ON THE MEROMORPHIC SOLUTIONS OF A CERTAIN TYPE OF NONLINEAR DIFFERENCE-DIFFERENTIAL EQUATION

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Abstract. The main objective of this paper is to give the specific forms of the meromorphic solutions of the nonlinear difference-differential equation

\[ f^n(z) + P_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}, \]

where \( P_d(z, f) \) is a difference-differential polynomial in \( f(z) \) of degree \( d \leq n - 1 \) with small functions of \( f(z) \) as its coefficients, \( p_1, p_2 \) are nonzero rational functions and \( \alpha_1, \alpha_2 \) are non-constant polynomials. More precisely, we find out the conditions for ensuring the existence of meromorphic solutions of the above equation.

Keywords: nonlinear differential equation; differential polynomial; Nevanlinna’s value distribution theory

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1. INTRODUCTION, DEFINITIONS AND RESULTS

In the paper, a meromorphic function means a function meromorphic in the open complex plane \( \mathbb{C} \). We use the standard notations of Nevanlinna theory, e.g., \( N(r, f) \), \( m(r, f) \), \( T(r, f) \), \( N(r, a; f) \), \( \overline{N}(r, a; f) \), \( m(r, a; f) \), etc. (see [2]). We denote by \( S(r, f) \) a quantity, not necessarily the same at each of its occurrence, that satisfies the condition \( S(r, f) = o\{T(r, f)\} \) as \( r \to \infty \) except possibly a set of finite linear measure.

A meromorphic function \( a = a(z) \) is called a small function of a meromorphic function \( f \) if \( T(r, a) = S(r, f) \). Let us denote by \( S(f) \) the class of all small functions of \( f \). Clearly \( \mathbb{C} \subset S(f) \) and if \( f \) is a transcendental function, then every rational function is a member of \( S(f) \).

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The order and hyper-order of a meromorphic function $f(z)$ are denoted and defined by

$$\varrho(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \varrho_2(f) = \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r},$$

respectively. It is clear that if $\varrho(f) < \infty$, then $\varrho_2(f) = 0$.

Let $k \in \mathbb{N}$ and $a \in \mathbb{C} \cup \{\infty\}$. We use the notations $N_k(r; a; f)$ and $N_{k+1}(r; a; f)$ to denote the counting function of $a$-points of $f$ with multiplicity not greater than $k$ and the counting function of $a$-points of $f$ with multiplicity greater than $k$, respectively. Similarly, $\overline{N}_k(r; a; f)$ and $\overline{N}_{k+1}(r; a; f)$ are their reduced functions, respectively.

By a differential polynomial $P_d(z, f)$ in $f(z)$ of degree $d$, we mean a polynomial in $f(z)$ and its derivatives of a total degree at most $d$ with small functions of $f(z)$ as coefficients. When the coefficients are polynomials, we call $P_d(z, f)$ an algebraic differential polynomial.

By a difference-differential polynomial $P_d(z, f)$ in $f(z)$ of degree $d$, we mean a polynomial in $f(z)$, its shifts and their derivatives of a total degree at most $d$ with small functions of $f(z)$ as coefficients.

It is always an interesting and quite difficult problem to prove the existence of the entire or meromorphic solutions $f(z)$ of a given differential equation and to find out the solutions if they exist. A special type of nonlinear differential equation

$$f^n(z) + P_d(z, f) = h(z),$$

where $h(z)$ is a given entire or meromorphic function and $P_d(z, f)$ is a differential polynomial in $f(z)$ of degree $d$, has become a matter of increasing interest among the researchers.

It is easy to show that the function $f_1(z) = \sin z$ is a solution of the nonlinear differential equation $4f^3(z) + 3f''(z) = -\sin 3z$. In [3], it was proved that $f_2(z) = -\frac{\sqrt{3}}{2} \cos z - \frac{1}{2} \sin z$ is also a solution of this equation. In 2004, Yang and Li (see [10]) proved that this equation admits exactly three entire solutions, namely $f_1(z)$, $f_2(z)$ and $f_3(z) = \frac{\sqrt{3}}{2} \cos z - \frac{1}{2} \sin z$. Since the function $-\sin 3z$ is a linear combination of $e^{3z}$ and $e^{-3z}$, so it is interesting to find all entire solutions of the general equation

$$(1.1) \quad f^n(z) + P_d(z, f) = p_1 e^{\lambda z} + p_2 e^{-\lambda z},$$

where $p_1$, $p_2$ and $\lambda$ are nonzero constants and $P_d(z, f)$ denotes a differential polynomial in $f(z)$ of degree $d \leq n - 1$.

In 2004, Yang and Li (see [10]) answered the above question partially and obtained the following result.
**Theorem A** ([10]). Let \( n \in \mathbb{N} \setminus \{1, 2\} \), \( P_d(z, f) \) be a differential polynomial in \( f \) of degree \( d \leq n - 3 \), \( b \in S(f) \) and \( \lambda, p_1, p_2 \) be three nonzero constants. Then the differential equation
\[
f^n(z) + P_d(z, f) = b(z)(p_1 e^{\lambda z} + p_2 e^{-\lambda z})
\]
has no transcendental entire solution \( f(z) \).

In 2006, Li and Yang (see [6]) derived similar conclusion when the term on the right-hand side of equation (1.1) was replaced by \( p_1(z) e^{\alpha_1 z} + p_2(z) e^{\alpha_2 z} \), where \( p_1(z), p_2(z) \) are nonzero polynomials, \( \alpha_1, \alpha_2 \) are two constants with \( \alpha_1 / \alpha_2 \not\in \mathbb{Q} \), and presented their result as follows.

**Theorem B** ([6]). Let \( n \in \mathbb{N} \setminus \{1, 2, 3\} \) and \( P_d(z, f) \) denote an algebraic differential polynomial in \( f(z) \) of degree \( d \leq n - 3 \). Let \( p_1(z), p_2(z) \) be two nonzero polynomials, \( \alpha_1 \) and \( \alpha_2 \) be two nonzero constants with \( \alpha_1 / \alpha_2 \not\in \mathbb{Q} \). Then the differential equation
\[
f^n(z) + P_d(z, f) = p_1(z) e^{\alpha_1 z} + p_2(z) e^{\alpha_2 z}
\]
has no transcendental entire solutions.

In 2011, Li derived the possible forms of solutions of equation (1.1) when \( d \leq n - 2 \), and obtained the following result (see [5]).

**Theorem C** ([5]). Let \( n \in \mathbb{N} \setminus \{1\} \), \( P_d(z, f) \) be a differential polynomial in \( f(z) \) of degree \( d \leq n - 2 \) and \( p_1, p_2, \alpha_1, \alpha_2 \) be nonzero constants and \( \alpha_1 \neq \alpha_2 \). If \( f(z) \) is a transcendental meromorphic solution of the equation
\[
f^n(z) + P_d(z, f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}
\]
satisfying \( N(r, \infty; f) = S(r, f) \), then one of the following holds:

(i) \( f(z) = c_0(z) + c_1 e^{\alpha_1 / nz} \),
(ii) \( f(z) = c_0(z) + c_2 e^{\alpha_2 / nz} \),
(iii) \( f(z) = c_1 e^{\alpha_1 / nz} + c_2 e^{\alpha_2 / nz} \) and \( \alpha_1 + \alpha_2 = 0 \),

where \( c_0 \in S(f) \) and \( c_1, c_2 \in \mathbb{C} \setminus \{0\} \) such that \( c_i^n = p_i, i = 1, 2 \).

In 2013, Liao, Yang and Zhang (see [7]) extended the above results by considering that \( h(z) \) is a meromorphic function of integer order and improved the results of Theorems B and C. Actually, they obtained the following result.
Theorem D ([7]). Let $n \in \mathbb{N} \setminus \{1, 2\}$ and $P_d(z, f)$ be a differential polynomial in $f(z)$ of degree $d$ with rational functions as its coefficients. Suppose that $p_1$, $p_2$ are nonzero rational functions and $\alpha_1$, $\alpha_2$ are polynomials. If $d \leq n - 2$, the differential equation

$$f^n(z) + P_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}$$

admits a meromorphic function $f(z)$ with finitely many poles. Then $\alpha_1'/\alpha_2'$ is a rational number. Furthermore, only one of the following four cases holds:

1. $f(z) = q(z)e^{p(z)}$ and $\alpha_1'(z)/\alpha_2'(z) = 1$, where $q(z)$ is a nonzero rational function and $p(z)$ is a polynomial with $np'(z) = \alpha_1'(z) = \alpha_2'(z)$;
2. $f(z) = q(z)e^{p(z)}$ and either $\alpha_1'(z)/\alpha_2'(z) = k/n$ or $\alpha_1'(z)/\alpha_2'(z) = n/k$, where $q(z)$ is a nonzero rational function, $k \in \mathbb{N}$ with $1 \leq k \leq d$ and $p(z)$ is a polynomial with $np'(z) = \alpha_1'(z)$ or $np'(z) = \alpha_2'(z)$;
3. $f(z)$ satisfies the first order linear differential equation $f'(z) = n^{-1}(p_1'(z)/p_2(z) + \alpha_2'(z))f(z) + \psi(z)$ and $\alpha_1'(z)/\alpha_2'(z) = (n-1)/n$ or $f(z)$ satisfies the first order linear differential equation $f'(z) = n^{-1}(p_1'(z)/p_2(z) + \alpha_1'(z))f(z) + \psi(z)$ and $\alpha_1'(z)/\alpha_2'(z) = n/(n-1)$, where $\psi(z)$ is a rational function;
4. $f(z) = \gamma_1(z)e^{\beta_1(z)} + \gamma_2(z)e^{-\beta_1(z)}$ and $\alpha_1'(z)/\alpha_2'(z) = -1$, where $\gamma_1(z)$, $\gamma_2(z)$ are nonzero rational functions and $\beta_1(z)$ is a polynomial with $n\beta_1'(z) = \alpha_1'(z)$ or $n\beta_1'(z) = \alpha_2'(z)$.

Now it is interesting to find out all the meromorphic solutions of the following nonlinear differential-difference equation:

$$f^n(z) + P_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)},$$

where $P_d(z, f)$ is a differential-difference polynomial in $f(z)$ of degree $d \leq n - 1$ with small functions of $f(z)$ as its coefficients, $p_1(z)$, $p_2(z)$ are nonzero rational functions and $\alpha_1(z)$, $\alpha_2(z)$ are non-constant polynomials.

In 2018, Lü, Wu, Wang and Yang (see [8]) derived the possible forms of the solutions of equation (1.2) when $n = 3$, $d = 1$, and obtained the following result.

Theorem E ([8]). Let $P_d(z, f)$ denote a difference-differential polynomial in $f(z)$ of degree one with small functions as its coefficients such that $P_d(z, 0) \equiv 0$ and let $p_1$, $p_2$, $\alpha_1$, $\alpha_2$ be nonzero constants such that $\alpha_1 \neq \alpha_2$. If $f(z)$ is an entire solution with $\varphi_2(f) < 1$ to equation

$$f^3(z) + P_d(z, f) = p_1e^{\alpha_1z} + p_2e^{\alpha_2z},$$

then one of the following relations holds:
(1) \( f(z) = c_1 \exp(\frac{1}{3} \alpha_1 z) + c_2 \exp(\frac{1}{3} \alpha_2 z) \), where \( c_1, c_2 \in \mathbb{C} \setminus \{0\} \) satisfying \( c_1^3 = p_1, c_2^3 = p_2 \) and \( \alpha_1 + \alpha_2 = 0 \),

(2) \( f^3(z) = (p_1 - c_1) \exp(\alpha_1 z) \) and \( P_d(z, f) = c_1 \exp(\alpha_1 z) + p_2 \exp(\alpha_2 z) \), where \( c_1 \) is a constant,

(3) \( f^3(z) = (p_2 - c_2) \exp(\alpha_2 z) \) and \( P_d(z, f) = p_1 \exp(\alpha_1 z) + c_2 \exp(\alpha_2 z) \), where \( c_2 \) is a constant.

For further study, it is quite natural to ask the following questions.

**Question 1.** What happens if \( f^3(z) \) is replaced by \( f^n(z) \), where \( n \in \mathbb{N} \), in Theorem E?

**Question 2.** What will happen if we delete the condition \( P_d(z, 0) \equiv 0 \) in Theorem E?

**Question 3.** How to find the solutions of equation (1.2) under the condition \( n \geq d + 2 \)?

The main objective of this paper is to find out the possible answers to the above questions. The following theorem is the main result of the paper.

**Theorem 1.1.** Let \( P_d(z, f) \) be a difference-differential polynomial in \( f(z) \) of degree \( d \in \mathbb{N} \cup \{0\} \) with small functions of \( f(z) \) as its coefficients and \( n \in \mathbb{N} \) such that \( n \geq d + 2 \). Suppose that \( p_1(z), p_2(z) \) are non-zero rational functions and \( \alpha_1(z), \alpha_2(z) \) are non-constant polynomials. If \( f(z) \) is a meromorphic solution to the difference-differential equation

\[
(1.3) \quad f^n(z) + P_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}
\]

satisfying \( q_2(f) < 1 \) and \( N(r, \infty; f) = S(r, f) \), then one of the following cases holds:

1. \( f(z) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)} \) such that \( q_2^2(f) < 1 \) and \( N(r, \infty; f) = S(r, f) \), then one of the following cases holds:

1. \( f(z) = q(z)e^{\alpha_2(z)/n} + \alpha_1(z) \equiv \alpha_2(z) \), where \( q(z) \) is a non-zero rational function such that \( q^n(z) = c_0p_2(z) \), where \( c_0 \in \mathbb{C} \setminus \{0\} \);

2. \( f(z) = q(z)e^{\alpha_1(z)/n} + \alpha_1 \equiv \alpha_2(z) \), where \( q(z) \) is a non-zero rational function such that \( q^n(z) = p_1(z) + c_1p_2(z) \), where \( c_1 \in \mathbb{C} \);

3. \( T(r, e^{(kn_{\alpha_1} - \alpha_2)/(n_{\alpha_1} + 1)}) = S(r, f) \), where \( k \in \{0, 1, 2, \ldots, d\} \). In this case, \( f(z) = q(z)e^{\alpha_1(z)/n} \), where \( q(z) \) is a non-zero rational function such that \( q^n(z) = p_1(z) \);

4. \( T(r, e^{(k_{\alpha_2} - \alpha_1)/(n_{\alpha_2} + 1)}) = S(r, f) \), where \( k \in \{0, 1, 2, \ldots, d\} \). In this case, \( f(z) = q(z)e^{\alpha_2(z)/n} \), where \( q(z) \) is a non-zero rational function such that \( q^n(z) = p_2(z) \);

5. \( T(r, e^{(n_{\alpha_1} - \alpha_2)/(n_{\alpha_2} + 1)}) = S(r, f) \). In this case, \( f(z) = u_1(z)e^{\alpha_1(z)/n} - v_1(z) \), where \( u_1(z) \) and \( v_1(z) \) are non-zero small functions of \( f(z) \) such that \( u_1^2(z) = p_1(z) \);

6. \( T(r, e^{(n_{\alpha_2} - \alpha_1)/(n_{\alpha_2} + 1)}) = S(r, f) \). In this case, \( f(z) = u_2(z)e^{\alpha_2(z)/n} - v_2(z) \), where \( u_2(z) \) and \( v_2(z) \) are non-zero small functions of \( f(z) \) such that \( u_2^2(z) = p_2(z) \);
(7) \( T(r, e^{\alpha_1-\alpha_2}) = S(r, f) \). In this case, \( f(z) = q(z)e^{\alpha_1/n} \) and \( P_d(z, f) \equiv 0 \), where \( q(z) \) and \( \varphi(z) \) are nonzero small functions of \( f(z) \) such that \( q^n(z) = p_1(z) + \varphi(z)p_2(z) \);

(8) \( T(r, e^{\alpha_1+\alpha_2}) = S(r, f) \). In this case, \( f(z) = \delta_1(z)e^{\gamma(z)} + \delta_2(z)e^{-\gamma(z)} \), where \( \delta_1(z) \), \( \delta_2(z) \) are nonzero small functions of \( f(z) \) and \( \gamma(z) \) is a non-constant polynomial such that either \( e^{\gamma(z)+\alpha_1(z)} \) is a small function of \( f(z) \) or \( e^{\alpha_1(z)+\alpha_2(z)} \) is a small function of \( f(z) \).

From Theorem 1.1 we have the following corollary.

**Corollary 1.1.** Equation (1.2) does not have any meromorphic solution \( f(z) \) satisfying \( N(r, \infty; f) = S(r, f) \), \( \varphi(f) = \infty \) and \( \varphi_2(f) < 1 \).

**Remark 1.1.** It is easy to see that conclusions (5) and (6) in Theorem 1.1 cannot be removed by the following examples.

**Example 1.1.** Let us consider the difference-differential equation

\[
f^3(z) + P_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)},
\]

where \( P_d(z, f) = -\frac{1}{3}f'(z) - \frac{2}{7}, \) \( p_1(z) = p_2(z) = 1, \) \( \alpha_1(z) = 3z \) and \( \alpha_2(z) = 2z \). Here \( n = 3 \) and \( d = 1 \). One can easily verify that \( f(z) = u_1(z)e^{\alpha_1(z)/3} - v_1(z) \), where \( u_1(z) = 1, \) \( v_1(z) = \frac{1}{3} \) is a solution of the given difference-differential equation.

**Example 1.2.** Let us consider the difference-differential equation

\[
f^4(z) + P_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)},
\]

where \( P_d(z, f) = f^2(z + c) - 3(f'(z))^2 - 4f''(z)f(z) - 2f(z + c), \) \( p_1(z) = 1, \) \( p_2(z) = 4, \) \( \alpha_1(z) = 4z, \) \( \alpha_2(z) = 3z \) and \( c \in \mathbb{C} \setminus \{0\} \) such that \( e^c = 1 \). Here \( n = 4 \) and \( d = 2 \). One can easily verify that \( f(z) = u_2(z)e^{\alpha_2(z)/4} - v_2(z) \), where \( u_2(z) = 1 \) and \( v_2(z) = -1 \) is a solution of the given difference-differential equation.

**Remark 1.2.** It is easy to see that conclusion (8) in Theorem 1.1 cannot be removed by the following examples.

**Example 1.3.** Let us consider the difference-differential equation

\[
f^2(z) + P_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)},
\]

where \( P_d(z, f) \equiv -2, \) \( p_1(z) = p_2(z) = 1, \) \( \alpha_1(z) = 2z \) and \( \alpha_2(z) = -2z \). Here \( n = 2 \) and \( d = 0 \). One can easily verify that \( f(z) = \delta_1(z)e^{\gamma(z)} + \delta_2(z)e^{-\gamma(z)} \) is a solution of the given difference-differential equation, where \( \delta_1(z) = \delta_2(z) = 1 \) and \( \gamma(z) = z \). Also we see that \( e^{n_\gamma(z) + \alpha_2(z)} \) is a small function of \( f(z) \).
Example 1.4. Let us consider the difference-differential equation
\[ f^3(z) + P_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}, \]
where \( P_d(z, f) = zf''(z) - f'(z) - (4z^3 + 3)f(z) \), \( p_1(z) = p_2(z) = 1 \), \( \alpha_1(z) = 3z^2 \) and \( \alpha_2(z) = -3z^3 \). Here \( n = 3 \) and \( d = 1 \). One can easily verify that
\[ f(z) = \delta_1(z)e^{\gamma(z)} + \delta_2(z)e^{-\gamma(z)} \]
is a solution of the given difference-differential equation, where \( \delta_1(z) = \delta_2(z) = 1 \) and \( \gamma(z) = z^2 \). Also we see that \( e^{\alpha_1(z)+\alpha_2(z)} \) is a small function of \( f(z) \).

2. Lemmas

The following lemmas are needful in the proof of our main result.

**Lemma 2.1** ([4]). Let \( f(z) \) be a transcendental meromorphic function and \( f^n(z)P(z, f) = Q(z, f) \), where \( P(z, f) \) and \( Q(z, f) \) are polynomials in \( f(z) \) and its derivatives with meromorphic coefficients, say \( \{a_\lambda(z): \lambda \in I\} \) such that \( m(r, a_\lambda) = S(r, f) \) for all \( \lambda \in I \). If the total degree of \( Q(z, f) \) as a polynomial in \( f(z) \) and its derivatives is less than or equal to \( n \), then \( m(r, P(z, f)) = S(r, f) \).

**Lemma 2.2** ([2]). Let \( f(z) \) be a non-constant meromorphic function and let \( a_i \in S(f), i = 1, 2 \). Then \( T(r, f) \leq N(r, \infty; f) + N(r, a_1; f) + N(r, a_2; f) + S(r, f) \).

**Lemma 2.3** ([9]). Let \( f(z) \) be a non-constant meromorphic function and let \( a_n(\neq 0), a_{n-1}, \ldots, a_0 \in S(f) \). Then \( T\left(r, \sum_{i=0}^{n} a_if^i\right) = nT(r, f) + S(r, f) \).

**Lemma 2.4** ([11]). Let \( f \) be a non-constant meromorphic function and \( k \in \mathbb{N} \). Then
\[ m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f) \]
and if \( f \) is of finite order of growth, then
\[ m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r). \]

**Lemma 2.5** ([1]). Let \( c \in \mathbb{C}\setminus\{0\}, \varepsilon > 0 \) and \( f(z) \) be a non-constant meromorphic function such that \( \varphi_2(f) < 1 \). Then
\[ m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r, f)}{r^{1-\varphi_2(f)-\varepsilon}}\right) \]
outside of an exceptional set of finite logarithmic measure.
Lemma 2.6. Let \(n \in \mathbb{N}\) and \(P_d(z, f)\) be a difference-differential polynomial in \(f(z)\) of degree \(d \leq n - 1\) with small functions of \(f(z)\) as its coefficients. Suppose that \(p_1(z)\), \(p_2(z)\) are nonzero rational functions and \(\alpha_1(z), \alpha_2(z)\) are non-constant polynomials. If \(f(z)\) is a meromorphic solution to the nonlinear difference-differential equation

\[
(f^n(z) + P_d(z, f)) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}
\]

satisfying \(q_2(f) < 1\) and \(N(r, \infty; f) = S(r, f)\), then \(f(z)\) is a transcendental meromorphic function of finite order.

Proof. Let \(f(z)\) be a rational function satisfying the differential-difference equation (2.1). Then clearly \(p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}\) is a rational function, say \(R_1(z)\), and so \(-p_1(z)e^{\alpha_1(z)} = p_2(z)e^{\alpha_2(z)} - R_1(z)\). This shows that \(p_2(z)e^{\alpha_2(z)} - R_1(z)\) has finitely many zeros. But from Lemma 2.2, one can easily conclude that \(p_2(z)e^{\alpha_2(z)} - R_1(z)\) has infinitely many zeros. Therefore we arrive at a contradiction. Consequently, any non-constant meromorphic solution of the difference-differential equation (2.1) must be transcendental.

A difference-differential polynomial \(P_d(z, f)\) in \(f(z)\) can be expressed as

\[
P_d(z, f) = \sum_{\mu} b_\mu(z)G_\mu(z, f),
\]

where \(b_\mu \in S(f)\) and

\[
G_\mu(z, f) = (f(z))^{p_\mu_0}(f'(z))^{p_\mu_1} \cdots (f^{(k)}(z))^{p_\mu_k}(f(z + c_0))^{q_\mu_0}(f(z + c_1))^{q_\mu_1} \cdots (f(z + c_k))^{q_\mu_k} \\
\times (f(z + c_\mu))^{l_\mu_0}(f'(z + c_\mu))^{l_\mu_1} \cdots (f^{(k)}(z + c_\mu))^{l_\mu_k},
\]

where \(p_\mu_0, p_\mu_1, \ldots, p_\mu_k, q_\mu_0, q_\mu_1, \ldots, q_\mu_k, l_\mu_0, l_\mu_1, \ldots, l_\mu_k \in \mathbb{N} \cup \{0\}\) such that \(\sum_{j=0}^k p_\mu_j + \sum_{j=0}^k q_\mu_j + \sum_{j=0}^k l_\mu_j = \mu \leq d\). Therefore we have

\[
P_d(z, f) = \sum_{\mu} b_\mu(z) \frac{G_\mu(z, f)}{f^\mu(z)} f^\mu(z).
\]

Now by Lemmas 2.4 and 2.5, we derive

\[
m \left( r, b_\mu(z) \frac{G_\mu(z, f)}{f^\mu(z)} \right) = m \left( r, b_\mu(z) \frac{f'(z)}{f(z)} \right) \cdots \frac{f^{(k)}(z)}{f(z)} \cdots \frac{f(z + c_\mu)}{f(z)} \cdots \frac{f^{(k)}(z + c_\mu)}{f(z)}
\]

\[= S(r, f). \]
Therefore (2.2) takes the form
\[
P_d(z, f) = c_d(z)f^d(z) + c_{d-1}(z)f^{d-1}(z) + \ldots + c_0(z),
\]
where \(c_d(z) \neq 0\) and \(m(r, c_i(z)) = S(r, f)\) for \(i = 0, 1, 2, \ldots, d\). Now by using the mathematical induction, it follows that \(m(r, P_d(z, f)) \leq dm(r, f) + S(r, f)\). Since \(N(r, \infty; f) = S(r, f)\), it follows that
\[
T(r, P_d(z, f)) \leq dT(r, f) + S(r, f).
\]

Now from (2.1) and (2.3) we have
\[
(2.3)\quad T(r, P_d(z, f)) \leq dT(r, f) + S(r, f).
\]

Therefore (2.2) takes the form
\[
T(p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}) = T(r, f^n(z) + P_d(z, f)) = nT(r, f) + S(r, f)
\]
and
\[
T(p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}) = T(r, f^n(z) + P_d(z, f))
\]
\[
\geq T(r, f^n(z)) - T(r, P_d(z, f))
\]
\[
\geq (n - d)T(r, f) + S(r, f).
\]

It follows from (2.4) and (2.5) that
\[
(n - d)T(r, f) + S(r, f) \leq T(p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}) \leq nT(r, f) + S(r, f),
\]
which implies that \(g(f) < \infty\). This completes the proof. \(\square\)

**Lemma 2.7** ([5]). Suppose that \(f(z)\) is a transcendental meromorphic function and \(q_1, q_2, q_3, a \in S(f)\) such that \(q_3a \neq 0\). If
\[
q_1 f^2 + q_2 ff' + q_3(f')^2 = a,
\]
then
\[
q_3(q_2^2 - 4q_1q_3)a' + q_2(q_2^2 - 4q_1q_3) - q_3(q_2^2 - 4q_1q_3)' + (q_2^2 - 4q_1q_3)q_3' \equiv 0.
\]

**Lemma 2.8** ([2]). Let \(f(z)\) be a non-constant meromorphic function and \(n \in \mathbb{N}\). Suppose that
\[
g(z) = f^n(z) + P_{n-1}(z, f),
\]
where \(P_{n-1}(z, f)\) is a differential polynomial in \(f(z)\) of degree at most \(n - 1\) with small functions of \(f(z)\) as its coefficients and
\[
N(r, f) + N\left(r, \frac{1}{g}\right) = S(r, f).
\]
Then \(g(z) = (f(z) + \gamma(z))^n\), where \(\gamma \in S(f)\).

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Lemma 2.9. Let \( f(z) \) be a non-constant meromorphic function and \( n \in \mathbb{N} \). Suppose that
\[
g(z) = f^{n+1}(z) + P_{n-1}(z, f),
\]
where \( P_{n-1}(z, f) \) is a differential polynomial in \( f(z) \) of degree at most \( n - 1 \) with small functions of \( f(z) \) as its coefficients and
\[
N(r, f) + N\left(r, \frac{1}{g}\right) = S(r, f).
\]
Then \( g(z) = f^{n+1}(z) \) and \( P_{n-1}(z, f) \equiv 0 \).

Proof. Firstly, from Lemma 2.8 we have \( g(z) = (f(z) + \gamma(z))^{n+1} \), where \( \gamma \in S(f) \). If possible, suppose that \( \gamma \not\equiv 0 \). Now from (2.6) we have
\[
(f(z) + \gamma(z))^{n+1} = f^{n+1}(z) + P_{n-1}(z, f)
\]
and so
\[
(n + 1) \gamma(z) f^n(z) + Q_{n-1}(z, f) = P_{n-1}(z, f),
\]
where \( Q_{n-1}(z, f) \) is a differential polynomial in \( f(z) \) of degree at most \( n - 1 \) with small functions of \( f(z) \) as its coefficients. Therefore we have
\[
f^{n-1}(z)(n + 1) \gamma(z) f(z) = P_{n-1}(z, f) - Q_{n-1}(z, f).
\]
Now by Lemma 2.1, we conclude that \( m(r, f) = S(r, f) \). Since \( N(r, \infty; f) = S(r, f) \), it follows that \( T(r, f) = S(r, f) \), which is impossible. Hence \( \gamma \equiv 0 \). Consequently, \( g(z) = f^{n+1}(z) \) and \( P_{n-1}(z, f) \equiv 0 \). This completes the proof. \( \square \)

3. Proof of the Theorem

Proof of Theorem 1.1. By the given condition, we have
\[
f^n + P_d = p_1 e^{\alpha_1} + p_2 e^{\alpha_2},
\]
where \( P_d = P_d(z, f) \). Let \( f \) be a meromorphic solution of equation (3.1). Then by Lemma 2.6, we can conclude that \( f \) is a transcendental meromorphic function of finite order. Now differentiating both sides of (3.1) once, we get
\[
n f^{n-1} f' + P'_d = (p_1 \alpha'_1 + p'_1) e^{\alpha_1} + (p_2 \alpha'_2 + p'_2) e^{\alpha_2}.
\]
Now by eliminating \( e^{\alpha_2} \) from (3.1) and (3.2), we have
\[
f^{n-1}(np_2 f' - (p_2 \alpha'_2 + p'_2) f) + p_2 P'_d - (p_2 \alpha'_2 + p'_2) P_d = A_1 e^{\alpha_1},
\]
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where \( A_1 = p_2(p_1\alpha' + p'_1) - p_1(p_2\alpha' + p'_2) \). Again by eliminating \( e^{\alpha_1} \) from (3.1) and (3.2), we have

\[
(3.4) \quad f^{n-1}(np_1f' - (p_1\alpha' + p'_1)f) + p_1 P_d' - (p_1\alpha' + p'_1) P_d = -A_1 e^{\alpha_2}.
\]

Suppose that \( A_1 \equiv 0 \). Then we have \( \alpha' - \alpha'' = p_2'/p_2 - p'_1/p_1 \) and so \( \alpha' \equiv \alpha'' \). Now from (3.3) we have

\[
(3.5) \quad f^{n-1}(np_2f' - (p_2\alpha' + p'_2)f) = (p_2\alpha' + p'_2) P_d - p_2 P_d'.
\]

Suppose that \( np_2f' - (p_2\alpha' + p'_2)f \neq 0 \). Then by Lemma 2.1, we have

\[
(3.6) \quad \begin{cases}
    m(r, np_2f' - (p_2\alpha' + p'_2)f) = S(r, f), \\
    m(r, np_2f f' - (p_2\alpha' + p'_2)f^2) = S(r, f).
\end{cases}
\]

Since \( N(r, \infty; f) = S(r, f) \), from (3.6) we conclude that

\[
T(r, f) \leq T(r, np_2f f' - (p_2\alpha' + p'_2)f^2) + T(r, np_2f f' - (p_2\alpha' + p'_2)f) + O(1) = S(r, f),
\]

which is impossible. Therefore \( np_2f' - (p_2\alpha' + p'_2)f \equiv 0 \) and so by integration, we get \( f^n = c_0 p_2 e^{\alpha_2} \), where \( c_0 \in \mathbb{C} \setminus \{0\} \). Therefore we let \( f(z) = q(z) e^{\alpha_2(z)/n} \), where \( q(z) \) is a nonzero rational function such that \( q^n(z) = c_0 p_2(z) \).

Next we suppose that \( A_1(z) \neq 0 \). Now differentiating (3.3) once, we get

\[
(3.7) \quad f^{n-2}(-(p_2\alpha' + p'_2)f^2 - np_2\alpha' f f' + (n - 1)np_2(f')^2 + np_2 f f'') + Q_d' = (A'_1 + A_1\alpha'_1)e^{\alpha_1},
\]

where

\[
(3.8) \quad Q_d = p_2 P_d' - (p_2\alpha' + p'_2) P_d.
\]

Eliminating \( e^{\alpha_1} \) from (3.3) and (3.7), we get

\[
(3.9) \quad f^{n-2}(h_{21} f^2 + h_{22} f f' + h_{23}(f')^2 + h_{24} f f'') = R_d,
\]

where

\[
(3.10) \quad \begin{cases}
    R_d = (A'_1 + A_1\alpha'_1) Q_d - A_1 Q_d', \\
    h_{21} = (p_2\alpha' + p'_2)(A'_1 + A_1\alpha'_1) - A_1(p_2\alpha' + p'_2)', \\
    h_{22} = -n(\alpha'_1 + \alpha''_2)p_2 A_1 - np_2 A'_1, \\
    h_{23} = n(n - 1)p_2 A_1 \neq 0, \\
    h_{24} = np_2 A_1 \neq 0.
\end{cases}
\]

Clearly, \( h_{2j} \) are rational functions for \( j = 1, 2, 3, 4 \).
First we suppose that $h_{21} \equiv 0$. Then we have
\[
\frac{(p_2\alpha_2' + p_2')'}{p_2\alpha_2' + p_2'} - \frac{A_1'}{A_1} \equiv \alpha_1'
\]
and so by integration we have $p_2\alpha_2' + p_2' = c_1 A_1 e^{\alpha_1}$, where $c_1 \in \mathbb{C} \setminus \{0\}$. This shows that $A_1 e^{\alpha_1} \in S(f)$. Then from (3.3) we have
\[
(3.11) \quad f^{n-1}(np_2 f' - (p_2\alpha_2' + p_2')f) = (p_2\alpha_2' + p_2')P_d - p_2P_d' + A_1 e^{\alpha_1}.
\]
In this case, one can also easily conclude that $f(z) = q(z)e^{\alpha_2(z)/n}$, where $q(z)$ is a nonzero rational function such that $q^n(z) = c_1 p_2(z)$, where $c_1 \in \mathbb{C} \setminus \{0\}$.

Next we suppose that $h_{21} \not\equiv 0$. Let
\[
(3.12) \quad h_{21} f^2 + h_{22} f f' + h_{23} (f')^2 + h_{24} f f'' = a.
\]
Now we consider the following two cases.

Case 1. Suppose that $a \equiv 0$. Then from (3.12) we have
\[
(3.13) \quad -h_{21} f^2 \equiv h_{22} f f' + h_{23} (f')^2 + h_{24} f f''.
\]
Let $z_1$ be a zero of $f$ of order $l_1$ such that $h_{2i}(z_1) \not\equiv 0$, for $i = 1, 2, 3, 4$. Clearly, $z_1$ is a zero with multiplicity $2l_1$ of the left-hand side of equation (3.13) and a zero with multiplicity $2l_1 - 2$ of the right-hand side of equation (3.13). Therefore we arrive at a contradiction from (3.13). Now from (3.13) we can easily conclude that $N(r, 0; f) = O(\log r)$. Since $a \equiv 0$, from (3.9) and (3.10) we have
\[
(3.14) \quad R_d \equiv 0, \quad \text{i.e.,} \quad (A_1' + A_1\alpha_1')Q_d \equiv A_1 Q_d'.
\]
First we suppose that $Q_d \equiv 0$. Then from (3.8) we have
\[
(3.15) \quad (p_2\alpha_2' + p_2')P_d \equiv p_2 P_d'.
\]
If $P_d \equiv 0$, then from (3.1) and (3.3) we have, respectively,
\[
(3.16) \quad f^n = p_1 e^{\alpha_1} + p_2 e^{\alpha_2}
\]
and
\[
(3.17) \quad f^{n-1}(np_2 f' - (p_2\alpha_2' + p_2')f) = A_1 e^{\alpha_1}.
\]
Now (3.17) gives
\[
(3.18) \quad np_2 \frac{f'}{f} - (p_2\alpha_2' + p_2') = A_1 e^{\alpha_1}.
\]
Using Lemma 2.4, one can easily conclude from (3.18) that \( m(r, e^{\alpha_1}/f^n) = O(\log r) \). Since \( N(r, 0; f) = O(\log r) \), we have \( T(r, e^{\alpha_1}/f^n) = O(\log r) \). Then by the first fundamental theorem, we have \( T(r, f^n/e^{\alpha_1}) = O(\log r) \). Also from (3.16) we have

\[
f^n e^{-\alpha_1} = p_1 + p_2 e^{\alpha_2 - \alpha_1}.
\]

This shows that \( T(r, e^{\alpha_2 - \alpha_1}) = O(\log r) \) and so \( e^{\alpha_2 - \alpha_1} \) is a nonzero constant. Let \( e^{\alpha_2 - \alpha_1} = c_2 \in \mathbb{C} \setminus \{0\} \). Clearly \( \alpha' = \alpha_2' \). Now from (3.16) we have \( f^n = \varphi_1 e^{\alpha_1} \), where \( \varphi_1 = p_1 + c_1 p_2 \) is a rational function. In this case we also have \( f(z) = q(z) e^{\alpha_1(z)/n} \), where \( q(z) \) is a nonzero rational function such that \( q^n(z) = p_1(z) + c_1 p_2(z) \).

Next we suppose that \( P_d \neq 0 \). Then from (3.15) we have

\[
P_d = c_3 p_2 e^{\alpha_2}.
\]

Integrating, we get \( P_d = c_3 p_2 e^{\alpha_2} \), where \( c_3 \in \mathbb{C} \setminus \{0\} \) and so from (3.1) we get

\[
f^n + \left( 1 - \frac{1}{c_3} \right) P_d = p_1 e^{\alpha_1}.
\]

If \( c_3 \neq 1 \), then by Lemma 2.9, we have \( f^n = p_1 e^{\alpha_1} \) and \( P_d \equiv 0 \), which contradicts the fact that \( P_d \neq 0 \). Therefore \( c_3 = 1 \) and so \( f^n = p_1 e^{\alpha_1} \) and \( P_d = p_2 e^{\alpha_2} \neq 0 \). In this case also, we have \( f(z) = q(z) e^{\alpha_1(z)/n} \), where \( q(z) \) is a nonzero rational function such that \( q^n(z) = p_1(z) \). Note that

\[
P_d(z, f) = \sum_{\mu} b_{\mu}(z) \frac{G_{\mu}(z, f)}{f^n(z)} f^{\mu}(z),
\]

where \( b_{\mu} \in S(f) \) and

\[
G_{\mu}(z, f) = (f(z))^{p_0^\mu} (f'(z))^{p_1^\mu} \cdots (f^{(k)}(z))^{p_k^2} 
\times (f(z + c_\mu))^{q_0^\mu} (f'(z + c_\mu))^{q_1^\mu} \cdots (f^{(k)}(z + c_\mu))^{q_k^\mu},
\]

\( p_0^\mu, p_1^\mu, \ldots, p_k^\mu, q_0^\mu, q_1^\mu, \ldots, q_k^\mu \in \mathbb{N} \cup \{0\} \) such that \( \sum_{j=0}^{k} p_j^\mu + \sum_{j=0}^{k} q_j^\mu = \mu \leq d \). Now by Lemmas 2.4 and 2.5, we derive \( m(r, G_{\mu}(z, f)/f^{\mu}(z)) = S(r, f) \). Since \( N(r, \infty; f) + N(r, 0; f) = S(r, f) \), it follows that \( T(r, G_{\mu}(z, f)/f^{\mu}(z)) = S(r, f) \). Therefore (3.20) takes the form \( P_d(z, f) = c_\mu(z) f^{d}(z) + c_{d-1}(z) f^{d-1}(z) + \ldots + c_0(z) \), where \( c_d(z) \neq 0 \) and \( c_i \in S(f) \) for \( i = 0, 1, 2, \ldots, d \). Now substituting \( f(z) = q(z) e^{\alpha_1(z)/n} \) into \( P_d(z, f) = p_2(z) e^{\alpha_2(z)} \), we get

\[
\sum_{k=0}^{d} a_{2k}(z) e^{k\alpha_1(z)/n} = p_2(z) e^{\alpha_2(z)},
\]

where \( a_{2k}(z) (k = 0, 1, \ldots, d) \) are small functions of \( f(z) \).
Since $T(r, f) = T(r, e^{\alpha_1/n}) + S(r, f)$, it follows that $a_{2k}(z)$, $k = 0, 1, \ldots, d$, are small functions of $e^{\alpha_1/n}$ and so $a_{2k}(z)$, $k = 0, 1, \ldots, d$, are small functions of $e^{k\alpha_1/n}$, where $k \in \{1, 2, \ldots, d\}$. Since $p_2 \neq 0$, from (3.21) we conclude that there exists at least one value of $k \in \{0, 1, \ldots, d\}$ such that $a_{2k} \not\equiv 0$. We now claim that there exists exactly one value of $k \in \{0, 1, \ldots, d\}$ such that $a_{2k} \not\equiv 0$. If $d = 0$, then our claim is true. Next we suppose that $d \geq 1$. If possible, suppose that there exist at least two values of $k \in \{0, 1, \ldots, d\}$ such that $a_{2k} \not\equiv 0$. For the sake of simplicity we may assume that $a_{2k} \not\equiv 0$ for $k \in \{0, 1, 2, \ldots, d\}$. Now by Lemma 2.3 we have

\[(3.22) \quad T\left(r, \sum_{k=1}^{d} a_{2k} e^{k\alpha_1/n}\right) = dT(r, e^{\alpha_1/n}) + S(r, e^{\alpha_1/n}).\]

Also from (3.21) we have

\[(3.23) \quad N\left(r, -a_{20}; \sum_{k=1}^{d} a_{2k} e^{k\alpha_1/n}\right) = N(r, 0; p_2) \leq S(r, e^{\alpha_1/n}).\]

Now from Lemmas 2.2, 2.3, (3.22) and (3.23) we have

\[dT(r, e^{\alpha_1/n}) \leq N\left(r, 0; \sum_{k=1}^{d} a_{2k} e^{k\alpha_1/n}\right) + N\left(r, \infty; \sum_{k=1}^{d} a_{2k} e^{k\alpha_1/n}\right)\]
\[+ N\left(r, -a_{20}; \sum_{k=1}^{d} a_{2k} e^{k\alpha_1/n}\right) + S(r, e^{\alpha_1/n})\]
\[\leq N\left(r, 0; \sum_{k=0}^{d-1} a_{2k} e^{k\alpha_1/n}\right) + S(r, e^{\alpha_1/n})\]
\[\leq T\left(r, \sum_{k=0}^{d-1} a_{2k} e^{k\alpha_1/n}\right) + S(r, e^{\alpha_1/n})\]
\[= (d - 1)T(r, e^{\alpha_1/n}) + S(r, e^{\alpha_1/n}),\]

which is impossible. Therefore there exists exactly one value of $k \in \{0, 1, \ldots, d\}$ such that $a_{2k} \not\equiv 0$ and so from (3.21) we conclude that there must exist exactly one value of $k \in \{0, 1, 2, \ldots, d\}$ such that $e^{(k\alpha_1-n\alpha_2)/n}$ is a small function of $f$.

Next we suppose that $Q_d \not\equiv 0$. Then from (3.14) we have

\[(3.24) \quad \frac{Q_d'}{Q_d} \equiv \frac{A_1'}{A_1} + \alpha_1'.\]

Integrating, we get $Q_d = c_4 A_1 e^{\alpha_1}$, where $c_4 \in \mathbb{C} \setminus \{0\}$ and so from (3.3) we get

\[f^{n-1}(np_2 f' - (p_2 \alpha_2 + p_2')f) \equiv \left(\frac{1}{c_4} - 1\right)Q_d.\]
Let \( \varphi_3 = np_2f' - (p_2a_2^2 + p_2^2)f \). If \( c_4 \neq 1 \), then by Lemma 2.1, we have \( m(r, \varphi_3) = S(r, f) \) and \( m(r, \varphi_3 f) = S(r, f) \). Since \( N(r, \infty; f) = S(r, f) \), it follows that \( T(r, \varphi_3) = S(r, f) \) and \( T(r, \varphi_3 f) = S(r, f) \). Note that

\[
T(r, f) \leq T(r, \varphi_3 f) + T\left( r, \frac{1}{\varphi_3} \right) + S(r, f) = S(r, f),
\]

which is impossible. Hence \( c_4 = 1 \) and so \( \varphi_3 \equiv 0 \). Then we have

\[
n\frac{f'}{f} = \frac{p_2'}{p_2} + \alpha_2.
\]

On integration, we get \( f^n = c_5p_2e^{\alpha_2} \), where \( c_5 \in \mathbb{C} \setminus \{0\} \). If \( c_5 \neq 1 \), then from (3.1) we have

\[
\left( 1 - \frac{1}{c_5^2} \right)f^n + P_d = p_1e^{\alpha_1}.
\]

Now by Lemma 2.9, we conclude that \( P_d \equiv 0 \) and so \( Q_d \equiv 0 \), which contradicts the fact that \( Q_d \neq 0 \). Hence \( c_5 = 1 \) and so \( f^n = p_2e^{\alpha_2} \). Also from (3.1) we have \( P_d = p_1e^{\alpha_1} \). In this case we have \( f(z) = q(z)e^{\alpha_2(z)/n} \), where \( q(z) \) is a nonzero rational function such that \( q^n(z) = p_2(z) \). Also there must exist exactly one \( k \in \{0, 1, 2, \ldots, d\} \) such that \( e^{(k\alpha_2 - m\alpha_1)/n} \) is a small function of \( f \).

Case 2. Suppose that \( a \neq 0 \). Then by Lemma 2.1, we can conclude that \( a \) is a small function of \( f \). Now from (3.12) we have

\[
(3.25) \quad \frac{1}{f^2} = \frac{h_{21}}{a} + \frac{h_{22}f'}{a} + \frac{h_{23}}{a}\left( \frac{f'}{f} \right)^2 + \frac{h_{24}}{a}f''.
\]

Therefore from Lemma 2.4 and (3.25) we conclude that \( m(r, 1/f^2) = S(r, f) \), i.e., \( m(r, 1/f) = S(r, f) \). Consequently, by the first fundamental theorem, we have \( T(r, f) = N(r, 0; f) + S(r, f) \). This shows that \( f \) has infinitely many zeros. Let \( z_2 \) be a multiple zero of \( f \) such that \( h_{2i}(z_2) \neq 0, \infty \) for \( i = 1, 2, 3, 4 \). Then from (3.12) we conclude that \( z_2 \) is a zero of \( a \). Therefore \( N_{(2)}(r, 0; f) \leq T(r, a) = S(r, f) \), i.e., \( N_{(2)}(r, 0; f) = S(r, f) \). Consequently, \( f \) has infinitely many simple zeros. Differentiating (3.12) once, we have

\[
(3.26) \quad a' = h_{21}'f^2 + (2h_{21} + h_{22}')ff' + (h_{22} + h_{23}')f'f'' + h_{24}'ff'' + (h_{22} + h_{24}')f'' + (2h_{23} + h_{24})f'f'' + h_{24}ff''.
\]

Now from (3.12) and (3.26) we have

\[
(3.27) \quad (ah_{21}' - a'h_{21})f^2 + (2ah_{21} + ah_{22}' - a'h_{22})ff' + (ah_{22} + ah_{23}' - a'h_{23})f'f'' + (ah_{22} + ah_{24}' - a'h_{24})f'' + a(2h_{23} + h_{24})f'f'' + ah_{24}ff''' \equiv 0.
\]
Let \( z_3 \) be a simple zero of \( f \) which is not a zero or pole of the coefficients in (3.27). Now from (3.27) we see that \( z_3 \) is a zero of \((2ah_{23} + ah_{24})f'' - (a'h_{23} - ah_{22} - ah'_{23})f'\). Let

\[
\alpha = \frac{(2ah_{23} + ah_{24})f'' - (a'h_{23} - ah_{22} - ah'_{23})f'}{f}.
\]

Since \( N(r, \infty; f) + N_1(r, 0; f) = S(r, f) \), from (3.28) we see that \( N(r, \infty; \alpha) = S(r, f) \). Also by Lemma 2.4, we have \( m(r, \alpha) = S(r, f) \) and so \( T(r, \alpha) = S(r, f) \). This shows that \( \alpha \) is a small function of \( f \). Therefore from (3.28) we have

\[
f'' = \frac{a'h_{23} - ah_{22} - ah'_{23}}{2ah_{23} + ah_{24}} f' + \frac{\alpha}{2ah_{23} + ah_{24}} f.
\]

Now from (3.12) and (3.29) we have

\[
a = q_1 f^2 + q_2 f f' + q_3 (f')^2,
\]

where

\[
q_1 = h_{21} - \frac{\beta}{2ah_{23} + ah_{24}}, \quad q_2 = h_{22} + \frac{a'h_{23} - ah_{22} - ah'_{23}}{2ah_{23} + ah_{24}} h_{24} \quad \text{and} \quad q_3 = h_{23}
\]

are small functions of \( f \). Also from (3.10) we see that

\[
q_3 = \frac{- \left( A'_1 + \frac{1}{a} - \frac{p'_2}{2n - 1} \right)}{2n - 1}.
\]

By Lemma 2.7, we have

\[
q_3(q_2^2 - 4q_1q_3)\frac{a'}{a} + q_2(q_2^2 - 4q_1q_3) - q_3(q_2^2 - 4q_1q_3)' + (q_2^2 - 4q_1q_3)q_3' \equiv 0.
\]

Let \( \delta = q_2^2 - 4q_1q_3 \). Clearly \( \delta \) is a small function of \( f \). Now we consider the following two sub-cases.

Sub-case 2.1. Suppose that \( \delta = q_2^2 - 4q_1q_3 \equiv 0 \). Then from (3.30) we have

\[
q_3 \left( f' + \frac{q_2}{2q_3} f \right)^2 = a.
\]

This shows that \( f' + q_2 f/(2q_3) \) is a small function of \( f \). Let \( b = f' + q_2 f/(2q_3) \). Since \( a \neq 0 \), it follows that \( b \neq 0 \). By substituting \( f' = b - q_2 f/(2q_3) \) into (3.3) and (3.4), we have, respectively,

\[
f^n \left( p_2a' + p'_2 + np_2 \frac{q_2}{2q_3} \right) - np_2bf^{n-1} + R_{1d} = A_1e^{\alpha_1}
\]

and

\[
f^n \left( p_1a' + p'_1 + np_1 \frac{q_2}{2q_3} \right) - np_1bf^{n-1} + R_{2d} = -A_1e^{\alpha_2},
\]

where \( R_{1d} = p_2P_d' - (p_2a'_2 + p'_2)P_d \) and \( R_{2d} = p_1P_d' - (p_1a'_1 + p'_1)P_d \).
Let
\[ \gamma_1 = p_2 \alpha'_2 + p'_2 + np_2 \frac{q_2}{2q_3}, \quad \gamma_2 = p_1 \alpha'_1 + p'_1 + np_1 \frac{q_2}{2q_3}. \]

First we suppose that \( \gamma_1 \equiv 0 \). Then using (3.31), we get
\[
\frac{p'_2}{p_2} + \alpha'_2 = \frac{n}{2n-1} \left( \alpha'_1 + \alpha'_2 + \frac{3 A'_1}{2 A_1} - \frac{1}{2} \frac{a'}{a} + \frac{1}{2} \frac{p'_2}{p_2} \right).
\]

Therefore by integrating, we get
\[
(p_2 e^{\alpha_2})^{2n-1} = c_6 A_1^{3n/2} p_2^{n/2} a^{n/2} e^{n(\alpha_1 + \alpha_2)},
\]
where \( c_6 \in \mathbb{C} \setminus \{0\} \). This shows that \( e^{(n-1)\alpha_2 - n\alpha_1} \) is a small function of \( f \). Next we suppose that \( \gamma_2 \equiv 0 \). Then using (3.31), we get
\[
\frac{p'_1}{p_1} + \alpha'_1 = \frac{n}{2n-1} \left( \alpha'_1 + \alpha'_2 + \frac{3 A'_1}{2 A_1} - \frac{1}{2} \frac{a'}{a} + \frac{1}{2} \frac{p'_2}{p_2} \right).
\]

Therefore by integrating, we get
\[
(p_1 e^{\alpha_1})^{2n-1} = c_7 A_1^{3n/2} p_2^{n/2} a^{n/2} e^{n(\alpha_1 + \alpha_2)},
\]
where \( c_7 \in \mathbb{C} \setminus \{0\} \). This shows that \( e^{(n-1)\alpha_1 - n\alpha_2} \) is a small function. Next we discuss the following four sub-cases.

**Sub-case 2.1.1.** Suppose that \( \gamma_1 \equiv 0 \) and \( \gamma_2 \equiv 0 \). Then both \( e^{(n-1)\alpha_2 - n\alpha_1} \) and \( e^{(n-1)\alpha_1 - n\alpha_2} \) are small functions of \( f \). Clearly \( e^{\alpha_1 + \alpha_2} \) is a small function of \( f \) and so \( e^{\alpha_2} = \varphi_4 e^{-\alpha_1} \), where \( \varphi_4 \) is a small function of \( f \). Now from (3.33) and (3.34) we have, respectively,
\[
-np_2 b f^{n-1} + R_{1d} = A_1 e^{\alpha_1}
\]
and
\[
-np_1 b f^{n-1} + R_{2d} = -A_1 \varphi_4 e^{-\alpha_1}.
\]

Eliminating \( e^{\alpha_1} \) and \( e^{-\alpha_1} \), from (3.35) and (3.36) we have
\[
f^{2n-3} (n^2 b^2 p_1 p_2 f) + R_{3d} = -A_1^2 \varphi_4,
\]
where \( R_{3d} = -np_2 b R_{2d} f^{n-1} - np_1 b R_{1d} f^{n-1} + R_{1d} R_{2d} \) is a differential polynomial in \( f \) of degree \( \leq 2n - 3 \) with small functions as its coefficients. Then by applying Lemma 2.1, we get from (3.37) that \( m(r, f) = S(r, f) \). Since \( N(r, \infty; f) = S(r, f) \), it follows that \( T(r, f) = S(r, f) \), which is impossible.
Sub-case 2.1.2. Suppose that $\gamma_1 \neq 0$ and $\gamma_2 \equiv 0$. Since $\gamma_2 \equiv 0$, we have that $e^{(n-1)\alpha_2-\alpha_1}$ is a small function of $f$ and so
\begin{equation}
(3.38) \quad e^{\alpha_2} = \varphi_5 e^{(n-1)\alpha_1/n}, \quad \text{where } \varphi_5 \in S(f).
\end{equation}

Now from (3.33) and Lemma 2.8, there exists a small function $v_1$ of $f$ such that
\begin{equation}
(3.39) \quad (f + v_1)^n = A_1 e^{\alpha_1}, \quad \text{i.e., } f = u_1 e^{\alpha_1/n} - v_1,
\end{equation}
where $u_1$ is a nonzero small function of $f$. Since $f$ has infinitely many zeros, it follows that $v_1 \neq 0$. Now from (3.1), (3.38) and (3.39) we have
\begin{equation*}
(u_1 e^{\alpha_1/n} - v_1)^n + P_d = p_1 e^{\alpha_1} + c_5 p_2 e^{(n-1)/n\alpha_1}.
\end{equation*}
Therefore by applying Lemma 2.4, we can conclude that $u_1^n(z) = p_1(z)$.

Sub-case 2.1.3. Suppose that $\gamma_1 \equiv 0$ and $\gamma_2 \neq 0$. Since $\gamma_1 \equiv 0$, we have that $e^{(n-1)\alpha_2-\alpha_1}$ is a small function of $f$ and so $e^{\alpha_1} = \varphi_6 e^{(n-1)/n\alpha_2}$, where $\varphi_6 \in S(f)$.

Now proceeding in the same way as in Sub-case 2.1.2, one can easily conclude that $f = u_2 e^{\alpha_2/n} - v_2$, where $u_2$ and $v_2$ are nonzero small functions of $f$ such that $u_2^n(z) = p_2(z)$.

Sub-case 2.1.4. Suppose that $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$. Now from (3.33) and (3.34) and Lemma 2.8, there exist two small functions $v_3$ and $v_4$ of $f$ such that
\begin{equation}
(f + v_3)^n = \frac{A_1}{\gamma_1} e^{\alpha_1} \quad \text{and} \quad (f + v_4)^n = -\frac{A_1}{\gamma_2} e^{\alpha_2}.
\end{equation}

From these we have, respectively,
\begin{equation}
(3.40) \quad f = u_3 e^{\alpha_1/n} - v_3 \quad \text{and} \quad f = u_4 e^{\alpha_2/n} - v_4,
\end{equation}
where $u_3^n = A_1/\gamma_1 \neq 0$ and $u_4^n = -A_1/\gamma_2 \neq 0$. Since $f$ has infinitely many zeros, it follows that $v_3 \neq 0$ and $v_4 \neq 0$.

First we suppose that $e^{\alpha_1-\alpha_2}$ is a small function of $f$. Then clearly $e^{\alpha_2} = \varphi_7 e^{\alpha_1}$, where $\varphi_7 \in S(f)$. Now from (3.1) we have
\begin{equation}
(3.41) \quad f^n + P_d = p_5 e^{\alpha_1},
\end{equation}
where $p_5 = p_1 + \varphi_7 p_2$. If $p_5 \equiv 0$, then from (3.41) we have $f^{n-1} f = -P_d$ and so by Lemma 2.1, we conclude that $m(r, f) = S(r, f)$. This shows that $T(r, f) = S(r, f)$, which is impossible. Next we suppose that $p_5 \neq 0$. Then by Lemma 2.9, we conclude that $f^n = p_5 e^{\alpha_1}$ and $P_d \equiv 0$. In this case we have $f(z) = q(z) e^{\alpha_1/n}$, where $q(z)$ is a nonzero small function of $f(z)$ such that $q^n(z) = p_1(z) + \varphi_7(z) p_2(z)$. 

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Next we suppose that $e^{\alpha_1 - \alpha_2}$ is not a small function of $f$. Note that $T(r, f) \leq T(r, e^{\alpha_1/n}) + S(r, f)$. Also

$$T(r, e^{\alpha_1/n}) \leq T(r, u_3 e^{\alpha_1/n}) + S(r, f) \leq T(r, u_3 e^{\alpha_1/n} - v_3) + S(r, f) = T(r, f) + S(r, f).$$

Combining these, we get $T(r, f) = T(r, u_3 e^{\alpha_1/n}) + S(r, f)$. Similarly, we have $T(r, f) = T(r, u_4 e^{\alpha_2/n}) + S(r, f)$. These show that $S(r, f) = S(r, u_3 e^{\alpha_1/n}) = S(r, u_4 e^{\alpha_2/n})$. Clearly $u_3, u_4, v_3$ and $v_4$ are small functions of both $e^{\alpha_1/n}$ and $e^{\alpha_2/n}$. On the other hand, from (3.40) we have

$$(3.42) \quad u_3 e^{\alpha_1/n} - u_4 e^{\alpha_2/n} = v_3 - v_4.$$ 

We claim that $v_3 \equiv v_4$. If not, suppose that $v_3 \neq v_4$. Now by Lemma 2.2, we get

$$T(r, f) = T(r, u_3 e^{\alpha_1/n}) + S(r, f) \leq N(r, 0; u_3 e^{\alpha_1/n}) + N(r, \infty; u_3 e^{\alpha_1/n}) + N(r, v_3 - v_4; u_3 e^{\alpha_1/n}) + S(r, u_3 e^{\alpha_1/n}) + S(r, f) = S(r, f),$$

which is a contradiction. Hence, $v_3 \equiv v_4$ and so from (3.42) we have

$$u_3 e^{\alpha_1/n} \equiv u_4 e^{\alpha_2/n}.$$ 

This shows that $e^{(\alpha_1 - \alpha_2)/n} = u_4/u_3$ and so $e^{\alpha_1 - \alpha_2} = (u_4/u_3)^n$. Consequently, $e^{\alpha_1 - \alpha_2}$ is a small function of $f$, which contradicts our assumption.

Sub-case 2.2. Suppose that $\delta = q_2^2 - 4q_1q_3 \neq 0$. Then from (3.32) we have

$$\frac{q_2}{q_3} \equiv \frac{\delta'}{\delta} - \frac{q_3'}{q_3} - \frac{a'}{a}.$$ 

Therefore from (3.10) and (3.31) we have

$$2(\alpha_1 + \alpha_2) \equiv (2n - 4) \frac{A_1'}{A_1} + (2n - 2) \frac{a'}{a} + (2n - 2) \frac{p_2'}{p_2} - (2n - 1) \frac{\delta'}{\delta}.$$ 

Integrating, we get

$$e^{2(\alpha_1 + \alpha_2)} = c_8 \frac{A_1^2 a^{2n-4} e^{2p_2n-2}}{\delta^{2n-1}},$$

where $c_8 \in \mathbb{C}$. This shows that $e^{\alpha_1 + \alpha_2}$ is a small function of $f$ and so $e^{\alpha_2} = \varphi_8 e^{-\alpha_1}$, where $\varphi_8 \in S(f)$. Now from (3.3) and (3.4), we have, respectively,

$$f^{n-1} (np_2f' - (p_2a_2' + p_2')f) + R_{1d} = A_1 e^{\alpha_1}$$

and

$$f^{n-1} (np_1f' - (p_1a_1' + p_1')f) + R_{2d} = -\varphi_8 A_1 e^{-\alpha_1}.$$ 

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Eliminating $e^{\alpha_1}$ and $e^{-\alpha_1}$, from (3.43) and (3.44) we have

$$f^{2n-2}(np_2f' - (p_2\alpha'_2 + p'_2)f)(np_1f' - (p_1\alpha'_1 + p'_1)f) + Q^*_d = -\varphi_8 A^2,$$

where

$$Q^*_d = f^{n-1}(np_2f' - (p_2\alpha'_2 + p'_2)f)R_{2d} + f^{n-1}(np_1f' - (p_1\alpha'_1 + p'_1)f)R_{1d} + R_{1d}R_{2d}$$

is a differential polynomial in $f$ of degree $\leq 2n - 2$ with small functions of $f$ as its coefficients. Now by Lemma 2.1, we conclude that $((p_1\alpha'_1 + p'_1)f - np_1f') \times ((p_2\alpha'_2 + p'_2)f - np_2f') = b_{11}$, where $b_{11}$ is a small function of $f$. If $b_{11} \equiv 0$, then we have either $(p_1\alpha'_1 + p'_1)f - np_1f' \equiv 0$ or $(p_2\alpha'_2 + p'_2)f - np_2f' \equiv 0$. Thus, in either case one can easily conclude that $N(r, 0; f) = S(r, f)$, which is impossible here. Hence $b_{11} \neq 0$. Therefore we can assume that

$$p_2\alpha'_2 + p'_2)f - np_2f' = b_1e^\gamma \quad \text{and} \quad (p_1\alpha'_1 + p'_1)f - np_1f' = b_2e^{-\gamma},$$

where $b_1, b_2$ are small functions of $f$ such that $b_1b_2 = b_{11}$ and $\gamma$ is an entire function. Since $f$ is of finite order, it follows that $\gamma$ is a polynomial.

First we suppose that $\gamma$ is a constant. Then from (3.46) we have

$$f' = \frac{1}{n}(\alpha'_2 + \frac{p'_2}{p_2})f - \frac{b_1e^\gamma}{np_2} \quad \text{and} \quad f' = \frac{1}{n}(\alpha'_1 + \frac{p'_1}{p_1})f - \frac{b_2e^{-\gamma}}{np_1}.$$ 

These imply that

$$\left(\alpha'_1 - \alpha'_2 + \frac{p'_1}{p_1} - \frac{p'_2}{p_2}\right)f = \frac{b_2e^{-\gamma}}{p_1} - \frac{b_1e^\gamma}{p_2}. \quad (3.47)$$

If $\alpha'_1 - \alpha'_2 + \frac{p'_1}{p_1} - \frac{p'_2}{p_2} \equiv 0$, then by integration, we have $e^{\alpha_1 - \alpha_2} = c_9p_2/p_1$, where $c_9 \in \mathbb{C} \setminus \{0\}$ and so $\alpha_1 - \alpha_2$ is a constant. Since $e^{\alpha_2} = \varphi_8 e^{-\alpha_1}$, it follows that $e^{\alpha_2}$ is a small function of $f$. Certainly $e^{\alpha_1}$ is also a small function of $f$. Now from (3.1) and Lemma 2.1, we conclude that $m(r, f) = S(r, f)$ and so $T(r, f) = S(r, f)$, which is impossible here. Therefore $\alpha'_1 - \alpha'_2 + \frac{p'_1}{p_1} - \frac{p'_2}{p_2} \neq 0$. Now from (3.47), it follows that $f$ is a small function of $f$, which is absurd.

Next we suppose that $\gamma$ is a non-constant polynomial. Now solving for $f$, we get from (3.46) that

$$(p_1p_2(\alpha'_2 - \alpha'_1) + p_1p'_2 - p'_1p_2)f = p_1b_1e^\gamma - p_2b_2e^{-\gamma}. \quad (3.48)$$

Using a similar argument, one can easily prove that $p_1p_2(\alpha'_2 - \alpha'_1) + p_1p'_2 - p'_1p_2 \neq 0$. Now from (3.48) we get $f(z) = \delta_1(z)e^{\gamma(z)} + \delta_2(z)e^{-\gamma(z)}$, where

$$\delta_1 = \frac{p_1b_1}{p_1p'_2 - p'_1p_2 - p_1p_2(\alpha'_1 - \alpha'_2)} \quad \text{and} \quad \delta_2 = \frac{-p_2b_2}{p_1p'_2 - p'_1p_2 - p_1p_2(\alpha'_1 - \alpha'_2)}.$$
Equation (3.46) can be rewritten as

\[(3.49) \quad A_2 f - np_2 f' = b_1 e^\gamma,\]

where \(A_2 = p_2 \alpha' + p'_2\). Differentiating (3.49) once, we get

\[(3.50) \quad A'_2 f + (A_2 - np_2') f' - np_2 f'' = (b'_1 + b_1 \gamma') e^\gamma.\]

Using (3.29), we get from (3.50) that

\[(3.51) \quad \left( A'_2 - \frac{1}{2n-1} \frac{\alpha}{A_1} \right) f + (A_2 - np_2' - \frac{1}{2n-1} (\alpha'_1 + \alpha'_2)p_2 - \frac{n(n-1) a'}{2n-1} A_1 p_2 + \frac{n(n-1) A'_1}{2n-1} p_2 + n\left( b'_1 + b_1 \gamma' \right) p_2) = (b'_1 + b_1 \gamma') e^\gamma.\]

Now from (3.10) and (3.51) we get

\[(3.52) \quad \left( A'_2 - \frac{1}{2n-1} \frac{\alpha}{A_1} \right) f + (A_2 - np_2' - \frac{1}{2n-1} (\alpha'_1 + \alpha'_2)p_2 - \frac{n(n-1) a'}{2n-1} A_1 p_2 + n\left( b'_1 + b_1 \gamma' \right) p_2) = (b'_1 + b_1 \gamma') e^\gamma.\]

Dividing (3.52) by (3.49), we get

\[(3.53) \quad \zeta_1 f + \zeta_2 f' \equiv 0,\]

where

\[\zeta_1 = A'_2 - \frac{1}{2n-1} \frac{\alpha}{A_1} - A_2 \left( \frac{b'_1}{b_1} + \gamma' \right)\]

and

\[\zeta_2 = A_2 - np_2' - \frac{1}{2n-1} (\alpha'_1 + \alpha'_2)p_2 - \frac{n(n-1) a'}{2n-1} A_1 p_2 + n\left( b'_1 + b_1 \gamma' \right) p_2.\]

Since \(ff' \neq 0\), it follows from (3.53) that either \(\zeta_1 \neq 0\) and \(\zeta_2 \neq 0\) or \(\zeta_1 \equiv 0\) and \(\zeta_2 \equiv 0\). First we suppose that \(\zeta_1 \neq 0\) and \(\zeta_2 \neq 0\). Then from (3.53), one can easily conclude that \(N(r,0;f) = S(r,f)\), which is a contradiction. Next we suppose that \(\zeta_1 \equiv 0\) and \(\zeta_2 \equiv 0\). Now \(\zeta_2 \equiv 0\) yields

\[\alpha'_2 - \frac{(n-1)^2 p'_2}{2n-1} \frac{1}{p_2} - \frac{1}{2n-1} (\alpha'_1 + \alpha'_2) - \frac{n(n-1) a'}{2n-1} A_1 - \frac{n(n-1) A'_1}{2n-1} A_1 + n\left( b'_1 + b_1 \gamma' \right) \equiv 0,\]

which implies that \(e^{(2n-1)(n\gamma + \alpha_2)} = c_{10} p_2^{(n-1)^2} e^{\alpha_1 + \alpha_2} (aA_1)^{n(n-1)} b_1^{-n}\), where \(c_{10} \in \mathbb{C} \setminus \{0\}\). Consequently, \(e^{n\gamma + \alpha_2}\) is a small function of \(f\). Therefore \(f(z) = \delta_1(z)e^{\gamma(z)} + \delta_2(z)e^{-\gamma(z)}\) and \(e^{(n\gamma + \alpha_2)}\) is a small function of \(f(z)\), where \(\delta_1(z), \delta_2(z)\) are nonzero small functions of \(f(z)\) and \(\gamma(z)\) is a non-constant polynomial such that either \(e^{\alpha_1 \gamma(z)} + \alpha_2(z)\) is a small function of \(f(z)\) or \(e^{n\gamma(z) + \alpha_1(z)}\) is a small function of \(f(z)\). \(\square\)

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4. An open problem

For further study, one may raise the following question as an open problem:

**Open Problem.** What will happen if we remove the condition $\varphi_2(f) < 1$ from Theorem 1.1?

**References**


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