THE LATTICE OF IDEALS OF A NUMERICAL SEMIGROUP AND ITS FROBENIUS RESTRICTED VARIETY ASSOCIATED

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Abstract. Let Δ be a numerical semigroup. In this work we show that $\mathcal{J}(\Delta)=\{I\cup\{0\}\colon I$ is an ideal of $\Delta\}$ is a distributive lattice, which in addition is a Frobenius restricted variety. We give an algorithm which allows us to compute the set $\mathcal{J}_a(\Delta)=\{S\in\mathcal{J}(\Delta)\colon \max(\Delta\setminus S)=a\}$ for a given $a\in\Delta$. As a consequence, we obtain another algorithm that computes all the elements of $\mathcal{J}(\Delta)$ with a fixed genus.

Keywords: numerical semigroup; ideal; Frobenius restricted variety; embedding dimension; Frobenius number; restricted Frobenius number; genus; multiplicity; Arf numerical semigroup; saturated semigroup

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1. Introduction

Let $\mathbb{Z}$ be the set of integer numbers and $\mathbb{N}=\{x\in\mathbb{Z}\colon x\geqslant 0\}$. A *numerical semigroup* is a subset S of $\mathbb{N}$, which is closed by the sum, $0\in S$ and $\mathbb{N}\setminus S=\{x\in\mathbb{N}\colon x\notin S\}$ is finite.

If A is a nonempty subset of $\mathbb{N}$, we denote by $\langle A\rangle$ the submonoid of $(\mathbb{N},+)$ generated by A, that is, $\langle A\rangle=\{\lambda_1a_1+\ldots+\lambda_na_n\colon n\in\mathbb{N}\setminus\{0\},\{a_1,\ldots,a_n\}\subseteq A$ and $\{\lambda_1,\ldots,\lambda_n\}\subseteq\mathbb{N}\}$. By Lemma 2.1 from [12], we know that $\langle A\rangle$ is a numerical semigroup if and only if $\gcd(A)=1$.

If S is a numerical semigroup and $S=\langle A\rangle$, then we say that A is a *system of generators* of S. Moreover, if $S\neq\langle B\rangle$ for all $B\supsetneq A$, then we say that A is a *minimal*

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system of generators of S. In [12], Corollary 2.8, it is shown that every numerical semigroup has a unique minimal system of generators which, in addition, is finite. We denote by $\mathrm{msg}(S)$ the minimal system of generators of S. The cardinality of $\mathrm{msg}(S)$ is called the *embedding dimension* of S and is denoted by ed(S).

If S is a numerical semigroup, then $\mathrm{F}(S) = \max(\mathbb{Z} \setminus S)$, $\mathrm{g}(S) = \sharp(\mathbb{N} \setminus S)$, where $\sharp A$ denotes the cardinality of set A, and $\mathrm{m}(S) = \min(S \setminus \{0\})$. They are three important invariants of S which we call *Frobenius number*, *genus* and *multiplicity* of S, respectively.

Let Δ be a numerical semigroup. An ideal of Δ is a nonempty subset I of Δ such that $I + \Delta = \{a + b\colon\ a \in I \text{ and } b \in \Delta\} \subseteq I$.

If I is an ideal of Δ, then $I \cup \{0\}$ is a numerical semigroup. This fact induces us to give the following definition. A numerical semigroup S is an I(Δ)-*semigroup* if $S \setminus \{0\}$ is an ideal of Δ. We denote $\mathcal{J}(\Delta) = \{S\colon\ S \text{ is an I}(\Delta)\text{-semigroup}\}$. The main aim of this manuscript is to study the set $\mathcal{J}(\Delta)$.

In Section 2, we recall some basic notions and results of the theory of ideals of numerical semigroups. In Section 3, we show that $\mathcal{J}(\Delta)$ is closed under union and intersection, and so it is a distributive lattice. Moreover, we show that if $S \in \mathcal{J}(\Delta)$ and $x = \max(\Delta \setminus S)$, then $S \cup \{x\} \in \mathcal{J}(\Delta)$ and consequentely, we have that $\mathcal{J}(\Delta)$ is a *Frobenius restricted variety.*

We say that an ideal I of a numerical semigroup Δ is *principal* if there exists $a \in \Delta$ such that $I = \{a\} + \Delta$. A P(Δ)-*semigroup* is a numerical semigroup with the form $(\{a\} + \Delta) \cup \{0\}$ and $a \in \Delta$. In Section 4, we illustrate that every $I(\Delta)$-semigroup can be expressed as a finite and irredundant union of P(Δ)-semigroups.

By Proposition 2.10 from [12], we know that if S is a numerical semigroup, then $\mathrm{ed(S)} \leqslant \mathrm{m(S)}$. A MED-*semigroup* is a numerical semigroup S such that $\mathrm{ed(S)} = \mathrm{m(S)}$. This class of numerical semigroups has been widely studied, see for instance [3]. In Section 4, we show that if S is an P(Δ)-semigroup and $S \neq \Delta$, then S is a MED-semigroup.

Inspired by [1], Lipman introduces and motivates in [6] the study of Arf rings. The characterization of these rings via their value semigroups yields the notion of *Arf numerical semigroup.* Every Arf numerical semigroup is a MED-semigroup. In Section 4, we show that Δ is an Arf numerical semigroup if and only if every P(Δ)-semigroup is an Arf numerical semigroup.

A particularly interesting type of numerical semigroups are called saturated numerical semigroups. The idea of saturation of singularities were introduced in three different ways by: Zariski in [13]–[15], Pham-Teissier in [9], and Campillo in [4]. As for the Arf property, saturated numerical semigroups come into scene after a characterization of saturated rings in terms of their value semigroups (see [5], [8]). In

Section 4, we show that a numerical semigroup Δ is satured if and only if there is at least one $\mathrm{P}(\Delta)$-satured semigroup.

Let Δ be a numerical semigroup. We say that an ideal is *irreducible* if it cannot be expressed as the intersection of two ideals properly containing it. If $a \in \Delta$, then we denote $\mathrm{B}(a) = \{s \in \Delta\colon\ a - s \in \Delta\}$. As a consequence of Lemma 3.1 from [2] we have that I is an irreducible ideal of Δ if and only if $I = \Delta$ or $I = \Delta \setminus \mathrm{B}(a)$ for some $a \in \Delta$. A $\mathrm{D}(\Delta)$-*semigroup* is a numerical semigroup with the form $(\Delta \setminus \mathrm{B}(a)) \cup \{0\}$ for some $a \in \Delta$. As a consequence of Theorem 3.3 from [2], we have that every $\mathrm{I}(\Delta)$-semigroup can be expressed as a unique, finite and irredundant intersection of $\mathrm{D}(\Delta)$-semigroups.

If $S \subsetneq T$ are numerical semigroups, then the *Frobenius number of S restricted* to T is $\mathrm{F}_T(S) = \max(T \setminus S)$. If Δ is a numerical semigroup and $a \in \Delta$, then we put $\mathcal{J}_a(\Delta) = \{S\colon\ S \text{ is an } \mathrm{I}(\Delta)\text{-semigroup and } \mathrm{F}_\Delta(S) = a\}$. In Section 5, we order the elements of the set $\mathcal{J}_a(\Delta)$ in the form of a tree with root $(\Delta \setminus \mathrm{B}(a)) \cup \{0\}$. This fact allows us in Section 6 to give an algorithm which computes all the elements of the set $\mathcal{J}_a(\Delta)$. Finally and based on the previous algorithm, we show another one, of different nature and with complexity not comparable to the algorithm presented in [7], that allows us to compute the set $\mathcal{J}(\Delta, k) = \{S\colon\ S \text{ is an } \mathrm{I}(\Delta)\text{-semigroup and } \mathrm{g}(\Delta) = \mathrm{g}(\Delta) + k\}$ for all $k \in \mathbb{N}$.

2. Basic concepts and results

Let Δ be a numerical semigroup. An *ideal* of Δ is a nonempty subset I of Δ such that $I + \Delta \subseteq I$. The following result has an easy proof.

Proposition 2.1. *If I and J are ideals of a numerical semigroup Δ, then $I \cup J$ and $I \cap J$ are also ideals of Δ.*

It is clear that if I is an ideal of Δ, then $\Delta \setminus I$ is finite. Therefore, if $I \neq \Delta$, then there exists $\max(\Delta \setminus I)$.

Proposition 2.2. *Let Δ be a numerical semigroup, let I be an ideal of Δ such that $I \neq \Delta$ and $x = \max(\Delta \setminus I)$. Then $I \cup \{x\}$ is an ideal of Δ.*

P r o o f. By maximility of x, we have $\{x\} + \Delta \subseteq I \cup \{x\}$. Therefore, $(I \cup \{x\}) + \Delta \subseteq I \cup \{x\}$. □

The following result is Proposition 1 from [7].

Proposition 2.3. *If Δ is a numerical semigroup and X is a nonempty subset of Δ, then $X + \Delta$ is an ideal of Δ. Moreover, every ideal of Δ has this form.*

If Δ is a numerical semigroup, then we define over $\mathbb{Z}$ the following order relation: $a \leqslant_\Delta b$ if and only if $b - a \in \Delta$. We say that a nonempty subset X of Δ is a *Δ-incomparable set* if $a - b \notin \Delta$ for all $(a, b) \in X \times X$ such that $a \neq b$. The following result is Theorem 5 from [7].

Theorem 2.4. *Let Δ be a numerical semigroup. Then the set*

$$\{X + \Delta \colon \ X \text{ is a } \Delta\text{-incomparable set}\}$$

is the set formed by all the ideals of S. Moreover, if X and Y are different Δ-incomparable sets, then $X + \Delta \neq Y + \Delta$.

If I is an ideal of a numerical semigroup Δ and $I = X + \Delta$, then we say that X is an *ideal system of generators* of I. Moreover, if X is a Δ-incomparable set, then we say that X is the *ideal minimal system of generators* of I. By Theorem 2.4, we know that every ideal I of Δ admits a unique ideal minimal system of generators. We denote this system by $\mathrm{imsg}_\Delta(I)$.

The following result is Proposition 2.6 from [7].

Proposition 2.5. *Let Δ be a numerical semigroup and let I be an ideal of Δ. Then* $\mathrm{imsg}_\Delta(I) = \mathrm{Minimals}_{\leqslant_\Delta}(\mathrm{I})$.

The following result is Proposition 7 from [7].

Proposition 2.6. *If Δ is a numerical semigroup and let X be a Δ-incomparable set, then X is finite.*

As a consequence of Propositions 2.5 and 2.6, the cardinal of $\mathrm{imsg}_\Delta(I)$ is an integer positive number. This number is called the *ideal dimension* of I in Δ and it is denoted by $\dim_\Delta(I)$.

The following result is Proposition 8 from [7].

Proposition 2.7. *If I is an ideal of Δ, then:*
(1) *$I = \Delta$ if and only if $0 \in I$,*
(2) *$I \cup \{0\}$ is a numerical semigroup.*

The following result is Proposition 9 from [7].

Proposition 2.8. *Let Δ be a numerical semigroup and let I be an ideal of Δ such that $I \neq \Delta$. Then* $\mathrm{imsg}_\Delta(I) = \mathrm{Minimals}_{\leqslant_\Delta}(\mathrm{msg}(\mathrm{I} \cup \{0\}))$.

As an immediate consequence of Proposition 2.8, we have the following result.

Corollary 2.9. *Let Δ be a numerical semigroup and let I be an ideal of Δ. Then* $\dim_\Delta(I) \leqslant \mathrm{ed}(\mathrm{I} \cup \{0\})$.

It is well known that if Δ is a numerical semigroup and $x \in \Delta$, then $\Delta \setminus \{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(\Delta)$.

The following result is easy to prove.

Proposition 2.10. *If I is an ideal of Δ and $x \in I$, then $I \setminus \{x\}$ is an ideal of Δ if and only if $x \in \mathrm{imsg}_\Delta(I)$.*

3. I(Δ)-semigroups

Let Δ be a numerical semigroup. By Proposition 2.7, we know that if I is an ideal of Δ, then $I \cup \{0\}$ is a numerical semigroup. An I(Δ)-*semigroup* is a numerical semigroup S such that $S \setminus \{0\}$ is an ideal of Δ. We put $\mathcal{J}(\Delta) = \{S \colon S$ is an I(Δ)-semigroup$\}$.

Example 3.1. It is clear that X is an $\mathbb{N}$-incomparable set if and only if $X = \{n\}$ for every $n \in \mathbb{N}$. Hence, by applying Theorem 2.4, $\mathcal{J}(\mathbb{N}) = \{\{0, n, \rightarrow\} \colon n \in \mathbb{N}\}$ (the symbol $\rightarrow$ means that every integer greater than n belongs to the set). The numerical semigroups with the form $\{0, n, \rightarrow\}$ are called *ordinary numerical semigroups*. So the concepts of I($\mathbb{N}$)-semigroup and ordinary numerical semigroup are equivalent.

If S and T are numerical semigroups and $S \subseteq T$, the *Frobenius number of S restricted* to T is $\mathrm{F}_T(S) = \max(T \setminus S)$. By definition $\mathrm{F}_T(T) = -1$.

By applying Propositions 2.1 and 2.2, we can easily deduce the following result.

Theorem 3.2. *Let Δ be a numerical semigroup. Then:*

(1) *If $\{S, T\} \subseteq \mathcal{J}(\Delta)$, then $\{S \cup T, S \cap T\} \subseteq \mathcal{J}(\Delta)$.*
(2) *Δ is the maximum element (with respect to set inclusion) of $\mathcal{J}(\Delta)$.*
(3) *If $S \in \mathcal{J}(\Delta)$ and $S \neq \Delta$, then $S \cup \{F_\Delta(S)\} \in \mathcal{J}(\Delta)$.*

A *lattice* is an algebraic structure $(L, \vee, \wedge)$ consisting of a set L and two binary operations $\vee$ and $\wedge$ over L satisfying the properties: commutative, associative, idempotent and absorption. If, in addition, it verifies the distributive property, then the lattice is called *distributive*. As an immediate consequence of Theorem 3.2, we have the following result.

Corollary 3.3. *If Δ is a numerical semigroup, then $(\mathcal{J}(\Delta), \cup, \cap)$ is a distributive lattice.*

A *Frobenius restricted variety* (see [10]) is a nonempty family $\mathcal{F}$ of numerical semigroups verifying the following conditions:

(1) $\mathcal{F}$ has a maximum element (and we denote it $\Delta(\mathcal{F})$).
(2) If $\{S, T\} \subseteq \mathcal{F}$, then $S \cap T \in \mathcal{F}$.
(3) If $S \in \mathcal{F}$ and $S \neq \Delta(\mathcal{F})$, then $S \cup \{\mathrm{F}_{\Delta(\mathcal{F})}(S)\} \in \mathcal{F}$.

As an immediate consequence of Theorem 3.2, we have the following result.

Corollary 3.4. *If Δ is a numerical semigroup, then $\mathcal{J}(\Delta)$ is a Frobenius restricted variety.*

4. P(Δ)-semigroups

In the rest of this work Δ denotes a numerical semigroup. An ideal I of Δ is *principal* if $\dim_\Delta(I) = 1$. So the set formed by all the principal ideals of Δ is $\{\{a\} + \Delta \colon a \in \Delta\}$.

Proposition 4.1. *If I is an ideal of Δ, then the next conditions are equivalent:*
(1) *I is a principal ideal.*
(2) *I cannot be expressed as the union of two ideals of Δ strictly contained in I.*

Proof. (1) $\Rightarrow$ (2): Let J and K be ideals of Δ such that $J \subseteq I$, $K \subseteq I$ and $I = J \cup K$. As I is a principal ideal of Δ, then there exits $a \in \Delta$ such that $I = \{a\} + \Delta$. Then $a \in I = J \cup K$ and hence $a \in J$ or $a \in K$. If $a \in J$, then $I = \{a\} + \Delta \subseteq J + \Delta \subseteq J$ and so $I = J$.

(2) $\Rightarrow$ (1): If I is not a principal ideal of Δ, then $\dim_\Delta(I) = n \geqslant 2$. Therefore, there exists $\{a_1, a_2, \ldots, a_n\}$ a Δ-incomparable set such that $\{a_1, \ldots, a_n\} + \Delta = I$. Let $J = \{a_1\} + \Delta$ and $K = \{a_2, \ldots, a_n\} + \Delta$. Then J and K are ideals of Δ such that $J \subseteq I$, $K \subseteq I$ and $I = J \cup K$. Moreover, applying that $\{a_1, a_2, \ldots, a_n\}$ is a Δ-incomparable set, we deduce that $J \subsetneq I$ and $K \subsetneq I$. □

A P(Δ)-*semigroup* is a numerical semigroup with the shape $(\{a\} + \Delta) \cup \{0\}$ for some $a \in \Delta$. We put $\mathcal{P}(\Delta) = \{S \colon S$ is a P(Δ)-semigroup$\}$.

Proposition 4.2. *Let Δ be a numerical semigroup.*
(1) *If $\{S_1, S_2, \ldots, S_n\} \subseteq \mathcal{P}(\Delta)$, then $S_1 \cup S_2 \cup \ldots \cup S_n \in \mathcal{J}(\Delta)$.*
(2) *If $S \in \mathcal{J}(\Delta)$ and $\dim_\Delta(S \setminus \{0\}) = n$, then there exists $\{S_1, S_2, \ldots, S_n\} \subseteq \mathcal{P}(\Delta)$ such that $S = S_1 \cup S_2 \cup \ldots \cup S_n$.*

Proof. (1) It is a consequence from Theorem 3.2.

(2) If $\dim_\Delta(S \setminus \{0\}) = n$, then there exists $\{x_1, x_2, \ldots, x_n\} \subseteq \Delta$ such that $S \setminus \{0\} = \{x_1, \ldots, x_n\} + \Delta$. For every $i \in \{1, \ldots, n\}$, let $S_i = (\{x_i\} + \Delta) \cup \{0\}$. It is clear that $S_i \in \mathcal{P}(\Delta)$ for all $i \in \{1, \ldots, n\}$ and $S = S_1 \cup \ldots \cup S_n$. □

We say that a union $\bigcup_{i\in\{1,\ldots,n\}} A_i$ of the sets A_i is *irredundant* if for every $j \in \{1,\ldots,n\}$, it is verified that $\bigcup_{i\in\{1,\ldots,n\}} A_i \neq \bigcup_{i\in\{1,\ldots,n\}\setminus\{j\}} A_i$. The following result has an easy proof.

Proposition 4.3. *Every* $\mathrm{I}(\Delta)$*-semigroup can be expressed in a unique way as a finite and irredundant union of* $\mathrm{P}(\Delta)$*-semigroups.*

The following result is deduced from [11], Proposition 2.

Proposition 4.4. *If* S *is a* $\mathrm{P}(\Delta)$*-semigroup and* $S \neq \Delta$*, then* S *is a* MED-*semigroup.*

The following result can be easily deduced from [11], Proposition 9.

Proposition 4.5. *If* $\Delta \neq \mathbb{N}$, $a \in \Delta\setminus\{0\}$ *and* $S = (\{a\}+\Delta)\cup\{0\}$*, then* $\mathrm{F}(S) = a + \mathrm{F}(\Delta)$, $\mathrm{g}(S) = a - 1 + \mathrm{g}(\Delta)$ *and* $\mathrm{m}(S) = a$.

As an immediate consequence of the previous proposition we have the following result.

Corollary 4.6. *If* $\{S,T\} \subseteq \mathcal{P}(\Delta)$*, then the following conditions are equivalent:*
(1) $S = T$,
(2) $\mathrm{m}(S) = \mathrm{m}(T)$,
(3) $\mathrm{F}(S) = \mathrm{F}(T)$,
(4) $\mathrm{g}(S) = \mathrm{g}(T)$.

Note that as a consequence of Proposition 4.5 and Corollary 4.6, the number of elements of $\mathcal{P}(\Delta)$ with Frobenius number F, genus g, multiplicity m, respectively, is 1 or 0 depending on whether there exists $a \in \Delta$ such that $\mathrm{F} = \mathrm{F}(\Delta)+a$, $\mathrm{g} = \mathrm{g}(\Delta)+a-1$, $\mathrm{m} = a$, respectively.

A numerical semigroup S is *Arf* if $x+y-z \in S$ for every $x,y,z \in S$ such that $z \leqslant y \leqslant x$. If S is an Arf numerical semigroup, then by [12], Proposition 3.12, we can deduce that S is a MED-semigroup.

The following result follows from [11], Corollary 38.

Proposition 4.7. Δ *is an Arf numerical semigroup if and only if all the elements of the set* $\mathcal{P}(\Delta)$ *are Arf numerical semigroups.*

If $A \subseteq \mathbb{N}$ and $a \in A\setminus\{0\}$, then we denote $\mathrm{d}_A(a) = \gcd\{x \in A\colon\ x \leqslant a\}$. A numerical semigroup is *saturated* if $s + \mathrm{d}_S(s) \in S$ for all $s \in S\setminus\{0\}$.

By Lemma 3.31 from [12], we know that every saturated numerical semigroup is an Arf numerical semigroup. The following result is deduced from [11], Corollary 43.

Proposition 4.8. *Δ is a saturated numerical semigroup if and only if $\mathcal{P}(\Delta)\setminus\{\Delta\}$ contains at least a saturated numerical semigroup.*

5. $\mathrm{D}(\Delta)$-semigroups

Let Δ be a numerical semigroup. An ideal is *irreducible* if it cannot be expressed as the intersection of two ideals properly containing it. If $a \in \Delta$, then we denote $\mathrm{B}(a) = \{s \in \Delta\colon\ a - s \in \Delta\}$. The following result follows from [2], Lemma 3.1.

Proposition 5.1. *I is an irreducible ideal of Δ if and only if $I = \Delta \setminus \mathrm{B}(a)$ for some $a \in \Delta$ or $I = \Delta$.*

A $\mathrm{D}(\Delta)$-*semigroup* is a numerical semigroup with the form $(\Delta\setminus\mathrm{B}(a))\cup\{0\}$ for some $a \in \Delta$. We put $\mathcal{D}(\Delta) = \{S\colon\ S \text{ is a } \mathrm{D}(\Delta)\text{-semigroup}\}$. We say that an intersection $\bigcap\limits_{i\in\{1,\dots,n\}} A_i$ of the sets A_i is *irredundant* if $\bigcap\limits_{i\in\{1,\dots,n\}} A_i \neq \bigcap\limits_{i\in\{1,\dots,n\}\setminus\{j\}} A_i$ for every $j \in \{1,\dots,n\}$.

The following result is deduced from [2], Theorem 3.3.

Proposition 5.2. *Every $\mathrm{I}(\Delta)$-semigroup can be expressed as a unique finite and irredundant intersection of $\mathrm{D}(\Delta)$-semigroups.*

If Δ is a numerical semigroup and $a \in \Delta$, then we put $\mathrm{S}(\Delta, a) = (\Delta \setminus \mathrm{B}(a)) \cup \{0\} \in \mathcal{D}(\Delta)$.

The following result has an easy proof.

Proposition 5.3. *If Δ is a numerical semigroup and $a \in \Delta \setminus \{0\}$, then $\mathrm{F}_\Delta(\mathrm{S}(\Delta, a)) = a$ and $\mathrm{g}(\mathrm{S}(\Delta, a)) = \mathrm{g}(\Delta) + \#\mathrm{B}(a) - 1$.*

Remark 5.4.

- ▷ Observe that $\mathrm{S}(\Delta, 0) = \Delta$ and so $\mathrm{F}_\Delta(\mathrm{S}(\Delta, 0)) = \mathrm{F}_\Delta(\Delta) = -1$. Therefore, $\{\mathrm{F}_\Delta(\mathrm{S}(\Delta, a))\colon\ a \in \Delta\} = (\Delta \setminus \{0\}) \cup \{-1\}$.
- ▷ We propose the study of the set $\{\#\mathrm{B}(a)\colon\ a \in \Delta\}$ as an open problem.

Theorem 5.5. *Let S be a numerical semigroup. Then S is a $\mathrm{D}(\Delta)$-semigroup if and only if S is a maximal element (with respect to set inclusion) of the set $\{T\colon$ is an $\mathrm{I}(\Delta)$-semigroup and $\mathrm{F}_\Delta(T) = \mathrm{F}_\Delta(S)\}$.*

Proof. *Necessity.* If S is not maximal, then there exists an $\mathrm{I}(\Delta)$-semigroup T such that $S \subsetneq T$ and $\mathrm{F}_\Delta(T) = \mathrm{F}_\Delta(S)$. By Theorem 3.2, we know that $S \cup \{\mathrm{F}_\Delta(S)\}$ is an $\mathrm{I}(\Delta)$-semigroup. Then $S = (S \cup \{\mathrm{F}_\Delta(S)\}) \cap T$ and so we have been able to write S as an intersection of two $\mathrm{I}(\Delta)$-semigroups properly containing S. Hence, S is not a $\mathrm{D}(\Delta)$-semigroup.

Sufficiency. Let $T = (\Delta \setminus \mathrm{B}(\mathrm{F}_\Delta(S))) \cup \{0\}$. It is clear that T is an $\mathrm{I}(\Delta)$-semigroup, $S \subseteq T$ and $\mathrm{F}_\Delta(T) = \mathrm{F}_\Delta(S)$. By applying the maximility of S, we obtain that $S = T$. Therefore, S is a $\mathrm{D}(\Delta)$-semigroup. □

As a consequence of Propositions 5.1 and 5.3, we deduce the following result.

Proposition 5.6. *If S is an $\mathrm{I}(\Delta)$-semigroup, then there exists a unique $\mathrm{D}(\Delta)$-semigroup T such that $S \subseteq T$ and $\mathrm{F}_\Delta(T) = \mathrm{F}_\Delta(S)$. Moreover, $T = \mathrm{S}(\Delta, \mathrm{F}_\Delta(S))$ if $S \neq \Delta$ and $T = \Delta$ if $S = \Delta$.*

If Δ is a numerical semigroup and $a \in \Delta \setminus \{0\}$, then we put $\mathcal{J}_a(\Delta) = \{S\colon S$ is an $\mathrm{I}(\Delta)$-semigroup and $\mathrm{F}_\Delta(S) = a\}$.

Proposition 5.7. *Let S be an $\mathrm{I}(\Delta)$-semigroup such that $S \neq \Delta$ and $\mathrm{F}_\Delta(S) = a$. Then $S = \mathrm{S}(\Delta, a)$ if and only if $\{h \in \Delta \setminus S\colon\ h \notin \mathrm{B}(a)\} = \emptyset$.*

Proof. If $S = \mathrm{S}(\Delta, a)$, then $S = (\Delta \setminus \mathrm{B}(a)) \cup \{0\}$ and so $\{h \in \Delta \setminus S\colon\ h \notin \mathrm{B}(a)\} = \emptyset$. Conversely, if $\{h \in \Delta \setminus S\colon\ h \notin \mathrm{B}(a)\} = \emptyset$, then it is clear that $S = \mathrm{S}(\Delta, a)$. □

If $S \in \mathcal{J}_a(\Delta)$ and $S \neq S(\Delta, a)$, then we put $\alpha(S) = \max\{h \in \Delta \setminus S\colon\ h \notin \mathrm{B}(a)\}$. By definition, $\alpha((S, a)) = 0$.

Proposition 5.8. *If $a \in \Delta$ and $S \in \mathcal{J}_a(\Delta)$, then $S \cup \{\alpha(S)\} \in \mathcal{J}_a(\Delta)$.*

Proof. By the maximality of $\alpha(S)$, we deduce that $\{\alpha(S)\} + \Delta \subseteq S \cup \{\alpha(S)\}$. From this result one easily deduces that $S \cup \{\alpha(S)\} \in \mathcal{J}_a(\Delta)$. □

If $S \in \mathcal{J}_a(\Delta)$, then the previous proposition can be used to define recursively the following sequence of elements of $\mathcal{J}_a(\Delta)$:

▷ $S_0 = S$,

▷ $S_{n+1} = S_n \cup \{\alpha(S_n)\}$ for all $n \in \mathbb{N}$.

As a consequence of Propositions 5.7 and 5.8, we have the following result.

Proposition 5.9. *If $a \in \Delta$ and $S \in \mathcal{J}_a(\Delta)$, then there is $p \in \mathbb{N}$ such that $S = S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_p = S(\Delta, a)$.*

A *graph* G is a pair (V, E) where V is a nonempty set and $E \subseteq \{(u, v) \in V \times V\colon u \neq v\}$. The elements of V and E are called *vertices* and *edges*, respectively. A *path of length* n connecting the vertices x and y of graph G is a sequence of different edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ such that $v_0 = x$ and $v_n = y$.

A graph is *a tree* if $G = (V, E)$, where there exists a vertex r (known as *the root* of G) such that for any other vertex x of G there exists a unique path connecting x and r. If (u, v) is an edge of a tree, then we say that u is a *child* of v.

If $a \in \Delta$, then we define the graph $G(\Delta, a)$ as follows: $\mathcal{J}_a(\Delta)$ is its set of vertices and $(S, T) \in \mathcal{J}_a(\Delta) \times \mathcal{J}_a(\Delta)$ is an edge if $T = S \cup \{\alpha(S)\}$ and $S \neq T$.

The following result is deduced from Proposition 5.9.

Theorem 5.10. *If $a \in \Delta$, then $G(\Delta, a)$ is a tree with root* $\mathrm{S}(\Delta, a)$.

A tree can be recurrently built starting from its root and connecting the vertex already built with its children, and this would be done with the help of an edge. Therefore, it is very interesting to know who are the children of an arbitrary vertex of a tree.

The following result is deduced from Proposition 2.10.

Lemma 5.11. *Let S be a* $\mathrm{I}(\Delta)$*-semigroup and $x \in S$. Then $S \setminus \{x\}$ is an* $\mathrm{I}(\Delta)$*-semigroup if and only if $x \in \mathrm{imsg}_\Delta(S \setminus \{0\})$.*

Proposition 5.12. *Let $a \in \Delta$ and $S \in \mathcal{J}_a(\Delta)$. Then the set formed by the children of S, in the tree $G(\Delta, a)$, is the set $\{S \setminus \{x\} \colon x \in \mathrm{imsg}_\Delta(S \setminus \{0\})$ and $\alpha(S) < x < a\}$.*

P r o o f. We have that: If T is a child of S, then $S = T \cup \{\alpha(T)\}$ and so $T = S \setminus \{\alpha(T)\}$. By applying Lemma 5.11, we have that $\alpha(T) \in \mathrm{imsg}_\Delta(S)$. And applying now the definition of $\alpha(T)$ and the fact that $S = T \cup \{\alpha(T)\}$, we easily deduce that $\alpha(S) < \alpha(T) < a$.

If $x \in \mathrm{imsg}_\Delta(S \setminus \{0\})$, then by Lemma 5.11, we know that $T = S \setminus \{x\}$ is an $\mathrm{I}(\Delta)$-semigroup. As $\alpha(S) < x < a$, then $\mathrm{F}_\Delta(T) = a$ and $\alpha(T) = x$. Therefore, $T \in \mathcal{J}_a(\Delta)$ and $S = T \cup \{\alpha(T)\}$. Hence, T is a child of S. □

E x a m p l e 5.13. Let $\Delta = \langle 4, 5 \rangle = \{0, 4, 5, 8, 9, 10, 12, \rightarrow\}$, (the symbol $\rightarrow$ means that every integer greater than 12 belongs to the set) and $a = 15$. We are going recurrently to build the tree $G(\Delta, 15)$. By Theorem 5.10, we know that the root of this tree is $\mathrm{S}(\Delta, 15) = (\Delta \setminus \mathrm{B}(15)) \cup \{0\} = (\Delta \setminus \{0, 5, 10, 15\}) \cup \{0\} = \{0, 4, 8, 9, 12, 13, 14, 16, \rightarrow\}$. By applying Proposition 5.12 and the previous comments of Theorem 5.10, we have following diagram (see Diagram 1).

Observe that $\begin{array}{c} P \\ \uparrow x \\ Q \end{array}$ means that $Q = P \setminus \{x\}$ and in addition $\alpha(Q) = x$.

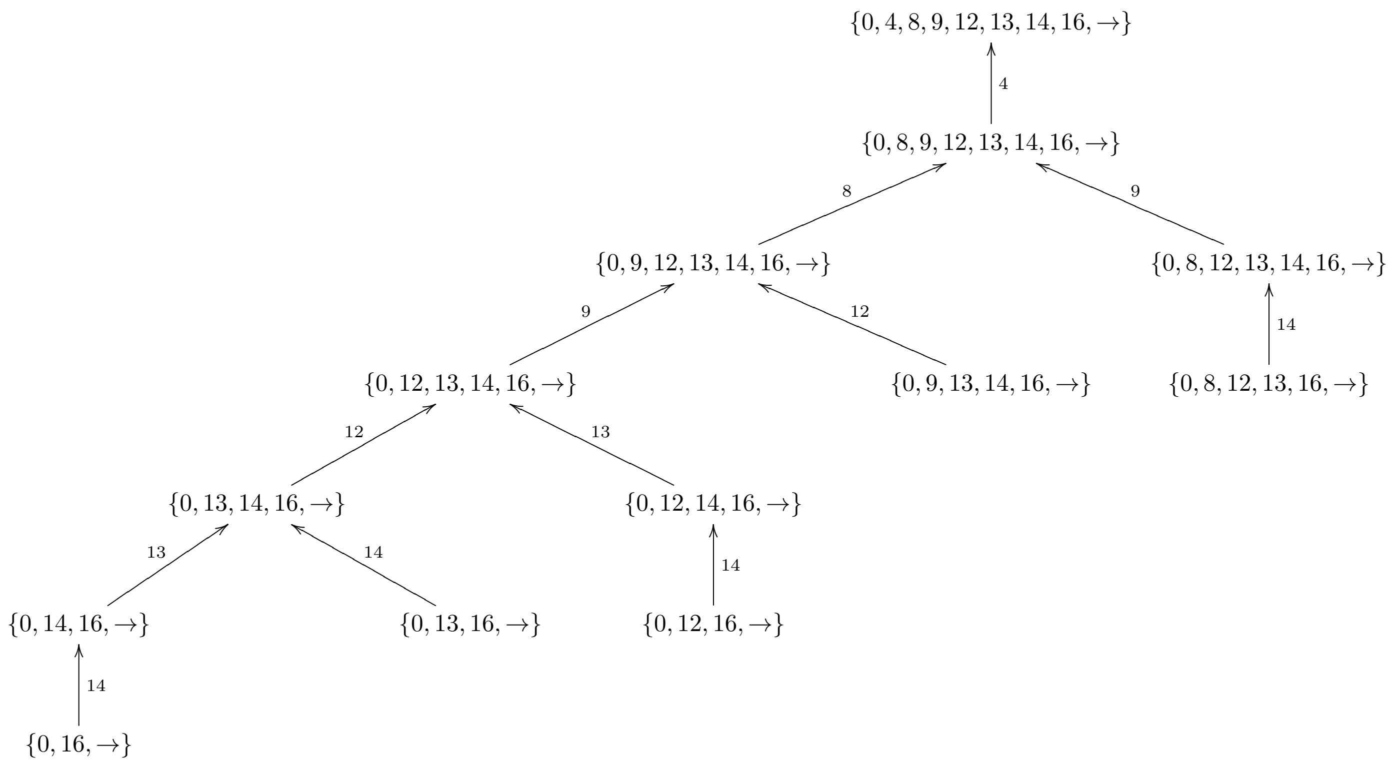
{0, 4, 8, 9, 12, 13, 14, 16, →}
4
{0, 8, 9, 12, 13, 14, 16, →}
8
9
{0, 9, 12, 13, 14, 16, →}
{0, 8, 12, 13, 14, 16, →}
9
12
14
{0, 12, 13, 14, 16, →}
{0, 9, 13, 14, 16, →}
{0, 8, 12, 13, 16, →}
12
13
{0, 13, 14, 16, →}
{0, 12, 14, 16, →}
13
14
14
{0, 14, 16, →}
{0, 13, 16, →}
{0, 12, 16, →}
14
{0, 16, →}

Diagram 1.

6. Algorithms to compute $\mathcal{J}_a(\Delta)$ and $\mathcal{J}(\Delta, k)$

Let Δ be a numerical semigroup and let $a \in \Delta$. Our first objective in this section is to give an algorithm which allows us to compute all the elements of the set $\mathcal{J}_a(\Delta)$. The idea of this algorithm is the construction of the tree $G(\Delta, a)$, as we have done in the previous section.

Algorithm 1

Input: A numerical semigroup Δ and $a \in \Delta \setminus \{0\}$.
Output: The set $\mathcal{J}_a(\Delta)$.

1: Compute $\mathrm{S}(\Delta, a)$.
2: $A = \{\mathrm{S}(\Delta, a)\}$.
3: $B = \{\mathrm{S}(\Delta, a)\}$.
4: For each $S \in A$ compute $C(S) = \{T\colon\ T \text{ is a child of } S\}$.
5: $D = \bigcup\limits_{S \in A} C(S)$.
6: If $D = \emptyset$, then return B.
7: $B = B \cup D$.
8: $A = D$ and go to 4.

We proceed to illustrate how the previous algorithm works with an example.

Example 6.1. Let $\Delta = \langle 4, 5\rangle$. We are going to built $\mathcal{J}_{15}(\Delta)$ using Algorithm 1.

▷ $\mathrm{S}(\Delta, 15) = \{0, 4, 8, 9, 12, 13, 14, 16, \rightarrow\}$.
▷ $A = \{\{0, 4, 8, 9, 12, 13, 14, 16, \rightarrow\}\}$ and $B = \{\{0, 4, 8, 9, 12, 13, 14, 16, \rightarrow\}\}$.
▷ $C(\{0, 4, 8, 9, 12, 13, 14, 16, \rightarrow\}) = \{\{0, 8, 9, 12, 13, 14, 16, \rightarrow\}\}$.
▷ $D = \{\{0, 8, 9, 12, 13, 14, 16, \rightarrow\}\}$.
▷ $B = \{\{0, 4, 8, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 8, 9, 12, 13, 14, 16, \rightarrow\}\}$.
▷ $A = \{\{0, 8, 9, 12, 13, 14, 16, \rightarrow\}\}$.
▷ $C(\{0, 8, 9, 12, 13, 14, 16, \rightarrow\}) = \{\{0, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 8, 12, 13, 14, 16, \rightarrow\}\}$.
▷ $D = \{\{0, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 8, 12, 13, 14, 16, \rightarrow\}\}$.
▷ $B = \{\{0, 4, 8, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 8, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 8, 12, 13, 14, 16, \rightarrow\}\}$.
▷ $A = \{\{0, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 8, 12, 13, 14, 16, \rightarrow\}\}$.
▷ $C(\{0, 9, 12, 13, 14, 16, \rightarrow\}) = \{\{0, 12, 13, 14, 16, \rightarrow\}, \{0, 9, 13, 14, 16, \rightarrow\}\}$ and $C(\{0, 8, 12, 13, 14, 16, \rightarrow\}) = \{\{0, 8, 12, 13, 16, \rightarrow\}\}$.
▷ $D = \{\{0, 12, 13, 14, 16, \rightarrow\}, \{0, 9, 13, 14, 16, \rightarrow\}, \{0, 8, 12, 13, 16, \rightarrow\}\}$.
▷ $B = \{\{0, 4, 8, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 8, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 8, 12, 13, 14, 16, \rightarrow\}, \{0, 12, 13, 14, 16, \rightarrow\}, \{0, 9, 13, 14, 16, \rightarrow\}, \{0, 8, 12, 13, 16, \rightarrow\}\}$.

▷ $A = \{\{0, 12, 13, 14, 16, \rightarrow\}, \{0, 9, 13, 14, 16, \rightarrow\}, \{0, 8, 12, 13, 16, \rightarrow\}\}$.

▷ $C(\{0, 12, 13, 14, 16, \rightarrow\}) = \{\{0, 13, 14, 16, \rightarrow\}, \{0, 12, 14, 16, \rightarrow\}\}$,
$C(\{0, 9, 13, 14, 16, \rightarrow\}) = \emptyset$ and $C(\{0, 8, 12, 13, 16, \rightarrow\}) = \emptyset$.

▷ $D = \{\{0, 13, 14, 16, \rightarrow\}, \{0, 12, 14, 16, \rightarrow\}\}$.

▷ $B = \{\{0, 4, 8, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 8, 9, 12, 13, 14, 16, \rightarrow\}$,
$\{0, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 8, 12, 13, 14, 16, \rightarrow\}, \{0, 12, 13, 14, 16, \rightarrow\}$,
$\{0, 9, 13, 14, 16, \rightarrow\}, \{0, 8, 12, 13, 16, \rightarrow\}, \{0, 13, 14, 16, \rightarrow\}, \{0, 12, 14, 16, \rightarrow\}\}$.

▷ $A = \{\{0, 13, 14, 16, \rightarrow\}, \{0, 12, 14, 16, \rightarrow\}\}$.

▷ $C(\{0, 13, 14, 16, \rightarrow\}) = \{\{0, 14, 16, \rightarrow\}, \{0, 13, 16, \rightarrow\}$ and $C(\{0, 12, 14, 16, \rightarrow\}) = \{\{0, 12, 16, \rightarrow\}\}$.

▷ $D = \{\{0, 14, 16, \rightarrow\}, \{0, 13, 16, \rightarrow\}, \{0, 12, 16, \rightarrow\}\}$.

▷ $B = \{\{0, 4, 8, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 8, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 9, 12, 13, 14, 16, \rightarrow\}$,
$\{0, 8, 12, 13, 14, 16, \rightarrow\}, \{0, 12, 13, 14, 16, \rightarrow\}, \{0, 9, 13, 14, 16, \rightarrow\}$,
$\{0, 8, 12, 13, 16, \rightarrow\}, \{0, 13, 14, 16, \rightarrow\}, \{0, 12, 14, 16, \rightarrow\}, \{0, 14, 16, \rightarrow\}$,
$\{0, 13, 16, \rightarrow\}, \{0, 12, 16, \rightarrow\}\}$.

▷ $A = \{\{0, 14, 16, \rightarrow\}, \{0, 13, 16, \rightarrow\}, \{0, 12, 16, \rightarrow\}\}$.

▷ $C(\{0, 14, 16, \rightarrow\}) = \{\{0, 16, \rightarrow\}\}$, $C(\{0, 13, 16, \rightarrow\}) = \emptyset$ and $C(\{0, 12, 16, \rightarrow\}) = \emptyset$.

▷ $D = \{\{0, 16, \rightarrow\}\}$.

▷ $B = \{\{0, 4, 8, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 8, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 9, 12, 13, 14, 16, \rightarrow\}$,
$\{0, 8, 12, 13, 14, 16, \rightarrow\}, \{0, 12, 13, 14, 16, \rightarrow\}, \{0, 9, 13, 14, 16, \rightarrow\}$,
$\{0, 8, 12, 13, 16, \rightarrow\}, \{0, 13, 14, 16, \rightarrow\}, \{0, 12, 14, 16, \rightarrow\}, \{0, 14, 16, \rightarrow\}$,
$\{0, 13, 16, \rightarrow\}, \{0, 12, 16, \rightarrow\}, \{0, 16, \rightarrow\}\}$.

▷ $A = \{\{0, 16, \rightarrow\}\}$.

▷ $C = (\{0, 16, \rightarrow\}) = \emptyset$.

▷ $D = \emptyset$.

▷ Return $\mathcal{J}_{15}(\langle 4, 5\rangle) = \{\{0, 4, 8, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 8, 9, 12, 13, 14, 16, \rightarrow\}$,
$\{0, 9, 12, 13, 14, 16, \rightarrow\}, \{0, 8, 12, 13, 14, 16, \rightarrow\}, \{0, 12, 13, 14, 16, \rightarrow\}$,
$\{0, 9, 13, 14, 16, \rightarrow\}, \{0, 8, 12, 13, 16, \rightarrow\}, \{0, 13, 14, 16, \rightarrow\}, \{0, 12, 14, 16, \rightarrow\}$,
$\{0, 14, 16, \rightarrow\}, \{0, 13, 16, \rightarrow\}, \{0, 12, 16, \rightarrow\}, \{0, 16, \rightarrow\}\}$.

Let Δ be a numerical semigroup and $k \in \mathbb{N}$. We now propose to give an algorithm that allows us to calculate $\mathcal{J}(\Delta, k)$. The following result appears in [12], Lemma 2.14.

Lemma 6.2. *If S is a numerical semigroup, then $\mathrm{F}(S) + 1 \leqslant 2\,\mathrm{g}(S)$.*

Proposition 6.3. *Let Δ be a numerical semigroup and $k \in \mathbb{N} \setminus \{0\}$. If $S \in \mathcal{J}(\Delta, k)$, then $S \in \mathcal{J}_a(\Delta)$ for some $a \in \Delta$ such that $a \leqslant 2\,\mathrm{g}(\Delta) + 2k - 1$ and $\#\mathrm{B}(a) \leqslant k + 1$.*

P r o o f. If $S \in \mathcal{J}(\Delta, k)$, then $\mathrm{g}(S) = \mathrm{g}(\Delta) + k$ and applying Lemma 6.2, we have that $\mathrm{F}(S) \leqslant 2\mathrm{g}(\Delta) + 2k - 1$. As $\mathrm{F}_\Delta(S) \leqslant \mathrm{F}(S)$, then $\mathrm{F}_\Delta(S) \leqslant 2\mathrm{g}(\Delta) + 2k - 1$. As $S \in \mathcal{J}_{\mathrm{F}_\Delta(S)}(\Delta)$, then $S \subseteq \mathrm{S}(\Delta, \mathrm{F}_\Delta(S))$ and so $\mathrm{g}(\mathrm{S}(\Delta, \mathrm{F}_\Delta(S))) \leqslant \mathrm{g}(S)$. By applying Proposition 5.3, we have that $\mathrm{g}(\Delta) + \#\mathrm{B}(\mathrm{F}_\Delta(S)) - 1 \leqslant \mathrm{g}(\Delta) + k$ and thus $\#\mathrm{B}(\mathrm{F}_\Delta(S)) \leqslant k + 1$. □

If $G = (V, E)$ is a tree and $v \in V$, then the *depth* of v, denoted by $\mathrm{d}(v)$, is the length of the only path connecting v with the root of the tree. The following result is easily deduced from Proposition 5.9 and Theorem 5.10.

Lemma 6.4. *If S is a vertex of $G(\Delta, a)$, then $\mathrm{g}(S) = \mathrm{g}(\mathrm{S}(\Delta, a)) + \mathrm{d}(S)$.*

Proposition 6.5. *Let Δ be a numerical semigroup, $k \in \mathbb{N} \setminus \{0\}$ and $a \in \Delta$. Then $S \in \mathcal{J}(\Delta, k) \cap \mathcal{J}_a(\Delta)$ if and only if S is a vertex of the tree $G(\Delta, a)$ with depth $k + 1 - \#\mathrm{B}(a)$.*

P r o o f. We have that $S \in \mathcal{J}(\Delta, k) \cap \mathcal{J}_a(\Delta)$ if and only if S is a vertex of $G(\Delta, a)$ and $\mathrm{g}(S) = \mathrm{g}(\Delta) + k$. By applying Lemma 6.4, we deduce that this occurs if and only if $\mathrm{g}(\mathrm{S}(\Delta, a)) + \mathrm{d}(S) = \mathrm{g}(\Delta) + k$. The proof concludes applying Proposition 5.3. □

If $G = (V, E)$ is a tree, we denote by $\mathrm{N}(G, n) = \{v \in V \colon \mathrm{d}(v) = n\}$. The following result has an immediate proof.

Proposition 6.6. *If $G = (V, E)$ is a tree and r is its root, then $\mathrm{N}(G, 0) = \{r\}$ and $\mathrm{N}(G, n + 1) = \{v \in V \colon v$ is a child of an element of $\mathrm{N}(G, n)\}$ for all $n \in \mathbb{N}$.*

We are already able to provide the algorithm to compute $\mathcal{J}(\Delta, k)$.

Algorithm 2

INPUT: A numerical semigroup Δ and $k \in \mathbb{N} \setminus \{0\}$.
OUTPUT: The set $\mathcal{J}(\Delta, k)$.
1: Compute $A = \{a \in \Delta \colon a \leqslant 2(\mathrm{g}(\Delta) + k) - 1\}$ and $\#\mathrm{B}(a) \leqslant k + 1$.
2: For all $a \in A$ compute $\mathrm{N}(a) = \mathrm{N}(G(\Delta, a), k + 1 - \#\mathrm{B}(a))$.
3: Return $\mathcal{J}(\Delta, k) = \bigcup_{a \in A} \mathrm{N}(a)$.

We proceed to illustrate how the previous algorithm works with an example.

E x a m p l e 6.7. Let $\Delta = \langle 4, 5 \rangle$ and $k = 2$. We are going to compute $\mathcal{J}(\langle 4, 5 \rangle, 2)$ using Algorithm 2.

(1) $A = \{a \in \Delta\colon\ a \leqslant 15 \text{ and } \#\mathrm{B}(a) \leqslant 3\} = \{0, 4, 5, 8, 10\}$.

(2) $\mathrm{N}(0) = \mathrm{N}(G(\Delta, 0), 2) = \emptyset$, $\mathrm{N}(4) = \mathrm{N}(G(\Delta, 4), 1) = \emptyset$, $\mathrm{N}(5) = \mathrm{N}(G(\Delta, 5), 1) = \{\{0, 8, 9, 10, 12, \rightarrow\}\}$, $\mathrm{N}(8) = \mathrm{N}(G(\Delta, 8), 0) = \{\mathrm{S}(\Delta, 8)\} = \{\{0, 5, 9, 10, 12, \rightarrow\}\}$, $\mathrm{N}(10) = \mathrm{N}(G(\Delta, 10), 0) = \{\mathrm{S}(\Delta, 10)\} = \{\{0, 4, 8, 9, 12, \rightarrow\}\}$.

(3) $\mathcal{J}(\langle 4, 5\rangle, 2) = \{\{0, 8, 9, 10, 12, \rightarrow\}, \{0, 5, 9, 10, 12, \rightarrow\}, \{0, 4, 8, 9, 12, \rightarrow\}\}$.

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