SOME EXTENSIONS OF CHU'S FORMULAS AND FURTHER COMBINATORIAL IDENTITIES

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Abstract. We present some extensions of Chu's formulas and several striking generalizations of some well-known combinatorial identities. As applications, some new identities on binomial sums, harmonic numbers, and the generalized harmonic numbers are also derived.

Keywords: partial fraction decomposition; polynomial; combinatorial identity; harmonic number; generalized harmonic number; complete Bell polynomial

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1. Introduction

First, let us recall that the generalized harmonic numbers denoted by $H_n^{(r)}$ are defined to be partial sums of the Riemann zeta series:

$$H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r} \quad \text{for } n, r = 1, 2, \ldots \tag{1.1}$$

When $r = 1$, these numbers reduce to the classical harmonic numbers, shortened as $H_n = H_n^{(1)}$.

Secondly, we recall that the complete Bell polynomials can be explicitly expressed as in [9]

$$\mathbf{B}_n(x_1, x_2, \ldots, x_n) = \sum_{m_1+2m_2+\ldots+nm_n=n} \frac{n!}{m_1!\, m_2! \ldots m_n!} \Big(\frac{x_1}{1!}\Big)^{m_1} \Big(\frac{x_2}{2!}\Big)^{m_2} \cdots \Big(\frac{x_n}{n!}\Big)^{m_n}.$$

Combinatorial identities is a classical topic in combinatorics that have always been of great importance since Euler's era. In [21], Karatsuba indicated that combinatorial

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identities are used in several combinatorial problems, number theory, probability, the construction of computational algorithms, and mathematical physics. For some specific references of these applications see, for example, [1], [2], [7], [11], [14], [18], [19], [20], [24], [29], [30], [31], [33], [36].

There are various formulas and identities involving binomial coefficients. One of these combinatorial identities is

$$\frac{n!}{x(x+1)\dots(x+n)} = \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{x+j},$$

which appeared, for example, in [12], page 3, [14], equation (1.41) and [16], page 188. In recent years, there has been considerable interest in providing simple probabilistic proofs for this identity (see, for example, [26], [27], [32], [34]).

The second identity is the formula

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^n = n!.$$

In the literature, this identity is usually called the Boole formula because it appears in Boole's classical book (see [6]). Actually, it goes back to Euler, so Gould (see [15]) renamed it to Euler's formula.

In recent decades, Euler's formula has received a regain of interest, therefore several papers have been devoted to provide new proofs. The interested reader can consult [1], [3], [4], [5], [13], [17], [22], [28].

Involving complex numbers, Katsuura (in [22]) generalized Euler's formula as

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (b+aj)^l = \begin{cases} 0 & \text{if} \quad 0 \leqslant l < n, \\ (-1)^n a^n n! & \text{if} \quad l = n, \end{cases}$$

where a and b are two complex numbers.

Extending Katsuura's formula, Pohoata (see [28]) considered the following identity in terms of polynomials with real coefficients:

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} P(\alpha + \beta j) = \beta^n n!\, a_n,$$

where $P(x)$ is a polynomial of degree n with leading coefficient a_n.

In [8], among other results, Chu established for any two natural numbers λ and θ with $0 \leqslant \theta < \lambda(n+1)$ the partial fraction decompositions of the two rational functions

$$\frac{1}{x^\lambda (x+1)^\lambda \dots (x+n)^\lambda} \quad \text{and} \quad \frac{x^\theta}{x^\lambda (x+1)^\lambda \dots (x+n)^\lambda},$$

then he obtained several striking harmonic number identities and recovered a conjectured identity due to Weideman (see [35]):

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(3(H_k-H_{n-k})^2+(H_k^{(2)}+H_{n-k}^{(2)}))=0. \tag{1.2}$$

In [10], Driver and his collaborators confirmed this formula via computer algebra and symbolic calculus. It is important to note that Weideman (see [35]) declared that this formula is one of the hardest challenges among algebraic identities.

In [37], Zhu and Luo rewrote these two identities of Chu (see [8]) in another form as

$$\frac{1}{x^\lambda(x+1)^\lambda\dots(x+n)^\lambda}=\sum_{k=0}^{n}\frac{(-1)^{k\lambda}}{(n!)^\lambda}\binom{n}{k}^\lambda\sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_j(x_1,x_2,\dots,x_j)}{j!\,(x+k)^{\lambda-j}}, \tag{1.3}$$

and for $\lambda\leqslant M<\lambda(n+1)$,

$$\frac{x^M}{x^\lambda(x+1)^\lambda\dots(x+n)^\lambda}=\sum_{k=0}^{n}\frac{(-1)^{k\lambda+M}}{(n!)^\lambda}\binom{n}{k}^\lambda k^M\sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_j(x_1,x_2,\dots,x_j)}{j!\,(x+k)^{\lambda-j}}, \tag{1.4}$$

and gave a novel proof of these two main results of Chu (see [8]) using an appropriate contour integral and Cauchy's residue theorem.

Motivated by these results, our purpose is to establish the following general combinatorial identities which are a common generalization of these important works introduced before.

Let m and n be two positive integers. Let $P(x)=x^m(x+1)^m(x+2)^m\dots(x+n)^m$ and $Q(x)\in\mathbb{C}[x]$ be two polynomials such that $\deg(Q)<m(n+1)$. Then the following algebraic identity holds true:

$$\frac{(n!)^mQ(x)}{P(x)}=\sum_{j=0}^{n}(-1)^{jm}\binom{n}{j}^m\sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\frac{(-1)^k\mathbf{B}_k(x_1,x_2,\dots,x_k)Q^{(i)}(-j)}{i!\,k!\,(x+j)^{m-i-k}},$$

where

$$x_l=m(l-1)!\,(H_{n-j}^{(l)}+(-1)^lH_j^{(l)}).$$

Here and further, $Q^{(i)}(x)$ denotes the ith derivative of $Q(x)$.

In addition, if $Q(x)\in\mathbb{C}[x]$ is a polynomial of degree l with leading coefficient a_l, then we have the following identity:

$$\sum_{j=0}^{n}(-1)^{jm}\binom{n}{j}^m\sum_{i=0}^{m-1}\frac{(-1)^{m-1-i}\mathbf{B}_{m-1-i}(x_1,x_2,\dots,x_{m-1-i})Q^{(i)}(-j)}{i!\,(m-1-i)!}$$
$$=\begin{cases}0 & \text{if } 0\leqslant l<m(n+1)-1,\\ (n!)^ma_l & \text{if } l=m(n+1)-1.\end{cases}$$

Consequently, we obtain

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}^3\Big(\frac{Q^{(2)}(-j)}{2}-3Q^{'}(-j)(H_{n-j}-H_j)$$
$$+\frac{3Q(-j)}{2}(3(H_{n-j}-H_j)^2+(H_{n-j}^{(2)}+H_j^{(2)}))\Big)$$
$$=\begin{cases}0 & \text{if}\quad 0\leqslant l<3n+2,\\ (n!)^3a_l & \text{if}\quad l=3n+2.\end{cases}$$

Setting $Q(x)=1$, the last expression reduces to the conjectured identity of Weideman (see [35]):

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}^3(3(H_{n-j}-H_j)^2+(H_{n-j}^{(2)}+H_j^{(2)}))=0.$$

2. Preliminaries and the proof of the main identities

We first formulate the following important result.

Theorem 2.1. *Let $\alpha_1,\alpha_2,\dots,\alpha_s$ be distinct elements in the field of complex numbers $\mathbb{C}$. For a positive integer m, let $P(x)=(x-\alpha_1)^m(x-\alpha_2)^m\dots(x-\alpha_s)^m$. For any polynomial $Q(x)\in\mathbb{C}[x]$ with $\deg(Q)<\deg(P)$, we have*

$$\frac{Q(x)}{P(x)}=\sum_{j=1}^{s}\sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\frac{(-1)^k g_j(\alpha_j)\mathbf{B}_k(x_1,\dots,x_k)Q^{(i)}(\alpha_j)}{i!\,k!\,(x-\alpha_j)^{m-i-k}},\tag{2.1}$$

where

$$x_l=m(l-1)!\sum_{i=1,i\neq j}^{s}\frac{1}{(\alpha_j-\alpha_i)^l}\quad\textit{and}\quad g_j(x)=\prod_{i=1,i\neq j}^{s}(x-\alpha_i)^{-m_i}.$$

P r o o f. From [25], equation (4) we have

$$Q(x)=\sum_{j=1}^{s}\sum_{i=0}^{m-1}\frac{1}{i!}Q^{(i)}(\alpha_j)L_{ji}(x)[P],$$

where

$$L_{ji}(x)[P]=P_j(x)(x-\alpha_j)^i\sum_{k=0}^{m-1-i}\frac{1}{k!}g_j^{(k)}(\alpha_j)(x-\alpha_j)^k$$

and

$$P_j(x)=\prod_{i=1,i\neq j}^{s}(x-\alpha_i)^m=\frac{P(x)}{(x-\alpha_j)^m},\quad g_j(x)=(P_j(x))^{-1}.$$

As a consequence, we obtain the identity

$$L_{ji}(x)[P] = P(x) \sum_{k=0}^{m-1-i} \frac{g_j^{(k)}(\alpha_j)}{k!\,(x-\alpha_j)^{m-i-k}}.$$

Therefore, by combining these identities, we can write

$$\frac{Q(x)}{P(x)} = \sum_{j=1}^{s} \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \frac{g_j^{(k)}(\alpha_j) Q^{(i)}(\alpha_j)}{i!\,k!\,(x-\alpha_j)^{m-i-k}}.$$

On the other hand, we have

$$g_j(x) = \varphi(x) \circ f_j(x),$$

where $\varphi(x) = \exp(mx)$ and $f_j(x) = \ln\Big(\prod_{i=1, i\neq j}^{s} (x-\alpha_i)^{-1}\Big)$. It is clear that $\varphi^{(k)}(x) = m^k \exp(mx)$ and $f_j^{(k)}(x) = (-1)^k (k-1)!\,\mathcal{H}_{k,\alpha_s[j]}(x)$, where $\mathcal{H}_{l,\alpha_s[j]}(x) = \sum_{i=1,i\neq j}^{s} 1/(x-\alpha_i)^l$. Now by applying the Faà di Bruno formula [23], equation (1.13), we get

$$g_j^{(k)}(x) = (-1)^k g_j(x) \sum_{m_1+2m_2+\ldots+km_k=k} \frac{k!}{m_1!\,m_2!\,\ldots m_k!} \prod_{l=1}^{k} \Big(\frac{m(l-1)!\,\mathcal{H}_{l,\alpha_s[j]}(x)}{l!}\Big)^{m_l}.$$

It follows that

$$g_j^{(k)}(\alpha_j) = (-1)^k g_j(\alpha_j) \mathbf{B}_k(x_1, \ldots, x_k).$$

Therefore, the rest follows easily. □

Theorem 2.1 has the following corollary.

Corollary 2.1. *Let m and n be two positive integers. Let $P(x) = x^m \times (x-1)^m \ldots (x-n)^m$ and $Q(x) \in \mathbb{C}[x]$ be two polynomials such that $\deg(Q) < \deg(P)$. We have*

$$\frac{(n!)^m Q(x)}{P(x)} = \sum_{j=0}^{n} \binom{n}{j}^m \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \frac{(-1)^{m(n-j)+k} \mathbf{B}_k(x_1, x_2, \ldots, x_k) Q^{(i)}(j)}{i!\,k!\,(x-j)^{m-i-k}},$$

where

$$x_l = m(l-1)!\,(H_j^{(l)} + (-1)^l H_{n-j}^{(l)}).$$

In particular, we get

$$\frac{(n!)^m x^l}{P(x)} = \sum_{j=0}^{n} \binom{n}{j}^m \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \binom{l}{i} j^{l-i} \frac{(-1)^{m(n-j)+k} \mathbf{B}_k(x_1, x_2, \ldots, x_k)}{k!\,(x-j)^{m-i-k}},$$

where $1 \leqslant l < m(n+1)$, *and*

$$\frac{(n!)^m}{P(x)} = \sum_{j=0}^{n} \binom{n}{j}^m \sum_{k=0}^{m-1} \frac{(-1)^{m(n-j)+k} \mathbf{B}_k(x_1, x_2, \ldots, x_k)}{k!\,(x-j)^{m-k}}.$$

According to the expression of Theorem 2.1, we can easily obtain the following result.

Corollary 2.2. *Let* m *and* n *be two positive integers. Let* $P(x) = x^m (x+1)^m \times (x+2)^m \ldots (x+n)^m$ *and* $Q(x) \in \mathbb{C}[x]$ *be two polynomials such that* $\deg(Q) < m(n+1)$. *The following algebraic identity holds true:*

$$\frac{(n!)^m Q(x)}{P(x)} = \sum_{j=0}^{n} (-1)^{jm} \binom{n}{j}^m \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \frac{(-1)^k \mathbf{B}_k(x_1, x_2, \ldots, x_k) Q^{(i)}(-j)}{i!\,k!\,(x+j)^{m-i-k}}, \tag{2.2}$$

where

$$x_l = m(l-1)!\,(H_{n-j}^{(l)} + (-1)^l H_j^{(l)}).$$

By multiplying both sides of (2.2) by x and letting x to ∞, we obtain the following result.

Theorem 2.2. *Let* m *and* n *be two positive integers. Let* $Q(x) \in \mathbb{C}[x]$ *be a polynomial of degree* l *with leading coefficient* a_l. *Then we have the following identity:*

$$\sum_{j=0}^{n} (-1)^{jm} \binom{n}{j}^m \sum_{i=0}^{m-1} \frac{(-1)^{m-1-i} \mathbf{B}_{m-1-i}(x_1, x_2, \ldots, x_{m-1-i}) Q^{(i)}(-j)}{i!\,(m-1-i)!} = \begin{cases} 0 & \text{if } 0 \leqslant l < m(n+1)-1, \\ (n!)^m a_l & \text{if } l = m(n+1)-1, \end{cases}$$

where

$$x_l = m(l-1)!\,(H_{n-j}^{(l)} + (-1)^l H_j^{(l)}).$$

Setting $m = 1, 2, 3$ in Theorem 2.2, we gain the following identities.

Corollary 2.3. *Let n be a positive integer and $Q(x) \in \mathbb{C}[x]$ be a polynomial of degree l with leading coefficient a_l. Then we have*

(a) *for $m = 1$*

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} Q(-j) = \begin{cases} 0 & \text{if } 0 \leqslant l < n, \\ n!\, a_l & \text{if } l = n, \end{cases} \tag{2.3}$$

(b) *for $m = 2$*

$$\sum_{j=0}^{n} \binom{n}{j}^2 (Q'(-j) - 2(H_{n-j} - H_j)Q(-j)) = \begin{cases} 0 & \text{if } 0 \leqslant l < 2n+1, \\ (n!)^2 a_l & \text{if } l = 2n+1, \end{cases} \tag{2.4}$$

(c) *for $m = 3$*

$$\begin{aligned} \sum_{j=0}^{n} (-1)^j \binom{n}{j}^3 \Big(& \frac{Q^{(2)}(-j)}{2} - 3Q'(-j)(H_{n-j} - H_j) \\ & + \frac{3Q(-j)}{2} (3(H_{n-j} - H_j)^2 + (H_{n-j}^{(2)} + H_j^{(2)})) \Big) \\ & = \begin{cases} 0 & \text{if } 0 \leqslant l < 3n+2, \\ (n!)^3 a_l & \text{if } l = 3n+2. \end{cases} \end{aligned} \tag{2.5}$$

Remark 2.1. In the following, we derive Euler's formula, Katsuura's formula, and Pohoata's formula.

- ▷ When $Q(x) = x^n$, identity (2.3) gives Euler's formula.
- ▷ When $Q(x) = (b - ax)^l$, identity (2.3) reduces to Katsuura's formula.
- ▷ Setting $Q(x) = P(\alpha - \beta x)$ in identity (2.3), we obtain Pohoata's formula.

The following example is an illustration of (2.4).

Example 2.1. Choose $Q(x) = x$ and $Q(x) = x^{2n+1}$ in (2.4), we derive

$$\sum_{j=0}^{n} \binom{n}{j}^2 (1 + 2j(H_{n-j} - H_j)) = 0$$

and

$$\sum_{j=0}^{n} \binom{n}{j}^2 ((2n+1)j^{2n} + 2j^{2n+1}(H_{n-j} - H_j)) = (n!)^2,$$

respectively.

E x a m p l e 2.2. By choosing special polynomials in Corollary 2.3, we obtain interesting identities.

▷ Setting $Q(x)=1$, the expression of (2.5) becomes

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}^3(3(H_{n-j}-H_j)^2+(H_{n-j}^{(2)}+H_j^{(2)}))=0.$$

The last identity is declared as one of the hardest challenges among identities. It is conjectured by [35], equation (20) and proved in [10] by means of symbolic calculus and computer algebra package Sigma.

▷ When $Q(x)=x$, the formula of (2.5) reduces to the identity

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}^3(2(H_{n-j}-H_j)+j(3(H_{n-j}-H_j)^2+(H_{n-j}^{(2)}+H_j^{(2)})))=0.$$

When we set $Q(x)=1$ and $Q(x)=x^\theta$, where θ is a positive integer, in the formula (2.2), we can easily reformulate the two algebraic identities appeared in the work of Chu (see [8]), anticipated at the beginning of this paper, as follows.

Corollary 2.4. *Let m, n and θ be three positive integers with $0\leqslant\theta<m(n+1)$. Then*

$$\begin{aligned}(2.6)\qquad &\frac{(n!)^m x^\theta}{x^m(x+1)^m\dots(x+n)^m}\\ &\qquad=\sum_{j=0}^{n}(-1)^{jm}\binom{n}{j}^m\sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{\theta}{i}j^{\theta-i}\frac{(-1)^{k+\theta-i}\mathbf{B}_k(x_1,x_2,\dots,x_k)}{k!\,(x+j)^{m-i-k}},\end{aligned}$$

where

$$x_l=m(l-1)!\,(H_{n-j}^{(l)}+(-1)^lH_j^{(l)}).$$

In particular, we have

$$\frac{(n!)^m}{x^m(x+1)^m\dots(x+n)^m}=\sum_{j=0}^{n}(-1)^{jm}\binom{n}{j}^m\sum_{k=0}^{m-1}\frac{(-1)^k\mathbf{B}_k(x_1,x_2,\dots,x_k)}{k!\,(x+j)^{m-k}}. \tag{2.7}$$

According to (2.6), we can provide a list of identities, for example the following two examples.

E x a m p l e 2.3. Let n and θ be two positive integers with $\theta < n+1$. Then

$$\frac{n!\,x^{\theta}}{x(x+1)\dots(x+n)} = \sum_{j=0}^{n}\binom{n}{j}(-1)^{j+\theta}\frac{j^{\theta}}{(x+j)}.$$

The last identity appeared in the work of Chu (see [8], Example 1).

E x a m p l e 2.4. Let n and θ be two positive integers with $\theta < 2(n+1)$. Then

$$\frac{(n!)^2x^{\theta}}{x^2(x+1)^2\dots(x+n)^2} = \sum_{j=0}^{n}\binom{n}{j}^2\Big(\frac{(-1)^{\theta}j^{\theta}}{(x+j)^2} - (-1)^{\theta}\frac{j^{\theta}(H_{n-j}-H_j)+\theta j^{\theta-1}}{(x+j)}\Big)$$

and the corresponding harmonic number identity is

$$\sum_{j=0}^{n} j^{\theta-1}\binom{n}{j}^2(\theta-2j(H_j-H_{n-j})) = \begin{cases} 0 & \text{if } 0\leqslant\theta<2n+1,\\ (n!)^2 & \text{if } \theta=2n+1.\end{cases}$$

We note that the last formula has been conjectured by Weideman in [35], equation (11) and proved in [10], Theorem 1 and recovered by Chu in [8], Example 2.

According to (2.7), we can obtain several expansion expressions involving the generalized harmonic numbers, for example for $m=2,3$, one obtains the following two identities.

R e m a r k 2.2.

$$\frac{(n!)^2}{x^2(x+1)^2\dots(x+n)^2} = \sum_{j=0}^{n}\binom{n}{j}^2\Big(\frac{1}{(x+j)^2} - \frac{2}{(x+j)}(H_{n-j}-H_j)\Big),$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}^3\Big(\frac{1}{(x+j)^3} - \frac{3(H_{n-j}-H_j)}{(x+j)^2} + \frac{9(H_{n-j}-H_j)^2+3(H_{n-j}^{(2)}+H_j^{(2)})}{(x+j)}\Big) = \frac{(n!)^3}{x^3(x+1)^3\dots(x+n)^3}.$$

Corollary 2.5. *Let n be a positive integer. Let $Q(x)\in\mathbb{C}[x]$ be a polynomial such that* $\deg(Q)<n+1$. *We have*

$$\frac{n!\,Q(x)}{x(x+1)\dots(x+n)} = \sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^jQ(-j)}{(x+j)}. \tag{2.8}$$

When $Q(x) = x^l$, $l = 0, 1, \ldots, n$, formula (2.8) reduces to

$$\frac{n!\,x^l}{x(x+1)\ldots(x+n)} = \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^{j+l} j^l}{(x+j)}.$$

As a consequence, we recover the well-known identity (see, for example, [12], page 3, [14], equation (1.41), and [16], page 188):

$$\frac{n!}{x(x+1)\ldots(x+n)} = \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{x+j}.$$

3. Conclusion

In this concluding section, we encourage the interested reader to develop the results of this paper and examine other important algebraic identities.

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