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REMARK ON REGULARITY CRITERION FOR WEAK SOLUTIONS TO THE SHEAR THINNING FLUIDS

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Abstract. J. Q. Yang (2019) established a regularity criterion for the 3D shear thinning fluids in the whole space $\mathbb{R}^3$ via two velocity components. The goal of this short note is to extend this result in viewpoint of Lorentz space.

Keywords: shear thinning fluids; regularity criterion

MSC 2020: 76D05, 35Q35

1. Introduction

In this paper, we study the shear thinning fluids:

$$\begin{cases} u_t - \nabla\cdot(|Du|^{p-2}Du) + (u\cdot\nabla)u + \nabla\pi = f, \\ \operatorname{div} u = 0 \end{cases} \quad \text{in } Q_T := \mathbb{R}^3\times(0,T), \tag{1.1}$$

where $u = (u_1, u_2, u_3)\colon \mathbb{R}^3\times(0,T) \to \mathbb{R}^3$ is the flow velocity vector, $\pi\colon \mathbb{R}^3\times(0,T)\to\mathbb{R}$ is the pressure. Also, $Du = \frac{1}{2}(\nabla u + \nabla u^T)$. We consider the initial value problem of (1.1), which requires initial conditions

$$u(x,0) = u_0(x), \quad x\in\mathbb{R}^3. \tag{1.2}$$

We assume that the initial data $u_0(x)\in L^2(\mathbb{R}^3)$ hold the incompressibility, i.e., $\operatorname{div} u_0(x) = 0$.

The non-Newtonian flows is a fluid such that the relation between the shear stress and the shear strain rate is non-linear. Typical examples are as follows: honey, blood, paint, melted butter and corn starch (see e.g. [2], [5], [14]).

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We report shortly some known results related to the existence of solutions and regularity issues. The existence of weak solutions was firstly shown for $\frac{11}{5} \leqslant q$ in [9]–[11], and later, the result was improved up to $\frac{6}{5} < q$ in [7]. Here, weak solutions are meant to solve the equations in the sense of distributions and satisfy

$$\sup_{0\leqslant t\leqslant T} \|u(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^T \|\nabla u(t)\|_{L^q(\mathbb{R}^3)}^q \,\mathrm{d}t \leqslant C\|u_0\|_{L^2(\mathbb{R}^3)}^2.$$

Málek et al. proved in [13] that a strong solution exists globally in time in periodic domains for $q \geqslant \frac{11}{5}$ in $\mathbb{R}^3$ (see [17] for the whole space case). Here by strong solutions we mean solutions satisfying the following energy estimate:

$$\sup_{0\leqslant t\leqslant T} \|u(t)\|_{H^1(\mathbb{R}^3)}^2 + \int_0^T \int_{\mathbb{R}^3} |Du|^{q-2}|\nabla Du|^2 \leqslant C\|u_0\|_{H^1(\mathbb{R}^3)}^2.$$

Also, they established local existence of strong solution in time for $q > \frac{5}{3}$ in three dimensional periodic domains (refer to [4] for shear thinning case, $\frac{7}{5} < q < 2$).

When $\frac{8}{5} < p \leqslant 2$, in particular, for the regularity issue for (1.1), Bae et al. (see [3]) obtained the following regularity of weak solutions of (1.1)–(1.2) in the class

$$u \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)), \quad \frac{3}{\alpha} + \frac{5p-6}{2\beta} \leqslant \frac{5p-8}{2}, \quad \alpha > \frac{6}{5p-8}. \tag{1.3}$$

On the other hand, Yang in [19] improved regularity condition (1.3) by only the two-component velocity field. More precisely, for $\widetilde{u} = (u_1, u_2, 0)$ he proved the regularity of weak solutions in the class

$$\widetilde{u} \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)), \quad \frac{3}{\alpha} + \frac{5p-6}{2\beta} \leqslant \frac{5p-8}{2}, \quad \alpha > \frac{6}{5p-8}. \tag{1.4}$$

For $\alpha = \infty$, Ahmad et al. (see [1]) improved Yang's result (1.4). More precisely, for $\widetilde{u} = (u_1, u_2, 0)$ he proved the regularity of weak solutions in the class

$$\widetilde{u} \in L^{(5p-6)/(5p-8)}(0,T;\mathrm{BMO}(\mathbb{R}^3)).$$

In this direction, our main result is read as follows:

Theorem 1.1. *Suppose that u is a weak solution to* (1.1)–(1.2) *with* $\frac{8}{5} < p \leqslant 2$. *Then there exists a constant $\delta > 0$ such that u is a strong solution on $(0,T]$ provided that $u \in L^{\beta,\infty}(0,T;L^{\alpha,\infty}(\mathbb{R}^3))$ and*

$$\|u\|_{L^{\beta,\infty}(0,T;L^{\alpha,\infty}(\mathbb{R}^3))} \leqslant \delta \quad \textit{with } \frac{3}{\alpha} + \frac{5p-6}{2\beta} \leqslant \frac{5p-8}{2}, \ \alpha > \frac{6}{5p-8}.$$

Theorem 1.2. *Suppose that u is a weak solution to* (1.1)–(1.2) *with* $\frac{8}{5} < p \leqslant 2$. *Then there exists a constant $\delta > 0$ such that u is a strong solution on $(0, T]$ provided that $\widetilde{u} \in L^{\beta,\infty}(0, T; L^{\alpha,\infty}(\mathbb{R}^3))$ and*

$$\|\widetilde{u}\|_{L^{\beta,\infty}(0,T;L^{\alpha,\infty}(\mathbb{R}^3))} \leqslant \delta \quad \textit{with}\ \frac{3}{\alpha} + \frac{5p-6}{2\beta} \leqslant \frac{5p-8}{2},\ \alpha > \frac{6}{5p-8}.$$

R e m a r k 1.3. For a half space $\mathbb{R}^3_+$ with the slip boundary condition instead of $\mathbb{R}^3$, our results are also established due to Sobolev embedding (see e.g. [8], pages 215–216).

2. Preliminaries

In this subsection we introduce the notation. For $1 \leqslant q \leqslant \infty$, we denote by $W^{k,q}(\Omega)$ the usual Sobolev spaces, namely $W^{k,q}(\Omega) = \{f \in L^q(\Omega)\colon D^\alpha f \in L^q(\Omega),\ 0 \leqslant |\alpha| \leqslant k\}$. The set of qth power Lebesgue integrable functions on Ω is denoted by $L^q(\Omega)$ and $L^q_{\rm loc}(\Omega)$ indicates the set of locally qth power Lebesgue integrable functions defined on Ω. Let $\mathcal{O} \subset \Omega$ and $J \subset I$, we denote $\|f\|_{L^{p,q}_{x,t}(\mathcal{O}\times J)} = \|\|f\|_{L^p(\mathcal{O})}\|_{L^q(J)}$. For vector fields u, v we write $(u_i v_j)_{i,j=1,2,3}$ as $u \otimes v$. We denote by $A : B = a_{ij} b_{ij}$ the 3×3 matrices $A = (a_{ij})$, $B = (b_{ij})$. The letter C is used to represent a generic constant, which may change from line to line. Before the proof, we will list some lemmas needed for the proof. First of all, we recall Korn's inequality in [17], Lemma 2.7.

Lemma 2.1. *Let $1 < q < \infty$. Assume that u is in $W^{1,q}(\mathbb{R}^3)$. Then*

$$\|\nabla u\|_{L^q(\mathbb{R}^3)} \leqslant C \|Du\|_{L^q(\mathbb{R}^3)},$$

where C is a positive constant depending on q.

We recall a key estimate for the stress tensor in [3] or [19].

Lemma 2.2. *Let $1 < p < 2$. Suppose that $u \in W^{2,s}(\mathbb{R}^3)$ for $1 < s < 2$. Then*

$$\int_{\mathbb{R}^3} |\nabla^2 u|^s \, \mathrm{d}x \leqslant C \bigg(\int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \nabla(|D(u)|^{p-2} D_{i,j}(u)) \cdot \nabla D_{i,j}(u) \, \mathrm{d}x \bigg)^{s/2} \times \bigg(\int_{\mathbb{R}^3} |\nabla u|^{(2-p)s/(2-s)} \, \mathrm{d}x \bigg)^{(2-s)/2}.$$

For $s=p$, in particular, we have

$$\int_{\mathbb{R}^3} |\nabla^2 u|^p \,\mathrm{d}x \leqslant \varepsilon \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \nabla(|D(u)|^{p-2} D_{i,j}(u)) \cdot \nabla D_{i,j}(u) \,\mathrm{d}x + C \int_{\mathbb{R}^3} |\nabla u|^p \,\mathrm{d}x.$$

Next, we present some basic facts on Lorentz spaces $L^{p,q}(\Omega)$. For $p, q \in [1,\infty]$ we define

$$L^{p,q}(\mathbb{R}^3) = \{f\colon\ f \text{ is a measurable function on } \mathbb{R}^3 \text{ and } \|f\|_{L^{p,q}(\mathbb{R}^3)} < \infty\}$$

with

$$\|f\|_{L^{p,q}(\mathbb{R}^3)} = \begin{cases} \left(p\displaystyle\int_0^\infty \alpha^q |\{x \in \mathbb{R}^3\colon\ |f(x)| > \alpha\}|^{q/p} \frac{\mathrm{d}\alpha}{\alpha}\right)^{1/q}, & q < \infty, \\ \sup\limits_{\alpha > 0} \alpha |\{x \in \mathbb{R}^3\colon\ |f(x)| > \alpha\}|^{1/p}, & q = \infty. \end{cases}$$

And also, followed [18], Lorentz space may be defined by real interpolation methods as

$$L^{p,q}(\mathbb{R}^3) = (L^{p_1}(\mathbb{R}^3), L^{p_2}(\mathbb{R}^3))_{\alpha,q}$$

with

$$\frac{1}{p} = \frac{1-\alpha}{p_1} + \frac{\alpha}{p_2}, \quad 1 \leqslant p_1 < p < p_2 \leqslant \infty.$$

We also remind the Hölder inequality in Lorentz spaces (see [15]).

Lemma 2.3. *Assume $1 \leqslant p_1,\ p_2 \leqslant \infty$, $1 \leqslant q_1,\ q_2 \leqslant \infty$ and $u \in L^{p_1,q_1}(\mathbb{R}^3)$, $v \in L^{p_2,q_2}(\mathbb{R}^3)$. Then $uv \in L^{p_3,q_3}(\mathbb{R}^3)$ with $1/p_3 = 1/p_1 + 1/p_2$ and $1/q_3 \leqslant 1/q_1 + 1/q_2$, and the inequality*

$$\|uv\|_{L^{p_3,q_3}(\mathbb{R}^3)} \leqslant C\|u\|_{L^{p_1,q_1}(\mathbb{R}^3)} \|v\|_{L^{p_2,q_2}(\mathbb{R}^3)}$$

is valid.

We recall the following useful Gronwall lemma required in our proof which was first shown by [6] (see e.g. [16], [12]).

Lemma 2.4. *Let φ be a measurable positive function defined on the interval $[0,T]$. Suppose that there exists $\kappa_0 > 0$ such that for all $0 < \kappa < \kappa_0$ and a.e. $t \in [0,T]$, φ satisfies the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi \leqslant \mu \lambda^{1-\kappa} \varphi^{1+2\kappa},$$

where $0 < \lambda \in L^{1,\infty}(0,T)$ and $\mu > 0$ with $\mu\|\lambda\|_{L^{1,\infty}(0,T)} < \frac{1}{2}$. Then φ is bounded on $[0,T]$.

3. Regularity criterion

Proof of Theorem 1.1. Testing $-\Delta u$ to the fluid equation of (1.1), respectively, and using integration by parts, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\nabla u(\tau)\|^2_{L^2_x(\mathbb{R}^3)} + \int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\nabla(|D(u)|^{p-2}D_{i,j}(u))\cdot\nabla D_{i,j}(u)\,\mathrm{d}x = \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\,\mathrm{d}x := A. \tag{3.1}$$

Now, the first term A is computed as follows. Since $\alpha > 6/(5p-8) > 3p/(4p-6) > p/(p-1)$, we have

$$\begin{aligned}
|A| &\leqslant \|u\|_{L^{\alpha,\infty}}\|\nabla u\|_{L^{p\alpha/(p\alpha-p-\alpha),p/(p-1)}}\|\nabla^2 u\|_{L^{p,6}}\\
&\leqslant \|u\|_{L^{\alpha,\infty}}\|\nabla u\|_{L^2}^{(8p\alpha-12\alpha-6p)/(5p-6)\alpha}\|\nabla u\|_{L^{3p/(3-p)}}^{3(2p+2\alpha-p\alpha)/(5p-6)\alpha}\|\nabla^2 u\|_{L^p}\\
&\leqslant \|u\|_{L^{\alpha,\infty}}\|\nabla u\|_{L^2}^{(8p\alpha-12\alpha-6p)/(5p-6)\alpha}\|\nabla^2 u\|_{L^p}^{(2p\alpha+6p)/(5p-6)\alpha}\\
&\leqslant \|u\|_{L^{\alpha,\infty}}^{\alpha(5p-6)/(5p\alpha-8\alpha-6)}\|\nabla u\|_{L^2}^{(8p\alpha-12\alpha-6p)/(5p\alpha-8\alpha-6)} + \varepsilon\|\nabla^2 u\|^p_{L^p}.
\end{aligned}$$

Due to Lemma 2.2, the nonlinear term A is estimated by

$$\begin{aligned}
|A| &\leqslant \|u\|_{L^{\alpha,\infty}}^{\alpha(5p-6)/(5p\alpha-8\alpha-6)}\|\nabla u\|_{L^2}^{(8p\alpha-12\alpha-6p)/(5p\alpha-8\alpha-6)}\\
&\quad + \varepsilon_1\varepsilon\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\nabla(|D(u)|^{p-2}D_{i,j}(u))\cdot\nabla D_{i,j}(u)\,\mathrm{d}x + \varepsilon_1 C\int_{\mathbb{R}^3}|\nabla u|^p\,\mathrm{d}x.
\end{aligned} \tag{3.2}$$

With the energy estimate and (3.2), (3.1) becomes

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}(\|u(t)\|^2_{L^2_x(\mathbb{R}^3)} + \|\nabla u(t)\|^2_{L^2_x(\mathbb{R}^3)})\\
&\qquad + \int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\nabla(|D(u)|^{p-2}D_{i,j}(u))\cdot\nabla D_{i,j}(u) + |\nabla u|^p\,\mathrm{d}x\\
&\qquad \leqslant \|u\|_{L^{\alpha,\infty}}^{\alpha(5p-6)/(5p\alpha-8\alpha-6)}\|\nabla u\|_{L^2}^{(8p\alpha-12\alpha-6p)/(5p\alpha-8\alpha-6)}\\
&\qquad \leqslant \|u\|_{L^{\alpha,\infty}}^{\alpha(5p-6)/(5p\alpha-8\alpha-6)}\\
&\qquad\quad \times \frac{1}{(1+\|\nabla u\|^2_{L^2})^{(4p\alpha-6\alpha-3p)/(5p\alpha-8\alpha-6)-1}}(1+\|\nabla u\|^2_{L^2})
\end{aligned}$$

$$\|u\|_{L^{\alpha,\infty}}^{\alpha(5p-6)/(5p\alpha-8\alpha-6)}(1+\|\nabla u\|^2_{L^2}). \tag{3.3}$$

Let $Y(t) := 1 + \|u(t)\|^2_{L^2_x(\mathbb{R}^3)} + \|\nabla u(t)\|^2_{L^2_x(\mathbb{R}^3)}$ and thus (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{dt}}Y(t) \leqslant C\|u\|_{L^{\alpha,\infty}}^{\alpha(5p-6)/(5p\alpha-8\alpha-6)}Y(t). \tag{3.4}$$

We use an argument similar to the one used in the work of Bosia et al.: For $\varepsilon > 0$, let $\beta_\varepsilon = \beta + 2p\varepsilon((5p-9)\beta - (5p-6))/(5p-6)$ and $(\alpha_\varepsilon, \beta_\varepsilon)$ be a Prodi-Serrin pair. We have

$$\|u\|^{\beta_\varepsilon}_{L^{\alpha_\varepsilon,\infty}(\mathbb{R}^3)} \leqslant \|u\|^{\beta(1-\varepsilon)}_{L^{\alpha,\infty}(\mathbb{R}^3)} \|u\|^{2p\varepsilon}_{L^{6,\infty}} \leqslant C\|u\|^{\beta(1-\varepsilon)}_{L^{\alpha,\infty}(\mathbb{R}^3)} \|\nabla u\|^{2p\varepsilon}_{L^2(\mathbb{R}^3)}, \tag{3.5}$$

where we use

$$\|f\|_{L^{p,q_2}(\mathbb{R}^3)} \leqslant \Big(\frac{q_1}{p}\Big)^{1/q_1 - 1/q_2} \|f\|_{L^{p,q_1}(\mathbb{R}^3)}, \quad 1 \leqslant p \leqslant \infty,\ 1 \leqslant q_1 < q_2 \leqslant \infty.$$

Since the pair $(\alpha_\varepsilon, \beta_\varepsilon)$ also meets $2/\alpha_\varepsilon + 3/\beta_\varepsilon = 2$, using (3.5), (3.4) becomes

$$\frac{\mathrm{d}}{\mathrm{dt}} Y(t) \leqslant C\|u\|^{\beta_\varepsilon}_{L^{\alpha_\varepsilon,\infty}(\mathbb{R}^3)} Y(t) \leqslant C\|u\|^{\beta(1-\varepsilon)}_{L^{\alpha,\infty}(\mathbb{R}^3)} Y(t)^{1+2\varepsilon}.$$

Applying Lemma 2.4, we obtain the desired result. □

P r o o f of Theorem 1.2. The proof is almost the same as that for Theorem 1.1. Indeed, from the proof of Theorem 1.1, convection term A is decomposed with three parts as follows:

$$\begin{aligned} -\sum_{i,j=1}^{3} \int_{\mathbb{R}^3} u_i \partial_i u_j \partial^2_{kk} u_j \,\mathrm{d}x &= \sum_{i=1}^{2} \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j \,\mathrm{d}x \\ &\quad + \sum_{j=1}^{2} \sum_{k=1}^{3} \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_j \partial_k u_j \,\mathrm{d}x \\ &\quad + \sum_{k=1}^{3} \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_3 \partial_k u_3 \,\mathrm{d}x \\ &:= A_1 + A_2 + A_3. \end{aligned}$$

For A_1 and A_2, by integration by parts, we have

$$\begin{aligned} A_1 &= -\sum_{i=1}^{2} \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} u_i \partial_i u_j \partial^2_{kk} u_j \,\mathrm{d}x - \sum_{i=1}^{2} \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} u_i \partial_k \partial_i u_j \partial_k u_j \,\mathrm{d}x, \\ A_2 &= -\sum_{i=1}^{2} \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} u_j \partial_3 u_j \partial^2_{kk} u_3 \,\mathrm{d}x - \sum_{i=1}^{2} \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} u_j \partial_3 \partial_k u_j \partial_k u_3 \,\mathrm{d}x. \end{aligned}$$

And thus, we know

$$|A_1|, |A_2| \leqslant C \int_{\mathbb{R}^3} |(\widetilde{u} \cdot \nabla u) : \nabla^2 u| \,\mathrm{d}x. \tag{3.6}$$

For A_3, using the divergence free condition, we get

$$\begin{aligned} A_3 &= -\sum_{k=1}^{3}\int_{\mathbb{R}^3} \partial_k u_3(\partial_1 u_1 + \partial_2 u_2)\partial_k u_3 \,\mathrm{d}x \\ &= -\sum_{k=1}^{3}\int_{\mathbb{R}^3} \partial_k u_3 \partial_1 u_1 \partial_k u_3 \,\mathrm{d}x - \sum_{k=1}^{3}\int_{\mathbb{R}^3} \partial_k u_3 \partial_2 u_2 \partial_k u_3 \,\mathrm{d}x \\ &= -\sum_{k=1}^{3}\int_{\mathbb{R}^3} u_1 \partial_1 \partial_k u_3 \partial_k u_3 \,\mathrm{d}x - \sum_{k=1}^{3}\int_{\mathbb{R}^3} u_1 \partial_k u_3 \partial_1 \partial_k u_3 \,\mathrm{d}x \\ &\quad - \sum_{k=1}^{3}\int_{\mathbb{R}^3} u_2 \partial_2 \partial_k u_3 \partial_k u_3 \,\mathrm{d}x - \sum_{k=1}^{3}\int_{\mathbb{R}^3} u_2 \partial_k u_3 \partial_2 \partial_k u_3 \,\mathrm{d}x \end{aligned}$$

and thus, we have

$$|A_3| \leqslant C\int_{\mathbb{R}^3} |(\widetilde{u}\cdot\nabla u) : \nabla^2 u| \,\mathrm{d}x. \tag{3.7}$$

Through (3.6) and (3.7), we can see that

$$|A| \leqslant C\int_{\mathbb{R}^3} |(u\cdot\nabla u) : \nabla^2 u| \,\mathrm{d}x.$$

Hence, with the same approach as in Theorem 1.1, we finally obtain the result in Theorem 1.2. The proof is complete. □

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