q} = q_0 + q_1 J + q_2 \varepsilon + q_3 J\varepsilon \) and \( \tilde{p} = p_0 + p_1 J + p_2 \varepsilon + p_3 J\varepsilon \in \tilde{\mathcal{Q}}_{\alpha\beta} \).

The analogue of the scalar product on \( \tilde{\mathcal{Q}}_{\alpha\beta} \) is defined as follows:
\[
\langle \tilde{q}, \tilde{p} \rangle_g = S_{q_0\bar{p}_0} + p S_{q_1\bar{p}_1} + (S_{q_0\bar{p}_1} + S_{q_1\bar{p}_0}) J + (S_{q_0\bar{p}_2} + S_{q_2\bar{p}_0} + p(2 + 3))\varepsilon + (S_{q_0\bar{p}_3} + S_{q_3\bar{p}_0} + S_{q_3\bar{p}_1}) J\varepsilon.
\]

## 3. Matrix representations in view of generalized quaternions

**and \( \mathcal{DG}C \) numbers**

In this section, we formulate the key concepts, including matrix correspondences. The following matrix approaches provide an alternative formulation of multiplication.

**Theorem 3.1.** Every generalized quaternion with \( \mathcal{DG}C \) number coefficients can be represented by a \( 4 \times 4 \) \( \mathcal{DG}C \) matrix.

**Proof.** Let us define the bijective linear map \( f_{\tilde{q}}: \tilde{\mathcal{Q}}_{\alpha\beta} \to \tilde{\mathcal{Q}}_{\alpha\beta} \) by \( f_{\tilde{q}}(\tilde{p}) = \tilde{q}\tilde{p} \) for every \( \tilde{p} \in \tilde{\mathcal{Q}}_{\alpha\beta} \). By using the equations:
\[
\begin{align*}
    f_{\tilde{q}}(1) &= \tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3, \\
    f_{\tilde{q}}(e_1) &= \tilde{q} e_1 = -\alpha a_1 + a_0 e_1 + \alpha a_3 e_2 - a_2 e_3, \\
    f_{\tilde{q}}(e_2) &= \tilde{q} e_2 = -\beta a_2 - \beta a_3 e_1 + a_0 e_2 + a_1 e_3, \\
    f_{\tilde{q}}(e_3) &= \tilde{q} e_3 = -\alpha \beta a_3 + \beta a_2 e_1 - \alpha a_1 e_2 + a_0 e_3,
\end{align*}
\]
a \( 4 \times 4 \) left \( \mathcal{DG}C \) matrix representation of \( \tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \) concerning the standard basis \( \{1, e_1, e_2, e_3\} \) is given by
\[
\mathbf{A}^i_{\tilde{q}} = \begin{bmatrix}
    a_0 & -\alpha a_1 & -\beta a_2 & -\alpha \beta a_3 \\
    a_1 & a_0 & -\beta a_3 & \beta a_2 \\
    a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\
    a_3 & -a_2 & a_1 & a_0
\end{bmatrix}.
\]

(3.1)
Denote by $M$ the following subset of $\mathbb{M}_4(\mathbb{D}_C)$ as:
\[
M := \left\{ A^l_q \in \mathbb{M}_4(\mathbb{D}_C) : \ A^l_q = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha \beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \right\}.
\]

Hence it can be concluded that there exists a correspondence between $\tilde{Q}_{\alpha \beta}$ and $M$ via the bijective map $m : \tilde{Q}_{\alpha \beta} \rightarrow M, \tilde{q} \mapsto A^l_q$.

Similarly, by the bijective linear map $f_q(\tilde{p}) = \tilde{p}\tilde{q}$, $4 \times 4$ right $DGC$ matrix representation of $\tilde{q}$ is also computed as below:
\[
A^r_q = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha \beta a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ a_2 & -\alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}.
\]

Thus, there exists a correspondence between $\tilde{Q}_{\alpha \beta}$ and $N$ via the bijective map $N : \tilde{Q}_{\alpha \beta} \rightarrow N, \tilde{q} \mapsto A^r_q$ where
\[
N := \left\{ A^r_q \in \mathbb{M}_4(\mathbb{D}_C) : A^r_q = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha \beta a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ a_2 & -\alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix} \right\}.
\]

The proof is completed. $\square$

**Theorem 3.2.** Every $DGC$ number with generalized quaternion coefficients can be represented by a $4 \times 4$ generalized quaternion matrix.

**Proof.** By considering the bijective linear map $F_q : \tilde{Q}_{\alpha \beta} \rightarrow \tilde{Q}_{\alpha \beta}$ by $F_q(\tilde{p}) = \tilde{q}\tilde{p}$ for every $\tilde{p} \in \tilde{Q}_{\alpha \beta}$, we have:
\[
F_q(1) = \tilde{q} = q_0 + q_1J + q_2\varepsilon + q_3J\varepsilon,
F_q(J) = \tilde{q}J = pq_1 + q_0J + pq_3\varepsilon + q_2J\varepsilon,
F_q(\varepsilon) = \tilde{q}\varepsilon = q_0\varepsilon + q_1J\varepsilon,
F_q(J\varepsilon) = \tilde{q}J\varepsilon = pq_1\varepsilon + q_0J\varepsilon.
\]

Namely, a $4 \times 4$ generalized quaternion matrix representation of $\tilde{q} = q_0 + q_1J + q_2\varepsilon + q_3J\varepsilon$ for the standard basis $\{1, J, \varepsilon, J\varepsilon\}$ is
\[
(3.2) \quad B_{\tilde{q}} = \begin{bmatrix} q_0 & pq_1 & 0 & 0 \\ q_1 & q_0 & 0 & 0 \\ q_2 & pq_3 & q_0 & pq_1 \\ q_3 & q_2 & q_1 & q_0 \end{bmatrix}.
\]

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Hence there exists a correspondence between \( \tilde{Q}_{\alpha\beta} \) and the set \( K \) via the bijective map \( K: \tilde{Q}_{\alpha\beta} \rightarrow K, \tilde{q} \mapsto B_{\tilde{q}} \) where

\[
K := \left\{ B_{\tilde{q}} \in M_4(\tilde{Q}_{\alpha\beta}): B_{\tilde{q}} = \begin{bmatrix} q_0 & pq_1 & 0 & 0 \\ q_1 & q_0 & 0 & 0 \\ q_2 & pq_3 & q_0 & pq_1 \\ q_3 & q_2 & q_1 & q_0 \end{bmatrix} \right\}.
\]

\( \square \)

**Corollary 3.1.** Let \( \tilde{q} \in \tilde{Q}_{\alpha\beta} \). Then, the following statements can be given:

(i) The left DGC matrix representation of \( \tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \) can be determined in the following form:

\[
A^l_{\tilde{q}} = a_0 I_4 + a_1 E^l_1 + a_2 E^l_2 + a_3 E^l_3,
\]

where

\[
e_1 \mapsto E^l_1 = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad e_2 \mapsto E^l_2 = \begin{bmatrix} 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},
\]

\[
e_3 \mapsto E^l_3 = \begin{bmatrix} 0 & 0 & 0 & -\alpha \beta \\ 0 & 0 & -\beta & 0 \\ 0 & \alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.
\]

The right DGC matrix representation of \( \tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \) can be determined in the following form:

\[
A^r_{\tilde{q}} = a_0 I_4 + a_1 E^r_1 + a_2 E^r_2 + a_3 E^r_3,
\]

where

\[
e_1 \mapsto E^r_1 = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad e_2 \mapsto E^r_2 = \begin{bmatrix} 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},
\]

\[
e_3 \mapsto E^r_3 = \begin{bmatrix} 0 & 0 & 0 & -\alpha \beta \\ 0 & 0 & \beta & 0 \\ 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.
\]
The generalized quaternion matrix representation of \( \tilde{q} = q_0 + q_1 J + q_2 \varepsilon + q_3 J \varepsilon \), is also in the form

\[
B_{\tilde{q}} = q_0 I_4 + q_1 J + q_2 \varepsilon + q_3 J \varepsilon,
\]

where

\[
J \mapsto J = \begin{bmatrix} 0 & p & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \varepsilon \mapsto \varepsilon = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},
\]

\[
J \varepsilon \mapsto J \varepsilon = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.
\]

Proof. Taking into account the bijective linear maps \( f \) and \( F \), it is obvious that

\[
A_l^{\tilde{q}} = A_r^{\tilde{q}} = B_1 = I_4, \quad A_l^{\varepsilon} = E_1, \quad A_r^{\varepsilon} = E_1, \quad B_J = J, \quad B_\varepsilon = \varepsilon, \quad B_{J \varepsilon} = J \varepsilon \] where \( i = 1, 2, 3 \). \( \square \)

After this part, the representation \( A_l^{\tilde{q}} \) will be considered and similar computations can be given for \( A_r^{\tilde{q}} \).

Corollary 3.2. The column matrix representation of \( \tilde{p} = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_3 \in \tilde{\mathbb{Q}}_{\alpha \beta} \) with respect to the basis \( \{1, e_1, e_2, e_3\} \) is given by:

\[
\tilde{p} = [b_0 \ b_1 \ b_2 \ b_3]^\top.
\]

Using the matrix in equation (3.1), the multiplication of \( \tilde{q}, \tilde{p} \in \tilde{\mathbb{Q}}_{\alpha \beta} \) can also be expressed by:

\[
\tilde{q} \tilde{p} = A_l^{\tilde{q}} \tilde{p} = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha \beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -\alpha a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}.
\]

Moreover, using equation (3.2) and \( \tilde{p} = [p_0 \ p_1 \ p_2 \ p_3]^\top \), we have:

\[
\tilde{q} \tilde{p} = \begin{bmatrix} q_0 & p q_1 & 0 & 0 \\ q_1 & q_0 & 0 & 0 \\ q_2 & p q_3 & q_0 & p q_1 \\ q_3 & q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}.
\]
By writing $\tilde{q} \in \tilde{Q}_{\alpha\beta}$ as in the form $\tilde{q} = (a_0 + a_1 e_1) + (a_2 + a_3 e_1)e_2$, we can state the following:

**Proposition 3.1.** Let $\tilde{q} = (a_0 + a_1 e_1) + (a_2 + a_3 e_1)e_2 \in \tilde{Q}_{\alpha\beta}$. Then, we have

$$\sigma A^I_\tilde{q} \sigma = A^I_{\tilde{q}^*},$$

where $\sigma = \text{diag}(1, 1, -1, -1)$ and $\tilde{q}^* = (a_0 + a_1 e_1) - (a_2 + a_3 e_1)e_2 \in \tilde{Q}_{\alpha\beta}$.

**Proof.** It is clear that

$$\sigma A^I_\tilde{q} \sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} a_0 & -\alpha a_1 & \beta a_2 & \alpha a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ -a_2 & -\alpha a_3 & a_0 & -\alpha a_1 \\ -a_3 & a_2 & a_1 & a_0 \end{bmatrix}.$$

Hence, the last matrix corresponds to $A^I_{\tilde{q}^*}$. \[\square\]

Standard elementary matrix operations establish the following theorems.

**Theorem 3.3.** For any $\tilde{q}, \tilde{p} \in \tilde{Q}_{\alpha\beta}$ and $\lambda \in \mathbb{R}$, the following properties are satisfied:

(i) $\tilde{q} = \tilde{p} \iff A^I_{\tilde{q}} = A^I_{\tilde{p}}$,

(ii) $A^I_{\lambda \tilde{q}} = \lambda (A^I_{\tilde{q}})$,

(iii) $A^I_{\tilde{q} \tilde{p}} = A^I_{\tilde{q}} A^I_{\tilde{p}}$,

(iv) $\tilde{q} = \tilde{p} \iff B_{\tilde{q}} = B_{\tilde{p}}$,

(v) $B_{\lambda \tilde{q}} = \lambda (B_{\tilde{q}})$,

(vi) $B_{\tilde{q} \tilde{p}} = B_{\tilde{q}} B_{\tilde{p}}$.

**Theorem 3.4.** Let $\tilde{q}$ be the conjugate and $\tilde{q}^{-1}$ be the inverse of non-null $\tilde{q} \in \tilde{Q}_{\alpha\beta}$. Then,

$$A^I_{\tilde{q}^{-1}} = \frac{1}{\sqrt{\det(A^I_{\tilde{q}})}} A^I_{\tilde{q}},$$

where $\det(A^I_{\tilde{q}}) = (a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2)^2 = N_{\tilde{q}}^2$.

**Proof.** By considering Definition 2.4 and Theorem 3.3 item (ii), the proof is clear. \[\square\]
Let us define a valuable construction of the vector representation of $\tilde{q}$ and give its properties.

**Definition 3.1.** Let $\tilde{q} = q_0 + q_1J + q_2\varepsilon + q_3J\varepsilon \in \widetilde{Q}_{\alpha\beta}$. The vector representation of $\tilde{q}$ is defined as:

$$\tilde{q} = \begin{bmatrix} q_0 & \tilde{q}_1^T & \tilde{q}_2^T & \tilde{q}_3^T \end{bmatrix}^T \in M_{16 \times 1}(\mathbb{R}),$$

where $q_{i-1} = x_{0i} + x_{1i}e_1 + x_{2i}e_2 + x_{3i}e_3 \in Q_{\alpha\beta}$ and $\tilde{q}_{i-1} = (x_{0i}, x_{1i}, x_{2i}, x_{3i})^T$ are vectors for $1 \leq i \leq 4$.

**Theorem 3.5.** Let $\tilde{q} = q_0 + q_1J + q_2\varepsilon + q_3J\varepsilon \in \widetilde{Q}_{\alpha\beta}$. Then,

(i) $X\tilde{q} = \tilde{q}^{11}$,

(ii) $Y\tilde{q} = \tilde{q}^{12}$,

(iii) $Z\tilde{q} = \tilde{q}^{13}$,

where

$$X = \text{diag}(1,1,1,1,-1,-1,-1,1,1,1,1,1,1,1,1,-1) \in M_{16}(\mathbb{R}),$$

$$Y = \text{diag}(1,1,1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,1,-1) \in M_{16}(\mathbb{R}),$$

$$Z = \text{diag}(1,1,1,1,-1,-1,-1,1,1,1,1,1,1,1,1,1,-1) \in M_{16}(\mathbb{R}).$$

**Proof.** For $1 \leq i \leq 4$, let us consider $\tilde{q} = q_0 + q_1J + q_2\varepsilon + q_3J\varepsilon \in \widetilde{Q}_{\alpha\beta}$, where $q_{i-1} = x_{0i} + x_{1i}e_1 + x_{2i}e_2 + x_{3i}e_3 \in Q_{\alpha\beta}$ and $\tilde{q}_{i-1} = (x_{0i}, x_{1i}, x_{2i}, x_{3i})^T$. Taking $Y = \text{diag}(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,-1)$, we have:

$$Y\tilde{q} = \begin{bmatrix} I_4 & 0 & 0 & 0 \\
0 & I_4 & 0 & 0 \\
0 & 0 & -I_4 & 0 \\
0 & 0 & 0 & -I_4 \end{bmatrix} \begin{bmatrix} x_{01} & x_{11} & x_{21} & x_{31} \\
x_{02} & x_{12} & x_{22} & x_{32} \\
x_{03} & x_{13} & x_{23} & x_{33} \\
x_{04} & x_{14} & x_{24} & x_{34} \end{bmatrix}^T.$$

It is clear that multiplication gives $\tilde{q}^{13}$. The other items can be proved similarly. $\square$

**Theorem 3.6.** Every DGDC number with generalized quaternion coefficients can be represented by an $8 \times 8$ generalized complex number matrix.
Proof. Let us consider \( \tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \tilde{Q}_{\alpha\beta} \), where \( a_i = z_{i1} + z_{i2} \varepsilon \in \mathbb{D}C_p \) and \( z_{i1}, z_{i2} \in C_p \) for \( i = 0, 1, 2, 3 \). Applying the bijective map

\[
\gamma(a_i) = \begin{bmatrix}
z_{i1} & 0 \\
z_{i2} & z_{i1}
\end{bmatrix},
\]

which is from \( DGC \) numbers to the subset of \( 2 \times 2 \) generalized complex number matrices, into equation (3.1), we can write:

\[
\begin{bmatrix}
\gamma(a_0) & \gamma(-\alpha a_1) & \gamma(-\beta a_2) & \gamma(-\alpha\beta a_3) \\
\gamma(a_1) & \gamma(a_0) & \gamma(-\beta a_3) & \gamma(\beta a_2) \\
\gamma(a_2) & \gamma(\alpha a_3) & \gamma(a_0) & \gamma(-\alpha a_1) \\
\gamma(a_3) & \gamma(-a_2) & \gamma(a_1) & \gamma(a_0)
\end{bmatrix}.
\]

(3.3)

It follows that

\[
\begin{bmatrix}
z_{01} & 0 & -\alpha z_{11} & 0 & -\beta z_{21} & 0 & -\alpha\beta z_{31} & 0 \\
z_{02} & -\alpha z_{12} & -\alpha z_{11} & -\beta z_{22} & -\beta z_{21} & -\alpha\beta z_{32} & -\alpha\beta z_{31} & 0 \\
z_{11} & 0 & z_{01} & 0 & \beta z_{31} & 0 & \beta z_{21} & 0 \\
z_{12} & z_{11} & z_{02} & z_{01} & -\beta z_{32} & \beta z_{31} & \beta z_{22} & \beta z_{21} \\
z_{21} & 0 & \alpha z_{31} & 0 & z_{01} & 0 & -\alpha z_{11} & 0 \\
z_{22} & z_{21} & \alpha z_{32} & \alpha z_{31} & z_{02} & z_{01} & -\alpha z_{12} & -\alpha z_{11} \\
z_{31} & 0 & -z_{21} & 0 & z_{11} & z_{01} & 0 & 0 \\
z_{32} & z_{31} & -z_{22} & -z_{21} & z_{12} & z_{11} & z_{02} & z_{01}
\end{bmatrix}.
\]

(3.4)

This is a representation of \( \tilde{q} \) with respect to the base \( \{1, \varepsilon, e_1, \varepsilon e_1, e_2, \varepsilon e_2, e_3, \varepsilon e_3\} \). It is called the left generalized complex matrix representation of \( \tilde{q} \) and denoted by \( C_l^{\tilde{q}} \).

Theorem 3.7. Every \( DGC \) number with generalized quaternion coefficients can be represented by a \( 16 \times 16 \) real matrix.

Proof. Let us consider \( \tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \tilde{Q}_{\alpha\beta} \), where \( a_i = z_{i1} + z_{i2} \varepsilon \in \mathbb{D}C_p \) and \( z_{i1} = x_{i1} + x_{i2} J \in C_p \) for \( i = 0, 1, 2, 3 \). We have the following \( 16 \times 16 \) real matrix:

\[
\begin{bmatrix}
\Gamma(z_{01}) & \Gamma(0) & \Gamma(-\alpha z_{11}) & \Gamma(0) & \Gamma(-\beta z_{21}) & \Gamma(0) & \Gamma(-\alpha\beta z_{31}) & \Gamma(0) \\
\Gamma(z_{02}) & \Gamma(z_{01}) & \Gamma(-\alpha z_{12}) & \Gamma(-\alpha z_{11}) & \Gamma(-\beta z_{22}) & \Gamma(-\beta z_{21}) & \Gamma(-\alpha\beta z_{32}) & \Gamma(-\alpha\beta z_{31}) \\
\Gamma(z_{11}) & \Gamma(0) & \Gamma(z_{01}) & \Gamma(0) & \Gamma(\beta z_{31}) & \Gamma(0) & \Gamma(\beta z_{21}) & \Gamma(0) \\
\Gamma(z_{12}) & \Gamma(z_{11}) & \Gamma(z_{02}) & \Gamma(z_{01}) & \Gamma(-\beta z_{32}) & \Gamma(\beta z_{31}) & \Gamma(\beta z_{22}) & \Gamma(\beta z_{21}) \\
\Gamma(z_{21}) & \Gamma(0) & \Gamma(\alpha z_{31}) & \Gamma(0) & \Gamma(z_{01}) & \Gamma(0) & \Gamma(-\alpha z_{11}) & \Gamma(0) \\
\Gamma(z_{22}) & \Gamma(z_{21}) & \Gamma(\alpha z_{32}) & \Gamma(\alpha z_{31}) & \Gamma(z_{01}) & \Gamma(z_{01}) & \Gamma(-\alpha z_{12}) & \Gamma(-\alpha z_{11}) \\
\Gamma(z_{31}) & \Gamma(0) & \Gamma(-z_{21}) & \Gamma(0) & \Gamma(z_{11}) & \Gamma(0) & \Gamma(z_{01}) & \Gamma(0) \\
\Gamma(z_{32}) & \Gamma(z_{31}) & \Gamma(-z_{22}) & \Gamma(-z_{21}) & \Gamma(z_{12}) & \Gamma(z_{11}) & \Gamma(z_{02}) & \Gamma(z_{01})
\end{bmatrix}.
\]

(3.5)
which is computed by applying the bijective map $\Gamma(z_{i1}) = \begin{bmatrix} x_{i1} & p x_{i2} \\ x_{i2} & x_{i1} \end{bmatrix}$ into equation (3.4). Here $\Gamma$ is defined from generalized complex numbers to the subset of $2 \times 2$ real matrices. Equation (3.5) is representation of $\tilde{q}$ with respect to the base

\[
\{1, J, \varepsilon, J\varepsilon, e_1, Je_1, \varepsilon e_1, Je e_1, e_2, Je_2, \varepsilon e_2, e_3, Je_3, \varepsilon e_3, J\varepsilon e_3\}.
\]

It is called the left real matrix representation of $\tilde{q}$ and denoted by $\mathcal{D}_{\tilde{q}}^l$.

\[\] Example 3.1. Let

\[
\tilde{q} = (18 + 6J + 8\varepsilon) + (-2 + 9\varepsilon)e_1 + (-7 + 3\varepsilon + 8J\varepsilon)e_2 + (19 + J - \varepsilon + 3J\varepsilon)e_3
\]

be a generalized quaternion in $\mathbb{Q}_{32}$ and $p = \frac{1}{2}$. Then,

\[
\mathcal{A}_{\tilde{q}}^l = \begin{bmatrix}
18 + 6J + 8\varepsilon & -3(-2 + 9\varepsilon) & -2(-7 + 3\varepsilon + 8J\varepsilon) & -6(19 + J - \varepsilon + 3J\varepsilon) \\
-2 + 9\varepsilon & 18 + 6J + 8\varepsilon & -2(19 + J - \varepsilon + 3J\varepsilon) & 2(-7 + 3\varepsilon + 8J\varepsilon) \\
-7 + 3\varepsilon + 8J\varepsilon & 3(19 + J - \varepsilon + 3J\varepsilon) & 18 + 6J + 8\varepsilon & -3(-2 + 9\varepsilon) \\
19 + J - \varepsilon + 3J\varepsilon & -7 + 3\varepsilon + 8J\varepsilon & -2 + 9\varepsilon & 18 + 6J + 8\varepsilon
\end{bmatrix},
\]

\[
\mathcal{B}_{\tilde{q}}^l = \begin{bmatrix}
18 - 2e_1 - 7e_2 + 19e_3 & \frac{1}{2}(6 + e_3) & 0 & 0 \\
6 + e_3 & 18 - 2e_1 - 7e_2 + 19e_3 & 0 & 0 \\
8 + 9e_1 + 3e_2 - e_3 & \frac{1}{2}(8e_2 + 3e_3) & 18 - 2e_1 - 7e_2 + 19e_3 & \frac{1}{2}(6 + e_3) \\
8e_2 + 3e_3 & 8 + 9e_1 + 3e_2 - e_3 & 6 + e_3 & 18 - 2e_1 - 7e_2 + 19e_3
\end{bmatrix},
\]

\[
\mathcal{C}_{\tilde{q}}^l = \begin{bmatrix}
18 + 6J & 0 & 6 & 0 & 14 & 0 & 0 & -114 - 6J & 0 \\
8 & 18 + 6J & -27 & 6 & -6 - 16J & 14 & 6 - 18J & -114 - 6J \\
-2 & 0 & 18 + 6J & 0 & -38 - 2J & 0 & -14 & 0 \\
-7 & 0 & 57 + 3J & 0 & 18 + 6J & 0 & 6 & 0 \\
3 + 8J & -7 & -3 + 9J & 57 + 3J & 8 & 18 + 6J & -27 & 6 \\
19 + J & 0 & 7 & 0 & -2 & 0 & 18 + 6J & 0 \\
-1 + 3J & 19 + J & -3 - 8J & 7 & 9 & -2 & 8 & 18 + 6J
\end{bmatrix},
\]

\[
\mathcal{D}_{\tilde{q}}^l = \begin{bmatrix}
18 & 3 & 0 & 0 & 6 & 0 & 0 & 0 & 14 & 0 & 0 & 0 & -114 - 6 & 0 & 0 \\
6 & 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -114 - 6 \\
8 & 0 & 18 & 3 & -27 & 0 & 6 & 0 & -6 - 16 & 14 & 0 & 6 & -9 & -114 - 3 \\
0 & 0 & 6 & 18 & 0 & 0 & 0 & 0 & -16 & -6 & 0 & 0 & -18 & 6 & -6 & -114 \\
-2 & 0 & 0 & 0 & 18 & 3 & 0 & 0 & -38 & -1 & 0 & 0 & -14 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 18 & 0 & 0 & -2 & -38 & 0 & 0 & 0 & 0 & 0 & 0 \\
9 & 0 & -2 & 0 & 8 & 0 & 18 & 3 & 2 & -3 & -38 & -1 & 6 & 8 & -14 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 18 & -6 & 2 & -2 & -38 & 16 & 6 & 0 & 0 \\
-7 & 0 & 0 & 0 & 57 & \frac{2}{3} & 0 & 0 & 18 & 3 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 57 & 0 & 0 & 6 & 18 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 4 & -7 & 0 & -3 & \frac{2}{3} & 57 & 1 & 8 & 0 & 18 & 3 & -27 & 0 & 6 & 0 \\
8 & 3 & 0 & 0 & 9 & -3 & 3 & 57 & 0 & 0 & 6 & 18 & 0 & 0 & 0 & 0 \\
19 & \frac{1}{2} & 0 & 0 & 7 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 18 & 3 & 0 & 0 \\
1 & 19 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 18 \\
-1 & \frac{2}{3} & 19 & \frac{2}{3} & -3 & -4 & 7 & 0 & 9 & 0 & -2 & 0 & 8 & 0 & 18 & 3 \\
3 & -1 & 1 & 19 & -8 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 18
\end{bmatrix}.
\]
Moreover, the matrix $A_{\tilde{q}^{-1}}$ is as follows:

\[
\frac{1}{\sqrt{\det(A_{\tilde{q}})}} \begin{bmatrix}
18 + 6J + 8\varepsilon & 3(-2 + 9\varepsilon) & 2(-7 + 3\varepsilon + 8J\varepsilon) & 6(19 + J - \varepsilon + 3J\varepsilon) \\
2 - 9\varepsilon & 18 + 6J + 8\varepsilon & 2(19 + J - \varepsilon + 3J\varepsilon) & -2(-7 + 3\varepsilon + 8J\varepsilon) \\
7 - 3\varepsilon - 8J\varepsilon & -3(19 + J - \varepsilon + 3J\varepsilon) & -18 - 6J - 8\varepsilon & 3(-2 + 9\varepsilon) \\
-19 - J + \varepsilon - 3J\varepsilon & (-7 + 3\varepsilon + 8J\varepsilon) & 18 + 6J + 8\varepsilon & 18 + 6J + 8\varepsilon
\end{bmatrix},
\]

where

\[
\det(A_{\tilde{q}}) = (2621 + 444J - 114\varepsilon + 544J\varepsilon)^2.
\]

Also, the vector representation of $\tilde{q}^{\dagger_2}$ is computed by:

\[
\tilde{q}^{\dagger_2} = \mathcal{Y}\tilde{q} = \begin{bmatrix}
I_4 & 0 & 0 & 0 \\
0 & I_4 & 0 & 0 \\
0 & 0 & -I_4 & 0 \\
0 & 0 & 0 & -I_4
\end{bmatrix} \begin{bmatrix}
[18 & -2 & -7 & 19]^T \\
[6 & 0 & 0 & 1]^T \\
[8 & 9 & 3 & -1]^T \\
[0 & 0 & 8 & 3]^T
\end{bmatrix}
= [18 & -2 & -7 & 19 & 6 & 0 & 0 & 1 & 8 & 9 & 3 & -1 & 0 & 0 & 8 & 3]^T.
\]

4. Conclusion

This study develops the theory of generalized quaternions with DGC number coefficients for any real number $p$. With this purpose, the algebraic structures and properties are investigated by considering them as a generalized quaternion and as a DGC number. In addition, different matrix representations are investigated and examples are presented. The crucial part of this paper is that one can find the different types of generalized quaternions included in the following Table 1:

<table>
<thead>
<tr>
<th>Type of components</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = -1$</td>
<td>dual-complex</td>
</tr>
<tr>
<td>$p = 0$</td>
<td>hyper-dual</td>
</tr>
<tr>
<td>$p = 1$</td>
<td>dual-hyperbolic</td>
</tr>
</tbody>
</table>

Table 1. Classification of generalized quaternions regarding components.

Moreover, it is worth pointing out that real, split, semi, split semi, and quasi quaternions are also obtained in this study by taking special values for $\alpha$ and $\beta$.

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References


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