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$C^{1,\alpha}$ REGULARITY FOR ELLIPTIC EQUATIONS WITH THE GENERAL NONSTANDARD GROWTH CONDITIONS

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Abstract. We study elliptic equations with the general nonstandard growth conditions involving Lebesgue measurable functions on Ω. We prove the global $C^{1,\alpha}$ regularity of bounded weak solutions of these equations with the Dirichlet boundary condition. Our results generalize the $C^{1,\alpha}$ regularity results for the elliptic equations in divergence form not only in the variable exponent case but also in the constant exponent case.

Keywords: nonstandard growth; $C^{1,\alpha}$ regularity; Hölder continuity; bounded weak solution; partial differential equations

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1. Introduction

Various mathematical problems with nonstandard growth conditions have been investigated by many authors in the last two decades. We refer to the overview paper [24] and the books [4], [13] and [31] .

Partial differential equations and variational problems with nonstandard growth conditions arise from elastic mechanics, electro-rheological fluids and image restoration; see, e.g., [7], [32], [33], [40]. Many results on the regularity for the nonstandard growth problems have been obtained (see, e.g., [2], [8], [15], [17], [18], [23], [25], [26], [29], [30], [34], [37], [39]). In particular, in [1], [5], [9]–[11], [16], [19], [20], [36], [38] the authors have obtained the $C^{1,\alpha}$ regularity results for integral functionals or elliptic equations with nonstandard growth conditions. In this paper we deal with the $C^{1,\alpha}$ regularity of bounded solutions to the quasilinear elliptic equation

$$-\operatorname{div} A(x,u,\nabla u) + B(x,u,\nabla u) = 0 \quad \text{in } \Omega \tag{1.1}$$

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with the Dirichlet boundary condition

$$u = g \quad \text{in } \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^n, n \geqslant 2$, and $\partial\Omega$ is its boundary.

We assume that A and B satisfy the variable exponent growth conditions (see Assumptions (H2) and (H3)), where the variable exponent $p\colon \overline{\Omega} \to \mathbb{R}$ satisfies the condition

$$1 < p^- := \inf_{\Omega} p(x) \leqslant p^+ := \sup_{\Omega} p(x) < \infty. \tag{1.3}$$

As well known, assuming that the variable exponent $p(x)$ is log-Hölder continuous, allows to prove Hölder continuity of weak solutions and minima. However, $C^{1,\alpha}$ regularity results in fact require that $p(x)$ is Hölder continuous rather than merely log-Hölder continuous; see for example [24], Theorem 8.1. The following assumptions will be used.

(H1) The function p is Hölder continuous on $\overline{\Omega}$, which is denoted by $p \in C^{0,\beta_1}(\overline{\Omega})$, that is, there exist a positive constant L_1 and exponent $\beta_1 \in (0,1)$ such that

$$|p(x^1) - p(x^2)| \leqslant L_1 |x^1 - x^2|^{\beta_1} \quad \text{for } x^1, x^2 \in \overline{\Omega}.$$

(H2) $A = (A_1, A_2, \ldots, A_n) \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. For every $(x,z) \in \overline{\Omega} \times \mathbb{R}$, $A(x,z,\cdot) \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$ and there exist a non-negative constant $k \geqslant 0$, non-increasing continuous function $\lambda\colon [0,\infty) \to (0,\infty)$ and non-decreasing continuous function $\Lambda\colon [0,\infty) \to (0,\infty)$ such that for all $x, x^1, x^2 \in \overline{\Omega}$, $z, z^1, z^2 \in \mathbb{R}$, $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{R}^n \setminus \{0\}$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$ the following conditions are satisfied:

$$|A(x,z,0)| \leqslant \Lambda(|z|) b(x), \tag{1.4}$$

where b is a non-negative function in the appropriate variable exponent Lebesgue space, see (1.10) for details,

$$\sum_{i,j=1}^{n} \frac{\partial A_j}{\partial \eta_i}(x,z,\eta)\xi_i\xi_j \geqslant \lambda(|z|)(k + |\eta|^2)^{(p(x)-2)/2}|\xi|^2, \tag{1.5}$$

$$\sum_{i,j=1}^{n} \Big|\frac{\partial A_j}{\partial \eta_i}(x,z,\eta)\Big| \leqslant \Lambda(|z|)(k + |\eta|^2)^{(p(x)-2)/2}, \tag{1.6}$$

$$\begin{aligned} &|A(x^1,z^1,\eta) - A(x^2,z^2,\eta)| \\ &\quad \leqslant \Lambda(\max\{|z^1|,|z^2|\})(|x^1 - x^2|^{\beta_1} + |z^1 - z^2|^{\beta_2}) \\ &\qquad \times [((k + |\eta|^2)^{(p(x^1)-2)/2} + (k + |\eta|^2)^{(p(x^2)-2)/2}) \\ &\qquad \times (1 + |\log(k + |\eta|^2)|)|\eta| + 1], \end{aligned} \tag{1.7}$$

where $\beta_1, \beta_2 \in (0,1)$ are given numbers.

(H3) $B\colon \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is a Carathéodory function satisfying the structural condition

$$|B(x,z,\eta)|\leqslant\Lambda(|z|)(|\eta|^{p(x)}+d(x)), \tag{1.8}$$

where Λ is as in Assumption (H2) and d is a non-negative Lebesgue measurable function satisfying some assumption; see (1.10) for details.

Without loss of generality we may assume that $L_1\geqslant 1$, $0\leqslant k\leqslant 1$, $\lambda(z)\leqslant 1$ and $\Lambda(z)\geqslant 1$ for all $z\in[0,\infty)$ in Assumptions (H1)–(H3).

A typical example of the function A satisfying Assumption (H2) is

$$A(x,z,\eta)=a(x,z)(k+|\eta|^2)^{(p(x)-2)/2}\eta+f(x,z),$$

where $a\colon \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ and $f\colon \overline{\Omega}\times\mathbb{R}\to\mathbb{R}^n$ are Hölder continuous in (x,z) and satisfy the inequalities

$$\lambda(|z|)\leqslant a(x,z)\leqslant\Lambda(|z|)\quad\text{and}\quad |f(x,z)|\leqslant\Lambda(|z|)b(x)\text{ for }(x,z)\in\overline{\Omega}\times\mathbb{R}.$$

The equation (1.1) with the structure conditions (H1)–(H3) is considered in [16] by Fan, while he proved the Hölder continuity of gradients of the bounded weak solutions of (1.1) under the conditions $b=0$, $d=\text{const}$, that is, $A(x,z,0)=0$ and $|B(x,z,\eta)|\leqslant\Lambda(|z|)(|\eta|^{p(x)}+1)$ instead of (1.4) and (1.8), respectively.

Zhang and Zhou (see [38]) and Yao (see [36]) have obtained local Hölder continuity for the gradients of weak solutions of (1.1), where the forms of A and B considered in [38] are $A(x,z,\eta)=|\eta|^{p(x)-2}\eta$ and $B(x,z,\eta)=|\eta|^{p(x)-2}\log(|\eta|)\eta\nabla p(x)$ but in [36] they are $A(x,z,\eta)=(a(x)\eta\cdot\eta)^{(p(x)-2)/2}a(x)\eta-|f(x)|^{p(x)-2}f(x)$, where $f(x)=(f_1(x),\dots,f_n(x))$, $f_i\in C^{0,\alpha}_{\rm loc}(\Omega)$ is a vector field and $a(x)=(a_{ij}(x))$, $a_{ij}\in C^{0,\alpha}_{\rm loc}(\Omega)$, is a symmetric matrix satisfying

$$\Lambda^{-1}|\eta|^2\leqslant a(x)\eta\cdot\eta\leqslant\Lambda|\eta|^2$$

with a positive constant Λ, and $B(x,z,\eta)=0$. To the best of the authors' knowledge, nothing is known on $C^{1,\alpha}$ regularity for the elliptic Dirichlet problem (1.1), (1.2) which satisfies Assumptions (H2) and (H3) when b and d are Lebesgue measurable functions. The aim of the present paper is to find the sharp conditions on $b(x)$ and $d(x)$ in (1.4) and (1.8) so as to ensure the Hölder continuity of gradients of the bounded weak solutions of (1.1). In this paper we use the variable exponent spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$, $W_0^{1,p(\cdot)}(\Omega)$ of which the definitions will be given in Section 2. The symbols of some common spaces used in this paper such as $L^\infty(\Omega)$, $W^{1,\infty}(\Omega)$, $C_0^1(\Omega)$, $C_0^\infty(\Omega)$, $C(\overline{\Omega})$, $C^{k,\alpha}(\partial\Omega)$, $C^{k,\alpha}_{\rm loc}(\Omega)$ and $C^{k,\alpha}(\overline{\Omega})$ with $k=0,1$ are standard.

Definition 1.1. (1) The function $u \in W^{1,p(\cdot)}(\Omega)$ is called a bounded weak solution of (1.1) if $u \in L^\infty(\Omega)$ and

$$\int_\Omega (A(x,u,\nabla u)\nabla\varphi + B(x,u,\nabla u)\varphi)\,\mathrm{d}x = 0 \quad \text{for every } \varphi \in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega). \tag{1.9}$$

(2) The function $u \in W^{1,p(\cdot)}(\Omega)$ is called a bounded weak solution of the Dirichlet problem (1.1), (1.2) if $u \in L^\infty(\Omega)$, $u - g \in W_0^{1,p(\cdot)}(\Omega)$ with $g \in W^{1,p(\cdot)}(\Omega)$ and (1.9) holds.

Our main results in this paper are the following Theorems 1.2 and 1.3.

Theorem 1.2. *Let p satisfy* (1.3) *and* (H1), *and let $u \in W^{1,p(\cdot)}(\Omega) \cap C^{0,\alpha_1}_{\mathrm{loc}}(\Omega)$ be a bounded weak solution of* (1.1) *with $\sup\limits_\Omega |u| \leqslant M$, where M is a given positive constant. Suppose that A and B satisfy Assumptions* (H2) *and* (H3) *with nonnegative functions b and d such that*

$$b \in L^{p'(\cdot)m(\cdot)}(\Omega), \quad d \in L^{m(\cdot)}(\Omega) \tag{1.10}$$

with $m \in C(\overline{\Omega})$ satisfying

$$m(x) > \frac{2n}{\beta} \quad \forall\, x \in \overline{\Omega}, \tag{1.11}$$

where

$$\beta = \min\{\beta_1, \alpha_1\beta_2\} \tag{1.12}$$

and β_1, β_2 are as in Assumption (H2), *and $p'(x) = p(x)/(p(x)-1)$. Then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ and for any open set $\Omega' \Subset \Omega$ the inequality*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega'})} \leqslant C \tag{1.13}$$

holds, where $\alpha \in (0,1)$ and C depends on n, $p(\cdot)$, $m(\cdot)$, $\lambda(M)$, $\Lambda(M)$, M, α_1, β_1 and β_2, and, moreover, C depends also on $\|b\|_{p'(\cdot)m(\cdot)}$, $\|d\|_{m(\cdot)}$ and $\operatorname{dist}(\Omega', \partial\Omega)$.

Theorem 1.3. *Let Ω be a bounded domain of $\mathbb{R}^n$ with C^{1,α_0} boundary $\partial\Omega$. Suppose that $g \in C^{1,\alpha_0}(\partial\Omega)$ and $u \in W^{1,p(\cdot)}(\Omega) \cap C^{0,\alpha_1}(\overline{\Omega})$ is a bounded weak solution of the Dirichlet problem* (1.1), (1.2) *with $\sup\limits_\Omega |u| \leqslant M$. Let p, A and B satisfy all the conditions of Theorem* 1.2. *Then $u \in C^{1,\alpha}(\overline{\Omega})$ and the inequality*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant C \tag{1.14}$$

holds, where $\alpha \in (0,1)$ and C depend on n, $p(\cdot)$, $m(\cdot)$, $\lambda(M)$, $\Lambda(M)$, M, α_1, β_1, β_2, $\|g\|_{C^{1,\alpha_0}(\partial\Omega)}$ and Ω, and, moreover, α depends also on α_0, C depends also on $\|b\|_{p'(\cdot)m(\cdot)}$ and $\|d\|_{m(\cdot)}$.

The proofs of Theorems 1.2 and 1.3 are similar to those of Fan (see [16]) who adapted the ideas used by Acerbi and Mingione in [1] for the $C^{1,\alpha}$ regularity of minimizer of the integral functional. However, in order to employ methods of [16], we need new results on the Hölder continuity and the higher integrability for bounded weak solutions of the Dirichlet problem (1.1), (1.2) satisfying (2.1), (2.2) with appropriate measurable functions a_2 and b_1 which arise from the conditions (1.4) and (1.8) because there are no such regularity results in literature. Indeed, Fan and Zhao in [17] and Fan in [16] obtained the Hölder continuity and the higher integrability for bounded weak solutions when a_2 and b_1 are constants, and Ri and Yu (see [37]) proved the boundedness and Hölder continuity for weak solutions of (1.1), (1.2) under the stronger structural conditions on A and B:

$$\begin{aligned} &A(x,z,\eta)\eta \geqslant a_0|\eta|^{p(x)} - a_1(x)|z|^{q(x)} - a_2(x),\\ &|A(x,z,\eta)| \leqslant b_0(|\eta|^{p(x)-1} + b_1(x)|z|^{q(x)/p'(x)} + b_2(x)),\\ &|B(x,z,\eta)| \leqslant c_0(x)|\eta|^{p(x)/q'(x)} + c_1(x)|z|^{q(x)-1} + c_2(x), \end{aligned}$$

where

$$p(x) \leqslant q(x) < p^*(x) := \begin{cases} \dfrac{np(x)}{n-p(x)} & \text{if } p(x) < n,\\ \infty & \text{if } p(x) \geqslant n \end{cases}$$

and a_0, b_0 are given positive constants, and a_1, a_2, b_1, b_2, c_0, c_1, c_2 are appropriate non-negative measurable functions; see [37] for details.

We prove in the next sections that if (2.1), (2.2) and (2.4) are satisfied and p is log-Hölder continuous in Ω, that is, there exists a constant $C_{\log}$ such that

$$-|p(x)-p(y)|\log|x-y| \leqslant C_{\log} \quad \forall\, x,y \in \Omega \text{ with } |x-y| \leqslant \frac{1}{2},$$

then a bounded weak solution of (1.1), (1.2) is Hölder continuous (see Theorems 2.5 and 2.8) and higher integrable (see Lemmas 3.1 and 4.1). Besides, our procedure for deriving the same estimates required for using Campanato's theorem to prove Hölder continuity of the gradients is slightly more difficult and sharper than that of [16] (see Lemmas 3.2–3.3 and Proposition 3.4).

Remark 1.4. We emphasize a sufficient condition for the Hölder continuity of bounded weak solutions to (1.1) or (1.1), (1.2) assumed in Theorems 1.2 and 1.3:

Let A and B satisfy (H2) (here we do not require (1.7)) and (H3) with b and d satisfying (1.10) for some $m \in C(\overline{\Omega})$ such that

$$m(x) > \max\Big\{1, \frac{n}{p(x)}\Big\} \quad \text{for any } x \in \overline{\Omega}. \tag{1.15}$$

Then the bounded weak solution of (1.1) belongs to $C^{0,\alpha_1}_{\rm loc}(\Omega)$ with some $\alpha_1 \in (0,1)$. Moreover, if Ω satisfies a uniform exterior cone condition on $\partial\Omega$ and u is a bounded weak solution of (1.1), (1.2) with $g \in C^{0,\alpha_0}(\partial\Omega)$, then u belongs to $C^{0,\alpha_1}(\overline{\Omega})$ with some $\alpha_1 \in (0,1)$.

The assertion of Remark 1.4 follows from Theorems 2.5, 2.8 and Remark 2.10.

Remark 1.5. It is clear that the constants b and d satisfy (1.10) for any $m \in C(\overline{\Omega})$ such as in (1.15). Therefore, it follows from Theorem 1.3 that bounded weak solutions of (1.1), (1.2) belong to $C^{1,\alpha}(\overline{\Omega})$ under the conditions of Theorem 1.3 (but we need not assume (1.10)–(1.12)). This shows that the results of the present paper generalize the $C^{1,\alpha}$ regularity results not only of [16] and [36] in the variable exponent case but also of [6], [27], [28], [35] in the constant exponent case where the authors assumed that b and d are all non-negative constants.

The rest of this paper is organized as follows. In Section 2, we introduce some known basic properties on the variable exponent Lebesgue space and Sobolev space, and prove the Hölder continuity of bounded weak solutions of (1.1) or (1.1), (1.2) by using the generalized De Giorgi classes introduced in [37] and the localization method. We prove the local $C^{1,\alpha}$ regularity for (1.1) in Section 3 and the global $C^{1,\alpha}$ regularity for the Dirichlet problem (1.1), (1.2) in Section 4. In the following sections the symbol C will be used as a generic symbol for a constant that may change from line to line or even within a line.

2. Preliminaries

Let E be a bounded open set in $\mathbb{R}^n, n \geqslant 2$, and $p\colon E \to [1,\infty)$ be a Lebesgue measurable function.

Define the variable exponent Lebesgue space $L^{p(\cdot)}(E)$ by

$$L^{p(\cdot)}(E) := \left\{u\colon\ u\colon E \to \mathbb{R} \text{ is a measurable function and } \int_E |u|^{p(x)}\,\mathrm{d}x < \infty\right\}$$

with the norm

$$\|u\|_{p(\cdot),E} := \|u\|_{L^{p(\cdot)}(E)} = \inf\left\{\lambda > 0\colon\ \int_E \left|\frac{u}{\lambda}\right|^{p(x)}\mathrm{d}x \leqslant 1\right\}$$

and the variable exponent Sobolev space $W^{1,p(\cdot)}(E)$ by

$$W^{1,p(\cdot)}(E) := \{u \in L^{p(\cdot)}(E)\colon\ |\nabla u| \in L^{p(\cdot)}(E)\}$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}(E)} = \|\nabla u\|_{p(\cdot),E} + \|u\|_{p(\cdot),E},$$

where $\|\nabla u\|_{p(\cdot),E} := \||\nabla u|\|_{p(\cdot),E}$.

Define $W_0^{1,p(\cdot)}(E)$ as the closure of $C_0^\infty(E)$ in $W^{1,p(\cdot)}(E)$. We point out that, when p is log-Hölder continuous in E, $\|\nabla u\|_{p(\cdot),E}$ is an equivalent norm on $W_0^{1,p(\cdot)}(E)$. All the spaces $L^{p(\cdot)}(E)$, $W^{1,p(\cdot)}(E)$ and $W_0^{1,p(\cdot)}(E)$ are Banach spaces.

For a measurable function $f\colon E \to \mathbb{R}$, we put

$$\sup_E f(x) := \operatorname{ess\,sup}_E f(x), \quad \inf_E f(x) := \operatorname{ess\,inf}_E f(x),$$
$$\operatorname{osc}_E f(x) := \sup_E f(x) - \inf_E f(x), \quad \|f\|_{\infty,E} = \|f\|_{L^\infty(E)}$$

and sometimes

$$f_E^+ := \sup_E f(x), \quad f_E^- := \inf_E f(x).$$

We denote by $|E|$ the n-Lebesgue measure of E. If $1 \leqslant p_E^- \leqslant p_E^+ < \infty$ and $u \in L^{p(\cdot)}(E)$, then there hold the inequalities

$$\min\{\|u\|_{p(\cdot),E}^{p_E^-}, \|u\|_{p(\cdot),E}^{p_E^+}\} \leqslant \int_E |u|^{p(x)}\,\mathrm{d}x \leqslant \max\{\|u\|_{p(\cdot),E}^{p_E^-}, \|u\|_{p(\cdot),E}^{p_E^+}\}$$

and if $0 \leqslant a_E^- \leqslant a_E^+ < \infty$, $r \in L^\infty(E)$, $1 \leqslant a(x)r(x)$, $r(x) \geqslant 1$ for a.e. $x \in E$ and $u \in L^{a(\cdot)r(\cdot)}(E)$ then there holds the inequality

$$\||u|^{a(x)}\|_{r(\cdot),E} \leqslant \max\{\|u\|_{a(\cdot)r(\cdot),E}^{a_E^-}, \|u\|_{a(\cdot)r(\cdot),E}^{a_E^+}\}.$$

Moreover, if $u \in L^{r(\cdot)}(E)$ and $1 \leqslant p(x) \leqslant r(x)$ for a.e. $x \in E$ then

$$\|u\|_{p(\cdot),E} \leqslant (1 + |E|)\|u\|_{r(\cdot),E}.$$

We refer to [12], [13] for the elementary properties of the spaces $L^{p(\cdot)}(E)$, $W^{1,p(\cdot)}(E)$ and $W_0^{1,p(\cdot)}(E)$.

Let $B_\varrho(x_0)$ be an open ball in $\mathbb{R}^n$ of radius ϱ centered at $x_0 \in \mathbb{R}^n$ and put

$$\omega_n := |B_1(x_0)|, \quad \Omega_\varrho(x_0) := \Omega \cap B_\varrho(x_0), \quad (\partial\Omega)_\varrho(x_0) := \partial\Omega \cap B_\varrho(x_0).$$

Now, we prove the Hölder continuity results of bounded weak solutions of (1.1) under Assumption (H3) and the condition that the coefficient $A\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function satisfying the structural conditions

$$A(x,z,\eta)\eta \geqslant a_0(|z|)|\eta|^{p(x)} - a_1(|z|)a_2(x), \tag{2.1}$$
$$|A(x,z,\eta)| \leqslant b_0(|z|)(|\eta|^{p(x)-1} + b_1(x)) \tag{2.2}$$

for a.e. $x \in \Omega$ and all $(z,\eta) \in \mathbb{R} \times \mathbb{R}^n$, where $a_0\colon [0,\infty) \to (0,\infty)$ is a non-increasing continuous function and $a_1, b_0\colon [0,\infty) \to (0,\infty)$ are non-decreasing continuous functions, and a_2, b_1 are non-negative measurable functions. Hölder continuity results for the bounded weak solutions of (1.1) have been obtained by Fan and

Zhao (see [17]) when a_2, b_1 and d in (2.1), (2.2) and Assumption (H3) are all constants. Recall the generalized De Giorgi's classes $\mathfrak{B}_{p(\cdot)}(B_R(y), M, \gamma, \gamma_1, \delta, 1/r)$ and $\mathfrak{B}_{p(\cdot)}((\partial\Omega)_R(z), M, \gamma, \gamma_1, \delta, 1/r)$ introduced in [37] to obtain the Hölder continuity of weak solutions.

Definition 2.1 (see [37]). Let M, γ, γ_1, δ, r be positive constants with $\delta \leqslant 2$, $r > 1$ and $B_R(y) \subset \Omega$. We say that a function u belongs to class $\mathfrak{B}_{p(\cdot)}(B_R(y), M, \gamma, \gamma_1, \delta, 1/r)$ if $u \in W^{1,p(\cdot)}(B_R)$, $\|u\|_{\infty,B_R} \leqslant M$ and the functions $\omega(x) = \pm u(x)$ satisfy the inequalities

$$\int_{A_{k,\tau}} |\nabla\omega|^{p(x)}\,\mathrm{d}x \leqslant \gamma \int_{A_{k,t}} \Big|\frac{\omega - k}{t-\tau}\Big|^{p(x)}\,\mathrm{d}x + \gamma_1 |A_{k,t}|^{1-1/r}$$

for arbitrary $0 < \tau < t \leqslant R$ and k such that $k \geqslant \sup\limits_{B_t(y)} \omega - \delta M$, where $A_{k,t} = \{x \in B_t\colon\ \omega(x) > k\}$.

Definition 2.2 (see [37]). Let M, γ, γ_1, δ, r, R be positive constants with $\delta \leqslant 2$, $r > 1$ and $z \in \partial\Omega$. We say that a function u belongs to the class $\mathfrak{B}_{p(\cdot)}((\partial\Omega)_R(z), M, \gamma, \gamma_1, \delta, 1/r)$ if $u \in W^{1,p(\cdot)}(\Omega_R)$, $\|u\|_{\infty,\Omega_R} \leqslant M$, $\sup\limits_{(\partial\Omega)_R} |u(x)| < \infty$ and the functions $\omega(x) = \pm u(x)$ satisfy the inequalities

$$\int_{A_{k,\tau}} |\nabla\omega|^{p(x)}\,\mathrm{d}x \leqslant \gamma \int_{A_{k,t}} \Big|\frac{\omega - k}{t-\tau}\Big|^{p(x)}\,\mathrm{d}x + \gamma_1 |A_{k,t}|^{1-1/r}$$

for arbitrary $0 < \tau < t \leqslant R$ and k such that $k \geqslant \max\Big\{\sup\limits_{\Omega_t} \omega - \delta M, \sup\limits_{(\partial\Omega)_t} \omega\Big\}$, where $A_{k,t} = \{x \in \Omega_t\colon\ \omega(x) > k\}$.

As mentioned in Introduction, the Hölder continuity results for weak solutions already known in literature do not allow us to use them to get the $C^{1,\alpha}$ regularity of bounded weak solutions of (1.1) or (1.1), (1.2). Therefore, we should study the Hölder continuity of bounded weak solutions under the conditions (2.1), (2.2).

Let M be a positive constant. If $p \in C(\overline{\Omega})$, then there exists a radius R_1 such that $M^{\operatorname{osc}_{\Omega_{R_1}} p} \leqslant 2$ for any B_{R_1} with $\Omega_{R_1} \neq \emptyset$. Note that if p is log-Hölder continuous in Ω then there exists a constant $L_2 > 0$ such that

$$R^{-\operatorname{osc}_{\Omega_R} p} \leqslant L_2 \quad \forall\, B_R \text{ with } \Omega_R \neq \emptyset; \tag{2.3}$$

see, e.g., [13], Lemma 4.1.6 for details.

We first provide the following interior Hölder continuity.

Lemma 2.3. *Let $p \in C(\overline{\Omega})$ satisfy (1.3) and be log-Hölder continuous in Ω and let M be a positive constant. Further, let $R_0 \in (0,1)$ be a number satisfying $R_0 \leqslant R_1$ and $\varepsilon_0 > 0$ and $r > 1$ be numbers such that $p_0 r \geqslant n + \varepsilon_0$, where $p_0 = p^-_{\Omega_{R_0}(x_0)}$ and*

$x_0 \in \overline{\Omega}$. *Let* $B_{R'}(y) \subset \Omega_{R_0}(x_0)$ *and* $u \in W^{1,p(\cdot)}(B_{R'}) \cap L^\infty(B_{R'})$, *and* $\sup\limits_{B_{R'}} |u| \leqslant M$. *Suppose that* $u \in \mathfrak{B}_{p(\cdot)}(B_R(y), M, \gamma, \gamma_1, \delta, 1/r)$ *for any* $R \in (0, R']$. *Then there exists a constant* $s = s(n, \gamma, p^+, p^-, L_2, \varepsilon_0) > 2$ *such that, for arbitrary* $0 < R \leqslant R'$,

$$\operatorname*{osc}_{B_R(y)} u \leqslant C R'^{-\alpha_1} R^{\alpha_1},$$

where L_2 *is as in* (2.3) *and*

$$C = 4 \max\Big\{\frac{(\omega_n + 1)(\gamma + 1) + \gamma_1}{\gamma} l 2^s R'^{\varepsilon_0/(n+\varepsilon_0)}, \operatorname*{osc}_{B_{R'}} u\Big\},$$
$$\alpha_1 = \min\Big\{\frac{\varepsilon_0}{n + \varepsilon_0}, -\log_4(1 - l^{-1}2^{-s})\Big\}, \quad l = \max\Big\{2, \frac{2}{\delta}\Big\}.$$

The proof of Lemma 2.3 is the same as that of Lemma 4.5 in [37], the only difference being that we must now replace $p_0\sigma_0$ by $n+\varepsilon_0$ and use the inequality $M^{p^+_{B_R} - p^-_{B_R}} \leqslant 2$ with $R \leqslant R_0$.

Lemma 2.4. *Let* $p \in C(\overline{\Omega})$ *satisfy* (1.3). *Suppose that* A *and* B *satisfy* (2.1), (2.2) *and* (1.8) *with non-negative measurable functions* a_2, b_1 *and* d *such that*

$$a_2, d \in L^{m(\cdot)}(\Omega), \quad b_1 \in L^{p'(\cdot)m(\cdot)}(\Omega), \tag{2.4}$$

where $m \in C(\overline{\Omega})$ *is as in* (1.15). *If* u *is a bounded weak solution of* (1.1) *such that* $\sup\limits_{\Omega} |u| \leqslant M$, *then*

$$u \in \mathfrak{B}_{p(\cdot)}\Big(B_R(y), M, \gamma, \gamma_1, \delta, \frac{1}{m^-_{B_R}}\Big) \quad \text{for any ball } B_R(y) \Subset \Omega \text{ with } |B_R| \leqslant 1,$$

where

$$\delta = \min\Big\{2, \frac{a_0(M)}{3M\Lambda(M)}\Big\}, \quad \gamma = \gamma(p^+, a_0(M), b_0(M)),$$
$$\gamma_1 = \gamma_1(p^+, p^-, a_0(M), a_1(M), b_0(M), \|a_2\|_{m(\cdot)}, \|b_1\|_{p'(\cdot)m(\cdot)}, \|d\|_{m(\cdot)}).$$

Proof. Let $0 < \tau < t \leqslant R$. Let $\eta \in C^1(\mathbb{R}^n)$ be a function such that $0 \leqslant \eta(x) \leqslant 1$, $|\nabla\eta(x)| \leqslant 2/(t - \tau)$ for $x \in \mathbb{R}^n$, $\eta(x) = 1$ for $x \in B_\tau(x_0)$ and $\operatorname{supp}\eta \subset B_t(x_0)$. Set $\omega^{(k)} = \max\{\omega - k, 0\}$ with $\omega = \pm u$ and $k \geqslant \sup\limits_{B_t} \omega - \delta M$. Then $\varphi = \eta^{p^+}\omega^{(k)} \in W_0^{1,p(\cdot)}(B_t) \cap L^\infty(B_t)$, so we can take φ as a test function in (1.9). Substituting φ into (1.9) we have

$$\int_{A_{k,t}} \eta^{p^+} A(x, u, \nabla u)\nabla\omega \,\mathrm{d}x = -p^+ \int_{A_{k,t}} \eta^{p^+-1}\nabla\eta A(x, u, \nabla u)(\omega - k)\,\mathrm{d}x - \int_{A_{k,t}} B(x, u, \nabla u)(\omega - k)\eta^{p^+}\,\mathrm{d}x. \tag{2.5}$$

Recalling (2.1), (2.2), (2.4), (1.8) and using the Young inequality and Hölder inequality and taking ε such that $2p^+b_0(M)\varepsilon = \frac{1}{3}a_0(M)$, we have

$$\pm \int_{A_{k,t}} \eta^{p^+} A(x,u,\nabla u)\nabla\omega \, \mathrm{d}x \geqslant a_0(M) \int_{A_{k,t}} \eta^{p^+} |\nabla\omega|^{p(x)} \, \mathrm{d}x - 2a_1(M)\|a_2\|_{m(\cdot)} |A_{k,t}|^{1-1/m^-_{B_R}}, \tag{2.6}$$

where the upper or lower sign be taken according to whether ω is $+u$ or $-u$, respectively,

$$\begin{aligned}\Big|p^+ \int_{A_{k,t}} & \eta^{p^+-1}\nabla\eta A(x,u,\nabla u)(\omega-k)\,\mathrm{d}x\Big| \\ &\leqslant 2p^+b_0(M)\int_{A_{k,t}} \Big(\varepsilon\eta^{p^+}|\nabla\omega|^{p(x)} + (\varepsilon^{1-p(x)}+1)\Big|\frac{\omega-k}{t-\tau}\Big|^{p(x)} + b_1^{p'(x)}(x)\Big)\,\mathrm{d}x \\ &\leqslant \frac{a_0(M)}{3}\int_{A_{k,t}} \eta^{p^+}|\nabla\omega|^{p(x)}\,\mathrm{d}x + C(p^+,a_0(M),b_0(M))\int_{A_{k,t}}\Big|\frac{\omega-k}{t-\tau}\Big|^{p(x)}\,\mathrm{d}x \\ &\quad + C(p^+,p^-,b_0(M),\|b_1\|_{p'(\cdot)m(\cdot)})|A_{k,t}|^{1-1/m^-_{B_R}}\end{aligned} \tag{2.7}$$

and using the fact that $0 \leqslant \omega(x)-k \leqslant \delta M \leqslant a_0(M)/(3\Lambda(M))$ for $x\in A_{k,t}$,

$$\Big|\int_{A_{k,t}} B(x,u,\nabla u)(\omega-k)\eta^{p^+}\,\mathrm{d}x\Big| \leqslant \frac{a_0(M)}{3}\int_{A_{k,t}}\eta^{p^+}|\nabla\omega|^{p(x)}\,\mathrm{d}x + C(a_0(M),\|d\|_{m(\cdot)})|A_{k,t}|^{1-1/m^-_{B_R}}. \tag{2.8}$$

Combining (2.6)–(2.8) with (2.5), we get

$$\int_{A_{k,\tau}} |\nabla\omega|^{p(x)}\,\mathrm{d}x \leqslant \gamma\int_{A_{k,t}}\Big|\frac{\omega-k}{t-\tau}\Big|^{p(x)}\,\mathrm{d}x + \gamma_1|A_{k,t}|^{1-1/m^-_{B_R}}.$$

The lemma is proved. □

Let m satisfy (1.15). Then, obviously, $p(x)m(x) > n$ holds for any $x\in\overline{\Omega}$. Therefore, if p and m are both continuous on $\overline{\Omega}$, then there exists a positive constant ε_0 such that

$$p(x)m(x) > n + 2\varepsilon_0$$

for any $x\in\overline{\Omega}$ and so we can take a $R_2>0$ satisfying

$$p^-_{\Omega_{R_2}} m^-_{\Omega_{R_2}} > n+\varepsilon_0 \tag{2.9}$$

for any Ω_{R_2} with $\Omega_{R_2}\neq\emptyset$.

Theorem 2.5. *Suppose that $p \in C(\overline{\Omega})$ is log-Hölder continuous in Ω. Let A and B satisfy all the conditions of Lemma 2.4 and let u be a bounded weak solution of (1.1) such that $\sup\limits_{\Omega} |u| \leqslant M$. Then $u \in C^{0,\alpha_1}_{\rm loc}(\Omega)$ and for any open set $\Omega' \Subset \Omega$ the estimate*

$$\|u\|_{C^{0,\alpha_1}(\overline{\Omega'})} \leqslant C$$

holds, where $\alpha_1 \in (0,1)$ and C depend on n, $p(\cdot)$, $m(\cdot)$, $a_0(M)$, $b_0(M)$ and M. Moreover, C depends also on $a_1(M)$, $\|a_2\|_{m(\cdot)}$, $\|b_1\|_{p'(\cdot)m(\cdot)}$, $\|d\|_{m(\cdot)}$ and $\operatorname{dist}(\Omega', \partial\Omega)$.

Proof. Let $R_0 \in (0,1)$ be so small that $R_0 \leqslant \min\{R_1, R_2\}$, $R_0 < \operatorname{dist}(\Omega', \partial\Omega)$ and $|B_{R_0}| \leqslant 1$. Then we have $p^-_{\Omega_{R_0}} m^-_{\Omega_{R_0}} > n + \varepsilon_0$ with ε_0 as in (2.9). Let $0 < R \leqslant \frac{1}{8}R_0$ and $\Omega'_R \neq \emptyset$, where $\Omega'_R = \Omega' \cap B_R(y)$. Since $B_R(y) \Subset \Omega$ and $|B_R(y)| \leqslant 1$, from Lemma 2.4 it follows that $u \in \mathfrak{B}_{p(\cdot)}(B_R(y), M, \gamma, \gamma_1, \delta, 1/m^-_{B_R})$, where γ, γ_1 and δ are as in Lemma 2.4. Therefore, using Lemma 2.3 with $R' = \frac{1}{8}R_0$, we get

$$\operatorname*{osc}_{\Omega'_R(y)} u \leqslant \operatorname*{osc}_{B_R(y)} u \leqslant C\Big(\frac{R_0}{8}\Big)^{-\alpha_1} R^{\alpha_1},$$

where α_1 and C are as in Lemma 2.3. The theorem is proved. $\square$

We next establish the global Hölder continuity. We say that Ω satisfies an exterior cone condition at a point $z \in \partial\Omega$ if there exists a finite right circular cone V_z with a vertex z such that $\overline{\Omega} \cap V_z = z$, in particular, say that Ω satisfies a uniform exterior cone condition on $\partial\Omega$ if Ω satisfies an exterior cone condition at every $z \in \partial\Omega$ and the cones V_z are all congruent to some fixed cone V (see [21] or [37]). We can obtain the following lemma by the same method used in the proof of Lemma 4.10 in [37], but with only minor modification as in Lemma 2.3.

Lemma 2.6. *Let a variable exponent p and numbers p_0, R_0, ε_0 and r be as in Lemma 2.3 and let Ω satisfy an exterior cone condition at $z \in \partial\Omega$. Let $B_{R'}(z) \subset B_{R_0}(x_0)$ and $u \in W^{1,p(\cdot)}(\Omega_{R'}(z)) \cap L^\infty(\Omega_{R'}(z))$ with $\sup\limits_{\Omega_{R'}} |u| \leqslant M$. Suppose that, for any $R \leqslant R'$, $u \in \mathfrak{B}_{p(\cdot)}((\partial\Omega)_R(z), M, \gamma, \gamma_1, \delta, 1/r)$ and $\operatorname*{osc}\limits_{(\partial\Omega)_R(z)} u \leqslant \beta_0 R^{\alpha_0}$, where α_0 and β_0 are given positive constants. Then there exists a constant $s = s(n, \gamma, p^+, p^-, L_2, \varepsilon_0, V_z) > 2$ such that, for arbitrary $0 < R \leqslant R'$,*

$$\operatorname*{osc}_{\Omega_R(z)} u \leqslant C R'^{-\alpha_1} R^{\alpha_1},$$

where L_2 is as in (2.3) and

$$C = 4\max\Big\{\frac{(\omega_n+1)(\gamma+1)+\gamma_1}{\gamma} l 2^s R'^{\varepsilon}, \operatorname*{osc}_{\Omega_{R'}} u, 4\beta_0 R'^{\varepsilon}\Big\}, \quad \varepsilon = \min\Big\{\alpha_0, \frac{\varepsilon_0}{n+\varepsilon_0}\Big\},$$

$$\alpha_1 = \min\Big\{\alpha_0, \frac{\varepsilon_0}{n+\varepsilon_0}, -\log_4(1 - l^{-1}2^{-s})\Big\}, \quad l = \max\Big\{4, \frac{2}{\delta}\Big\}.$$

Analogously to Lemma 2.4 we have the following lemma.

Lemma 2.7. *Let $p \in C(\overline{\Omega})$ satisfy* (1.3) *and be log-Hölder continuous in Ω and let Ω satisfy a uniform exterior cone condition on $\partial\Omega$. Let A and B satisfy all the conditions of Lemma* 2.4. *If u is a bounded solution of the Dirichlet problem* (1.1), (1.2) *with $g \in L^\infty(\partial\Omega)$ such that $\sup\limits_{\Omega}|u| \leqslant M$, then $u \in \mathfrak{B}_{p(\cdot)}((\partial\Omega)_R(z), M, \gamma, \gamma_1, \delta, 1/(m^-_{\Omega_R(z)}))$ for any $R > 0$ with $|B_R(z)| \leqslant 1$ and for all $z \in \partial\Omega$, where γ, γ_1, δ are the same as in Lemma* 2.4.

P r o o f. Let $0 < \tau < t \leqslant R$ and $z \in \partial\Omega$. Setting $\varphi = \eta^{p^+}\omega^{(k)}$, from the conditions on p and Ω we have $\varphi \in W_0^{1,p(\cdot)}(\Omega_t(z)) \cap L^\infty(\Omega_t(z))$ for $k \geqslant \max\Big\{\sup\limits_{\Omega_t(z)} \omega - \delta M, \sup\limits_{(\partial\Omega)_t(z)} \omega\Big\}$, where η is a function as in the proof of Lemma 2.4 when $x_0 = z$. Then we can take φ as a test function in (1.9) and the lemma is proved similarly to proof of Lemma 2.4. □

Next, by combining Theorem 2.5, Lemmas 2.6 and 2.7 we have the following global Hölder continuity for bounded weak solutions of the Dirichlet problem (1.1), (1.2).

Theorem 2.8. *Let all the conditions of Lemma* 2.7 *be satisfied and Ω satisfy a uniform exterior cone condition on $\partial\Omega$. If u is a bounded solution of the Dirichlet problem* (1.1), (1.2) *with $g \in C^{0,\alpha_0}(\partial\Omega)$, then $u \in C^{0,\alpha_1}(\overline{\Omega})$ and*

$$\|u\|_{C^{0,\alpha_1}(\overline{\Omega})} \leqslant C,$$

where α_1 and C depend on n, $p(\cdot)$, $m(\cdot)$, $\|u\|_{\infty,\Omega}$, $a_0(\|u\|_\infty)$, $b_0(\|u\|_\infty)$ and Ω and, moreover, α_1 depends also on α_0, and C depends also on $\|g\|_{C^{0,\alpha_0}(\partial\Omega)}$, $a_1(\|u\|_\infty)$, $\|a_2\|_{m(\cdot)}$, $\|b_1\|_{p'(\cdot)m(\cdot)}$ and $\|d\|_{m(\cdot)}$.

According to [35], we have the following lemma.

Lemma 2.9. *Let A satisfy Assumption* (H2) *except* (1.7) *and p satisfy* (1.3). *Then we have the inequalities* (2.1), (2.2) *with*

$$a_0(|z|) = \frac{1}{2}4^{-p^+}\lambda(|z|),$$
$$a_1(|z|) = \Big(2\Big(\frac{4^{p^+}}{\lambda(|z|)} + 1\Big)\Big)^{p^+} + \Big(2\Big(\frac{4^{p^+}}{\lambda(|z|)} + 1\Big)\Big)^{1/(p^- -1)}(\Lambda(|z|))^{p^-/(p^- -1)},$$
$$a_2(x) = b^{p'(x)}(x) + 1,$$
$$b_0(|z|) = \max\Big\{\frac{1}{(p^- - 1)}, 2^{p^+-2}\Big\}\Lambda(|z|), \quad b_1(x) = b(x) + 1,$$

and

$$(2.10)\qquad (A(x,z,\eta)-A(x,z,\eta'))(\eta-\eta') \geqslant \begin{cases} \lambda_1(|z|)|\eta-\eta'|^{p(x)} & \text{if } p(x)\geqslant 2,\\ \lambda_1(|z|)(k+|\eta|^2+|\eta'|^2)^{(p(x)-2)/2}|\eta-\eta'|^2 & \text{if } 1<p(x)<2 \end{cases}$$

with $\lambda_1(|z|) = 4^{-p^+}\lambda(|z|)$.

P r o o f. We first prove (2.10) similarly to the proof of [35], Lemma 1. Without loss of generality, we may suppose that $|\eta|\leqslant|\eta'|$. From (1.5) we have

$$\begin{aligned} A(x,z,\eta)-A(x,z,\eta')(\eta-\eta') &= \int_0^1 \sum_{i,j=1}^n \frac{\partial A_j}{\partial \eta_i}(x,z,\eta'+t(\eta-\eta'))(\eta_i-\eta_i')(\eta_j-\eta_j')\,\mathrm{d}t\\ &\geqslant \lambda(|z|)\int_0^{1/4}(k+|\eta'+t(\eta-\eta')|^2)^{(p(x)-2)/2}|\eta-\eta'|^2)\,\mathrm{d}t. \end{aligned}$$

Therefore, using

$$\frac{1}{16}|\eta-\eta'|^2\leqslant k+|\eta'+t(\eta-\eta')|^2\leqslant k+|\eta|^2+|\eta'|^2 \quad \forall\, t\in\Big[0,\frac14\Big],$$

we arrive at

$$(A(x,z,\eta)-A(x,z,\eta'))(\eta-\eta') \geqslant \begin{cases} \Big(\dfrac14\Big)^{p(x)-1}\lambda(|z|)|\eta-\eta'|^{p(x)} & \text{if } p(x)\geqslant 2,\\ \dfrac14\lambda(|z|)(k+|\eta|^2+|\eta'|^2)^{(p(x)-2)/2}|\eta-\eta'|^2 & \text{if } 1<p(x)<2, \end{cases}$$

from which follows (2.10) with $\lambda_1(|z|) = \min\{(\frac14)^{p^+-1},\frac14\}\lambda(|z|)$. In order to prove (2.1), putting $\eta'=0$ in (2.10), we have

$$A(x,z,\eta)\eta\geqslant \begin{cases} \lambda_1(|z|)|\eta|^{p(x)}+A(x,z,0)\eta & \text{if } p(x)\geqslant 2,\\ \lambda_1(|z|)(k+|\eta|^2)^{(p(x)-2)/2}|\eta|^2+A(x,z,0)\eta & \text{if } 1<p(x)<2. \end{cases}$$

Using (1.4) and the Young inequality, for any $\varepsilon>0$ we get

$$|A(x,z,0)\eta|\leqslant\varepsilon|\eta|^{p(x)}+\varepsilon^{-1/(p(x)-1)}(\Lambda(|z|))^{p^-/(p^--1)}b^{p'(x)}(x).$$

Therefore, we have

$$A(x,z,\eta)\eta\geqslant(\lambda_1(|z|)-\varepsilon)|\eta|^{p(x)}-\varepsilon^{-1/(p(x)-1)}(\Lambda(|z|))^{p^-/(p^--1)}b^{p'(x)}(x)$$

when $p(x) \geqslant 2$, and using the Young inequality,

$$\begin{aligned}A(x,z,\eta)\eta &\geqslant \lambda_1(|z|)(1+|\eta|^2)^{(p(x)-2)/2}|\eta|^2 - \varepsilon|\eta|^{p(x)} \\ &\quad - \varepsilon^{-1/(p(x)-1)}(\Lambda|z|)^{p^-/(p^- -1)}b^{p'(x)}(x) \\ &\geqslant \lambda_1(|z|)|\eta|^{p(x)}\frac{|\eta|}{\sqrt{1+|\eta|^2}} - \varepsilon|\eta|^{p(x)} \\ &\quad - \varepsilon^{-1/(p(x)-1)}(\Lambda(|z|))^{p^-/(p^- -1)}b^{p'(x)}(x) \\ &\geqslant \lambda_1(|z|)|\eta|^{p(x)} - \lambda_1(|z|)\frac{|\eta|^{p(x)}}{\sqrt{1+|\eta|^2}} - \varepsilon|\eta|^{p(x)} \\ &\quad - \varepsilon^{-1/(p(x)-1)}(\Lambda(|z|))^{p^-/(p^- -1)}b^{p'(x)}(x) \\ &\geqslant (\lambda_1(|z|)(1-\varepsilon)-\varepsilon)|\eta|^{p(x)} - \varepsilon^{-(p(x)-1)} \\ &\quad - \varepsilon^{-1/(p(x)-1)}(\Lambda(|z|))^{p^-/(p^- -1)}b^{p'(x)}(x)\end{aligned}$$

where $1 < p(x) < 2$. Taking ε such that $\lambda_1(|z|)(1-\varepsilon)-\varepsilon = \frac{1}{2}\lambda_1(|z|)$, from the above inequalities we arrive at (2.1). Using the equality

$$A_j(x,z,\eta) - A_j(x,z,0) = \int_0^1 \sum_{i=1}^n \frac{\partial A_j}{\partial \eta_i}(x,z,t\eta)\eta_i \,\mathrm{d}t,$$

condition (1.4), the Schwarz inequality and (1.6), we have

$$\begin{aligned}|A(x,z,\eta)| &\leqslant \Lambda(|z|)b(x) + \int_0^1 \sum_{i,j=1}^n \Big|\frac{\partial A_j}{\partial \eta_i}(x,z,t\eta)\Big| \,\mathrm{d}t|\eta| \\ &\leqslant \Lambda(|z|)b(x) + \Lambda(|z|)|\eta|\int_0^1 (k+t^2|\eta|^2)^{(p(x)-2)/2}\,\mathrm{d}t \\ &\leqslant \max\Big\{\frac{1}{p^- -1}, 2^{p^+ -2}\Big\}\Lambda(|z|)(|\eta|^{p(x)-1} + b(x) + 1),\end{aligned}$$

since

$$\int_0^1 (k+t^2|\eta|^2)^{(p(x)-2)/2}\,\mathrm{d}t|\eta| \leqslant \int_0^1 t^{p^- -2}\,\mathrm{d}t|\eta|^{p(x)-1} = \frac{|\eta|^{p(x)-1}}{p^- -1}$$

when $1 < p(x) < 2$ and

$$\begin{aligned}\int_0^1 (k+t^2|\eta|^2)^{(p(x)-2)/2}\,\mathrm{d}t|\eta| &\leqslant \int_0^1 (1+t^2|\eta|^2)^{(p(x)-2)/2}\,\mathrm{d}t(1+|\eta|^2)^{1/2} \\ &\leqslant (1+|\eta|^2)^{(p(x)-1)/2} \leqslant (1+|\eta|)^{p(x)-1} \\ &\leqslant 2^{p^+ -2}(|\eta|^{p(x)-1}+1)\end{aligned}$$

when $p(x) \geqslant 2$. Therefore, we obtain (2.2) and the lemma is proved. □

R e m a r k 2.10. Let Assumption (H2) except equation (1.7) be satisfied with $b \in L^{p'(\cdot)m(\cdot)}(\Omega)$. Then Lemma 2.9 shows that the coefficient A satisfies the conditions (2.1)–(2.4) with a_0, a_1, a_2, b_0 and b_1 as in Lemma 2.9.

R e m a r k 2.11. For given $\delta > 0$, from

$$\lim_{t\to\infty}\frac{\log t}{t^\delta} = 0 \quad \text{where} \lim_{t\to 0^+} t^\delta \log t = 0$$

it follows that there is a positive constant $C(\delta)$ depending only on δ such that for every $|\eta| > 0$

$$|\log(k+|\eta|^2)| \leqslant C(\delta) + (k+|\eta|^2)^\delta + (k+|\eta|^2)^{-\delta}. \tag{2.11}$$

Therefore, by (1.7) and (2.11) we obtain

$$\begin{aligned} &|A(x^1,z^1,\eta) - A(x^2,z^2,\eta)| \\ &\qquad \leqslant \Lambda_p(\max\{|z^1|,|z^2|\})(|x^1-x^2|^{\beta_1} + |z^1-z^2|^{\beta_2})(1+|\eta|^{\overline{p}-1+2\delta}), \end{aligned} \tag{2.12}$$

where $\Lambda_p(t) = C(p^-)\Lambda(t)$, $\overline{p} = \max\{p(x^1), p(x^2)\}$ and $\delta > 0$ is a number such that $\delta < \frac{1}{2}(p^- - 1)$.

3. Proof of Theorem 1.2

To prove Theorem 1.2, we first need a new result on the higher integrability for the bounded weak solutions of (1.1), which is stated in the following lemma, since the known results in this field are not applicable for the case of the conditions (2.1)–(2.2) and (2.4).

Lemma 3.1. *Let p be log-Hölder continuous in Ω and satisfy* (1.3) *and let the coefficients A and B satisfy* (2.1), (2.2), (2.4) *and* (H3). *Let u be a bounded weak solution of* (1.1) *with $\sup\limits_\Omega |u| \leqslant M$. Then, given an open set $\Omega' \Subset \Omega$, there exist positive constants $R_0 = R_0(n, p(\cdot), a_0(M), \Lambda(M), \operatorname{dist}(\Omega', \partial\Omega))$, $C = C(n, p(\cdot), a_0(M), a_1(M), b_0(M), \Lambda(M), M)$ and $\delta_0 = \delta_0(n, p(\cdot), m(\cdot), a_0(M), \Lambda(M), M) \in (0, m^- - 1]$ such that for every ball $B_{2R} \subset \Omega'$ with $R \in (0, R_0]$ and for any $\delta \in (0, \delta_0]$ it holds*

$$\fint_{B_R} |\nabla u|^{(1+\delta)p(x)}\,\mathrm{d}x \leqslant C\biggl(\biggl(\fint_{B_{2R}} |\nabla u|^{p(x)}\,\mathrm{d}x\biggr)^{1+\delta} + \fint_{B_{2R}} |f|^{1+\delta}\,\mathrm{d}x\biggr), \tag{3.1}$$

where $\fint_E \omega\,\mathrm{d}x = |E|^{-1}\int_E \omega\,\mathrm{d}x$, $f(x) = a_2(x) + b_1^{p'(x)}(x) + d(x) + 1$.

P r o o f. Since u is continuous on $\overline{\Omega'}$ by Theorem 2.5, there is $R_1 > 0$ such that

$$|u(x^1) - u(x^2)| \leqslant \frac{a_0(M)}{3\Lambda(M)} \quad \forall\, x^1, x^2 \in \Omega' \text{ with } |x^1 - x^2| < 4R_1. \tag{3.2}$$

Consider the concentric balls $B_R(x_0) \subset B_{2R}(x_0) \subset \Omega'$ with $R \leqslant R_1$. Take $\zeta \in C_0^1(B_{2R})$ such that $\zeta = 1$ on B_R and $0 \leqslant \zeta \leqslant 1$, $|\nabla\zeta| \leqslant 4/R$ on B_{2R}. Set $(u)_{2R} = |B_{2R}|^{-1} \int_{B_{2R}} u \,\mathrm{d}x$. Putting $\varphi = \zeta^{p^+}(u - (u)_{2R})$ and using (2.1), (2.2), (2.4) and the Young inequality, we obtain

$$\begin{aligned}
&\int_{B_{2R}} A(x,u,\nabla u)\nabla\varphi \,\mathrm{d}x \\
&\quad = \int_{B_{2R}} (\zeta^{p^+} A(x,u,\nabla u)\nabla u + p^+\zeta^{p^+-1}(u-(u)_{2R})\nabla\zeta A(x,u,\nabla u)) \,\mathrm{d}x \\
&\quad \geqslant a_0(M)\int_{B_{2R}} \zeta^{p^+}|\nabla u|^{p(x)} \,\mathrm{d}x - a_1(M)\int_{B_{2R}} a_2(x)\,\mathrm{d}x \\
&\qquad - p^+ b_0(M)\frac{4}{R}\int_{B_{2R}} |u-(u)_{2R}|\zeta^{p^+-1}|\nabla u|^{p(x)-1}\,\mathrm{d}x \\
&\qquad - p^+ b_0(M)\frac{4}{R}\int_{B_{2R}} |u-(u)_{2R}| b_1(x)\,\mathrm{d}x \\
&\quad \geqslant a_0(M)\int_{B_{2R}} \zeta^{p^+}|\nabla u|^{p(x)} \,\mathrm{d}x - a_1(M)\int_{B_{2R}} a_2(x)\,\mathrm{d}x \\
&\qquad - 4p^+ b_0(M)\int_{B_{2R}} \Big(\varepsilon\zeta^{p^+}|\nabla u|^{p(x)} + \varepsilon^{-(p(x)-1)}\Big|\frac{u-(u)_{2R}}{R}\Big|^{p(x)}\Big)\,\mathrm{d}x \\
&\qquad - 4p^+ b_0(M)\int_{B_{2R}} \Big(\Big|\frac{u-(u)_{2R}}{R}\Big|^{p(x)} + b_1^{p'(x)}(x)\Big)\,\mathrm{d}x.
\end{aligned}$$

Choosing ε so that $4p^+b_0(M)\varepsilon = \frac{1}{3}a_0(M)$, we get

$$\begin{aligned}
&\int_{B_{2R}} A(x,u,\nabla u)\nabla\varphi \,\mathrm{d}x \\
&\qquad \geqslant \frac{2}{3}a_0(M)\int_{B_{2R}} \zeta^{p^+}|\nabla u|^{p(x)}\,\mathrm{d}x - C(p^+,p^-,a_0(M),a_1(M),b_0(M)) \\
&\qquad\quad \times \int_{B_{2R}} \Big(1 + a_2(x) + b_1^{p'(x)}(x) + \Big|\frac{u-(u)_{2R}}{R}\Big|^{p(x)}\Big)\,\mathrm{d}x.
\end{aligned} \tag{3.3}$$

By (1.8) and (3.2), we have

$$\begin{aligned}
\Big|\int_{B_{2R}} B(x,u,\nabla u)\varphi \,\mathrm{d}x\Big| &\leqslant \int_{B_{2R}} \zeta^{p^+}|u-(u)_{2R}|\Lambda(M)(|\nabla u|^{p(x)} + d(x))\,\mathrm{d}x \\
&\leqslant \frac{1}{3}a_0(M)\Big(\int_{B_{2R}} \zeta^{p^+}|\nabla u|^{p(x)}\,\mathrm{d}x + \int_{B_{2R}} d(x)\,\mathrm{d}x\Big).
\end{aligned} \tag{3.4}$$

Taking the above φ as a test function in (1.9), by (3.3) and (3.4), we obtain

$$\fint_{B_R} |\nabla u|^{p(x)}\,\mathrm{d}x \leqslant C \fint_{B_{2R}} \Big(\Big|\frac{u-(u)_{2R}}{R}\Big|^{p(x)} + f(x)\Big)\,\mathrm{d}x. \tag{3.5}$$

Now, we choose $\varepsilon > 0$ so small that $\varepsilon < \min\{p^- - 1, 1/(n-1)\}$. By absolute continuity of the Lebesgue integral there exists $R_0 \in (0, R_1]$ such that for every ball $B_{2R} \subset \Omega$ with $0 < R \leqslant R_0$ it holds $\|\nabla u\|_{L^{p(\cdot)/(1+\varepsilon)}(B_{2R}} \leqslant 1$, where R_0 depends on n, $p(\cdot)$, $a_0(M)$, $\Lambda(M)$ and $\operatorname{dist}(\Omega', \partial\Omega)$. Therefore, from Proposition 8.2.11 of [13] we have

$$\fint_{B_{2R}} \Big|\frac{u-(u)_{2R}}{R}\Big|^{p(x)}\,\mathrm{d}x \leqslant C\Big(\fint_{B_{2R}} |\nabla u|^{p(x)/(1+\varepsilon)}\,\mathrm{d}x\Big)^{1+\varepsilon} + C, \tag{3.6}$$

where $C = C(n, p(\cdot))$. Substituting (3.6) into (3.5) we get

$$\fint_{B_R} |\nabla u|^{p(x)}\,\mathrm{d}x \leqslant C\Big(\fint_{B_{2R}} |\nabla u|^{p(x)/(1+\varepsilon)}\,\mathrm{d}x\Big)^{1+\varepsilon} + C\fint_{B_{2R}} f(x)\,\mathrm{d}x, \tag{3.7}$$

where $C = C(n, p(\cdot), a_0(M), a_1(M), b_0(M), \Lambda(M), M)$.

Since $f \in L^{m^-}(\Omega)$, by the Gehring lemma (see [14], Theorem 3.7) and (3.7) there exists a number $\delta_0 \in (0, m^- - 1]$ depending on n, C, ε and m^- such that

$$\Big(\fint_{B_R} |\nabla u|^{(1+\delta)p(x)}\,\mathrm{d}x\Big)^{1/(1+\delta)} \leqslant C\fint_{B_{2R}} |\nabla u|^{p(x)}\,\mathrm{d}x + C\Big(\fint_{B_{2R}} |f|^{1+\delta}\,\mathrm{d}x\Big)^{1/(1+\delta)}$$

for all $\delta \in (0, \delta_0]$, all $R \in (0, R_0]$, which completes the proof of Lemma 3.1. □

In the rest of this section we suppose that Assumptions (H1)–(H3), (1.10) and (1.3) are satisfied. Since $u \in C^{0,\alpha_1}_{\mathrm{loc}}(\Omega)$, for any open set $\Omega' \Subset \Omega$ there is $L_3 > 0$ depending on n, $p(\cdot)$, $m(\cdot)$, $\lambda(M)$, $\Lambda(M)$, M, $\|b\|_{p'(\cdot)m(\cdot)}$, $\|d\|_{m(\cdot)}$ and $\operatorname{dist}(\Omega', \partial\Omega)$ such that

$$|u(x^1) - u(x^2)| \leqslant L_3 |x^1 - x^2|^{\alpha_1} \quad \forall x^1, x^2 \in \Omega'. \tag{3.8}$$

Let $\Omega' \Subset \Omega$ and $B_{4R_1}(x_0) \subset \Omega'$. Let R_0 and δ_0 be as in Lemma 3.1. Without loss of generality, we may assume that $|B_{R_0}| \leqslant 1$ and $R_0 \leqslant 1$. Let $\delta \in (0, \delta_0]$ and R_1 be so small that $R_1 \leqslant \frac{1}{2}R_0$, $\int_{B_{2R_1}} |\nabla u|^{p(x)}\,\mathrm{d}x \leqslant 1$ and

$$p^+_{B_{2R_1}(x_0)}\Big(1 + \frac{\delta}{2}\Big) \leqslant p^-_{B_{2R_1}(x_0)}(1+\delta). \tag{3.9}$$

It is clear that R_1 is non-decreasing in δ. We have $|\nabla u| \in L^{p^+_{B_{2R_1}(x_0)}(1+\delta/2)}(B_{2R_1}(x_0))$ from Lemma 3.1 and (3.9). Let $B_R := B_R(x_c) \subset B_{2R}$ be two concentric balls in

$B_{2R_1}(x_0)$, not necessarily concentric with $B_{2R_1}(x_0)$. Put $p_*(R) = p^+_{B_{2R}}$ and let $x_* \in \overline{B}_{2R}$ be such that $p(x_*) = p_*(R)$. Define $\overline{A}(\eta) = A(x_*, u(x_*), \eta)$. Defining

$$h(t) = \lambda_0 (k + t^2)^{(p(x_*)-2)/2} t,$$

from (1.5) and (1.6) we have

$$\sum_{i,j=1}^{n} \frac{\partial \overline{A}_j}{\partial \eta_i}(\eta)\xi_i\xi_j \geqslant \frac{h(|\eta|)}{|\eta|}|\xi|^2, \quad \sum_{i,j=1}^{n} \Big|\frac{\partial \overline{A}_j}{\partial \eta_i}(\eta)\Big| \leqslant \frac{\Lambda_0}{\lambda_0}\frac{h(|\eta|)}{|\eta|},$$

where $\lambda_0 = \lambda(M)$, $\Lambda_0 = \Lambda(M)$, and

$$\min\{p(x_*) - 1, 1\} \leqslant \frac{th'(t)}{h(t)} \leqslant \max\{p(x_*) - 1, 1\}.$$

Setting

$$H(t) = \int_0^t h(\tau)\,\mathrm{d}\tau,$$

we have

$$H(t) = \frac{\lambda_0}{p(x_*)}((k + t^2)^{p(x_*)/2} - k^{p(x_*)/2}).$$

So, from Lemma 3.1 and (3.9) it holds $\int_{B_R} H(|\nabla u|)\,\mathrm{d}x < \infty$ and $u \in C^{0,\alpha_1}(\overline{B}_R)$, since $u \in C^{0,\alpha_1}_{\mathrm{loc}}(\Omega)$. Therefore, by Theorem 1.7 and Lemma 5.2 of [28] we have the following result for Dirichlet problem,

$$\begin{cases} \operatorname{div} \overline{A}(\nabla v) = 0 & \text{in } B_R, \\ v = u & \text{on } \partial B_R. \end{cases} \tag{3.10}$$

For brevity, we write p_* instead of $p(x_*)$.

Lemma 3.2. *There is a unique solution v of the problem* (3.10) *such that $v \in W^{1,p_*}(B_R) \cap C^{1,\alpha_2}_{\mathrm{loc}}(B_R) \cap C(\overline{B_R})$ and*

$$\sup_{B_{R/2}} |\nabla v|^{p_*} \leqslant CR^{-n}\Big(\int_{B_R} |\nabla v|^{p_*}\,\mathrm{d}x + R^n\Big), \tag{3.11}$$

$$\int_{B_\varrho} |\nabla v - (\nabla v)_\varrho|^{p_*}\,\mathrm{d}x \tag{3.12}$$

$$\leqslant \begin{cases} C\varrho^n\Big(\Big(\dfrac{\varrho}{R}\Big)^{\alpha_2 p_*/2}\Big(\displaystyle⨍_{B_R} |\nabla v - (\nabla v)_R|^{p_*}\,\mathrm{d}x\Big)^{p_*/2} \\ \qquad + \Big(\dfrac{\varrho}{R}\Big)^{\alpha_2}\displaystyle⨍_{B_R} |\nabla v - (\nabla v)_R|^{p_*}\,\mathrm{d}x\Big) & \textit{if } 1 < p_* < 2, \\ C\varrho^n\Big(\dfrac{\varrho}{R}\Big)^{\alpha_2}\Big(\displaystyle⨍_{B_R} |\nabla v - (\nabla v)_R|^{p_*}\,\mathrm{d}x + 1\Big) & \textit{if } p_* \geqslant 2 \end{cases}$$

for any $\varrho \in (0, R)$,

$$\int_{B_R} |\nabla v|^{p_*} \, \mathrm{d}x \leqslant C \int_{B_R} (|\nabla u|^{p_*} + 1) \, \mathrm{d}x, \tag{3.13}$$

$$\sup_{B_R} |u - v| \leqslant \operatorname*{osc}_{B_R} u, \tag{3.14}$$

where $\alpha_2 \in (0,1)$ *and* C *depends on* n, p_*, λ_0 *and* Λ_0.

Proof. By Lemma 1.1, Theorem 1.7 and Lemma 5.2 of [28] there is a unique solution $v \in W^{1,p_*}(B_R) \cap C^{1,\alpha_2}_{\mathrm{loc}}(B_R) \cap C(\overline{B}_R)$ of (3.10), which satisfies

$$\sup_{B_{R/2}} ((k + |\nabla v|^2)^{(p_*-2)/2} |\nabla v|^2) \leqslant C R^{-n} \int_{B_R} (k + |\nabla v|^2)^{(p_*-2)/2} |\nabla v|^2 \, \mathrm{d}x, \tag{3.15}$$

$$\begin{aligned} &\fint_{B_\varrho} (k + |\nabla v - (\nabla v)_\varrho|^2)^{(p_*-2)/2} |\nabla v - (\nabla v)_\varrho|^2 \, \mathrm{d}x \\ &\quad \leqslant C \Big(\frac{\varrho}{R}\Big)^{\alpha_2} \fint_{B_R} (k + |\nabla v - (\nabla v)_R|^2)^{(p_*-2)/2} \\ &\qquad \times |\nabla v - (\nabla v)_R|^2 \, \mathrm{d}x \quad \text{for } 0 < \varrho < R, \end{aligned} \tag{3.16}$$

$$\int_{B_R} (k + |\nabla v|^2)^{(p_*-2)/2} |\nabla v|^2 \, \mathrm{d}x \leqslant C \int_{B_R} (1 + (k + |\nabla u|^2)^{(p_*-2)/2} |\nabla u|^2) \, \mathrm{d}x, \tag{3.17}$$

and (3.14), where α_2 and C depend on n, p_*, λ_0, and Λ_0. In order to prove (3.11) we first assume that $1 < p_* < 2$. Obviously,

$$\int_{B_R} (k + |\nabla v|^2)^{(p_*-2)/2} |\nabla v|^2 \, \mathrm{d}x \leqslant \int_{B_R} |\nabla v|^{p_*} \, \mathrm{d}x. \tag{3.18}$$

Since $1 < p_* < 2$, we get

$$\begin{aligned} \sup_{B_{R/2}} ((k + |\nabla v|^2)^{(p_*-2)/2} |\nabla v|^2) &\geqslant \sup_{B_{R/2}} ((1 + |\nabla v|^2)^{(p_*-2)/2} |\nabla v|^2) \\ &= \sup_{B_{R/2}} \Big(\frac{1 + |\nabla v|^2}{(1 + |\nabla v|^2)^{(2-p_*)/2}} - \frac{1}{(1 + |\nabla v|^2)^{(2-p_*)/2}} \Big) \\ &\geqslant \sup_{B_{R/2}} (1 + |\nabla v|^2)^{p_*/2} - 1 > \sup_{B_{R/2}} |\nabla v|^{p_*} - 1. \end{aligned}$$

Therefore we arrive at (3.11) by (3.15) and (3.18) when $1 < p_* < 2$. Next, suppose that $p_* \geqslant 2$. Obviously,

$$\sup_{B_{R/2}} |\nabla v|^{p_*} \leqslant \sup_{B_{R/2}} ((k + |\nabla v|^2)^{(p_*-2)/2} |\nabla v|^2). \tag{3.19}$$

Putting $B_R^1 = \{x \in B_R \colon \ |\nabla v(x)| \geqslant 1\}$, we have

$$\begin{aligned}
(3.20) \qquad & \int_{B_R} (k + |\nabla v|^2)^{(p_*-2)/2} |\nabla v|^2 \, \mathrm{d}x \\
&\leqslant \int_{B_R^1} \Big(1 + \frac{1}{|\nabla v|^2}\Big)^{(p_*-2)/2} |\nabla v|^{p_*} \, \mathrm{d}x + \int_{B_R \setminus B_R^1} 2^{(p_*-2)/2} \, \mathrm{d}x \\
&\leqslant 2^{(p_*-2)/2} \Big(\int_{B_R} |\nabla v|^{p_*} \, \mathrm{d}x + |B_R| \Big).
\end{aligned}$$

Combining (3.15), (3.19) and (3.20), we get (3.11) when $p_* \geqslant 2$. Using (3.16), analogously above we obtain the inequality (3.12) in the case that $p_* \geqslant 2$.

In order to prove (3.12) in the case that $1 < p_* < 2$, we first note the obvious inequality

$$(3.21) \quad \fint_{B_R} (k + |\nabla v - (\nabla v)_R|^2)^{(p_*-2)/2} |\nabla v - (\nabla v)_R|^2 \, \mathrm{d}x \leqslant \fint_{B_R} |\nabla v - (\nabla v)_R|^{p_*} \, \mathrm{d}x.$$

Putting $B_\varrho^1 = \{x \in B_\varrho \colon \ |\nabla v(x) - (\nabla v)_\varrho| \geqslant 1\}$ and using (3.16), (3.21) and the Hölder inequality, we have

$$\begin{aligned}
& \int_{B_\varrho} |\nabla v - (\nabla v)_\varrho|^{p_*} \, \mathrm{d}x \\
&= \int_{B_\varrho^1} \Big(1 + \frac{k}{|\nabla v - (\nabla v)_\varrho|^2}\Big)^{(2-p_*)/2} (k + |\nabla v - (\nabla v)_\varrho|^2)^{(p_*-2)/2} |\nabla v - (\nabla v)_\varrho|^2 \, \mathrm{d}x \\
&\quad + \int_{B_\varrho \setminus B_\varrho^1} (k + |\nabla v - (\nabla v)_\varrho|^2)^{(2-p_*)p_*/4} \\
&\quad \times (k + |\nabla v - (\nabla v)_\varrho|^2)^{(p_*-2)p_*/4} |\nabla v - (\nabla v)_\varrho|^{p_*} \, \mathrm{d}x \\
&\leqslant 2 \int_{B_\varrho^1} (k + |\nabla v - (\nabla v)_\varrho|^2)^{(p_*-2)/2} |\nabla v - (\nabla v)_\varrho|^2 \, \mathrm{d}x \\
&\quad + \Big(\int_{B_\varrho \setminus B_\varrho^1} (k + |\nabla v - (\nabla v)_\varrho|^2)^{p_*/2} \, \mathrm{d}x \Big)^{1-p_*/2} \\
&\quad \times \Big(\int_{B_\varrho \setminus B_\varrho^1} (k + |\nabla v - (\nabla v)_\varrho|^2)^{(p_*-2)/2} |\nabla v - (\nabla v)_\varrho|^2 \, \mathrm{d}x \Big)^{p_*/2} \\
&\leqslant C|B_\varrho| \Big(\frac{\varrho}{R}\Big)^{\alpha_2} \fint_{B_R} (k + |\nabla v - (\nabla v)_R|^2)^{(p_*-2)/2} |\nabla v - (\nabla v)_R|^2 \, \mathrm{d}x \\
&\quad + C|B_\varrho| \Big(\frac{\varrho}{R}\Big)^{\alpha_2 p_*/2} \Big(\fint_{B_R} (k + |\nabla v - (\nabla v)_R|^2)^{(p_*-2)/2} |\nabla v - (\nabla v)_R|^2 \, \mathrm{d}x \Big)^{p_*/2}
\end{aligned}$$

$$\leqslant C\varrho^n\Big(\frac{\varrho}{R}\Big)^{\alpha_2 p_*/2}\Big(\fint_{B_R}|\nabla v-(\nabla v)_R|^{p_*}\,\mathrm{d}x\Big)^{p_*/2}$$
$$\times\Big(\Big(\Big(\frac{\varrho}{R}\Big)^{\alpha_2}\fint_{B_R}|\nabla v-(\nabla v)_R|^{p_*}\,\mathrm{d}x\Big)^{1-p_*/2}+1\Big).$$

Since the proof of (3.13) is similar to that of (3.11), we omit it. □

Lemma 3.3. *Let Assumptions* (H1)–(H3) *be fulfilled with b and d satisfying* (1.10)–(1.12) *and p satisfy* (1.3). *Let v be as mentioned in Lemma* 3.2 *and let $0<\delta\leqslant\delta_0$, where δ_0 is as in Lemma* 3.1. *Then we have*

$$\int_{B_R}|\nabla u-\nabla v|^{p_*}\,\mathrm{d}x\leqslant CR^{\beta/2}\int_{B_{2R}}(R^{-n\delta}|\nabla u|^{p(x)}+|f|^{1+\delta})\,\mathrm{d}x \tag{3.22}$$

for $B_R\subset B_{2R}\subset B_{2R_1}(x_0)$, where $C=C(n,p^+,p^-,\lambda_0,\Lambda_0,M,\mathrm{dist}(\Omega',\partial\Omega))$, and β, f and $B_{2R_1}(x_0)$ are as in (1.12), (3.1) *and* (3.9), *respectively.*

P r o o f. Put $I=\int_{B_R}(\overline{A}(\nabla u)-\overline{A}(\nabla v))(\nabla u-\nabla v)\,\mathrm{d}x$. Since v is a solution of (3.10), we have

$$\begin{aligned}I&=\int_{B_R}\overline{A}(\nabla u)(\nabla u-\nabla v)\,\mathrm{d}x\\&=\int_{B_R}(\overline{A}(\nabla u)-A(x,u,\nabla u))(\nabla u-\nabla v)\,\mathrm{d}x+\int_{B_R}A(x,u,\nabla u)(\nabla u-\nabla v)\,\mathrm{d}x\\&=I_1+I_2.\end{aligned}$$

Taking $\delta_1\in(0,1)$ such that $\delta_1<\min\{\frac12(p^--1),\frac14(p^--1)\delta\}$, by (3.9)

$$p_*\Big(1+\frac{2\delta_1}{p_*-1}\Big)\leqslant p^-_{B_{2R_1}(x_0)}(1+\delta)\leqslant p(x)(1+\delta)\quad\forall\,x\in B_{2R_1}(x_0). \tag{3.23}$$

By (2.12), (3.1), (3.8), (3.13) and (3.23), noting $\int_{B_{2R_1}(x_0)}|\nabla u|^{p(x)}\,\mathrm{d}x\leqslant 1$, we have

$$\begin{aligned}I_1&=\int_{B_R}(A(x_*,u(x_*),\nabla u(x))-A(x,u(x),\nabla u(x)))(\nabla u(x)-\nabla v(x))\,\mathrm{d}x\\&\leqslant CR^\beta\int_{B_R}(1+|\nabla u|^{p_*-1+2\delta_1})(|\nabla u|+|\nabla v|)\,\mathrm{d}x\\&\leqslant CR^\beta\int_{B_R}(|\nabla u|^{(1+\delta)p(x)}+1)\,\mathrm{d}x\leqslant CR^\beta\int_{B_{2R}}(R^{-n\delta}|\nabla u|^{p(x)}+|f|^{1+\delta})\,\mathrm{d}x.\end{aligned}$$

Since u is a bounded weak solution of (1.1), from (1.9) we have

$$I_2=\int_{B_R}A(x,u,\nabla u)(\nabla u-\nabla v)\,\mathrm{d}x=-\int_{B_R}B(x,u,\nabla u)(u-v)\,\mathrm{d}x.$$

Using (1.8), (3.8) and (3.14), we get

$$I_2 \leqslant \Lambda_0 \int_{B_R} (|\nabla u|^{p(x)} + d(x))\,\mathrm{d}x \operatorname*{osc}_{B_R} u \leqslant CR^{\alpha_1} \int_{B_R} (|\nabla u|^{p(x)} + |f|^{1+\delta})\,\mathrm{d}x,$$

and so

$$I \leqslant CR^{\beta} \int_{B_{2R}} (R^{-n\delta}|\nabla u|^{p(x)} + |f|^{1+\delta})\,\mathrm{d}x. \tag{3.24}$$

However, by (2.10) it holds

$$I \geqslant \begin{cases} \lambda_1(M) \displaystyle\int_{B_R} |\nabla u - \nabla v|^{p_*}\,\mathrm{d}x & \text{if } p_* \geqslant 2, \\ \lambda_1(M) \displaystyle\int_{B_R} (k + |\nabla u|^2 + |\nabla v|^2)^{(p_*-2)/2} |\nabla u - \nabla v|^2\,\mathrm{d}x & \text{if } 1 < p_* < 2. \end{cases} \tag{3.25}$$

Thus, it is clear that (3.22) holds when $p_* \geqslant 2$. Let $1 < p_* < 2$. By the Hölder inequality, (3.1), (3.9), (3.13), (3.24) and (3.25) we have

$$\begin{aligned} &\int_{B_R} |\nabla u - \nabla v|^{p_*}\,\mathrm{d}x \\ &\quad\leqslant \Big(\int_{B_R} (k + |\nabla u|^2 + |\nabla v|^2)^{(p_*-2)/2} |\nabla u - \nabla v|^2\,\mathrm{d}x\Big)^{1/2} \\ &\quad\quad\times \Big(\int_{B_R} (k + |\nabla u|^2 + |\nabla v|^2)^{(2-p_*)/2} |\nabla u - \nabla v|^{2(p_*-1)}\,\mathrm{d}x\Big)^{1/2} \\ &\quad\leqslant CR^{\beta/2} \Big(\int_{B_{2R}} (R^{-n\delta}|\nabla u|^{p(x)} + |f|^{1+\delta})\,\mathrm{d}x\Big)^{1/2} \Big(\int_{B_R} (|\nabla u|^{(1+\delta)p(x)} + 1)\,\mathrm{d}x\Big)^{1/2} \\ &\quad\leqslant CR^{\beta/2} \int_{B_{2R}} (R^{-n\delta}|\nabla u|^{p(x)} + |f|^{1+\delta})\,\mathrm{d}x. \end{aligned}$$

The proof of Lemma 3.3 is complete. □

Proposition 3.4. *Let A, B and p satisfy the conditions in Lemma 3.3. Let $B_{2R_1}(x_0)$ be as above and, moreover, R_1 be a number satisfying (3.9) with $\delta \in (0, \delta_0]$ such that*

$$\delta < \min\Big\{\frac{\beta(2n+\alpha_2)}{4n(n+\alpha_2)}, \frac{\beta m^-}{2n} - 1\Big\}. \tag{3.26}$$

Then, given $\tau \in (0, n)$, there exist positive constants $R_\tau < \frac{1}{16}R_1$ and C_τ depending on τ, n, $p(\cdot)$, $m(\cdot)$, $\lambda(M)$, $\Lambda(M)$, M, α_1, β_1, β_2, $\|b\|_{p'(\cdot)m(\cdot)}$, $\|d\|_{m(\cdot)}$ and $\operatorname{dist}(\Omega', \partial\Omega)$, such that

$$\int_{B_\varrho(x_c)} |\nabla u|^{p_*(\varrho)}\,\mathrm{d}x \leqslant C_\tau \varrho^{n-\tau} \quad \forall\, x_c \in B_{R_1/2}(x_0),\ \forall\, \varrho \in (0, R_\tau,$$

where $p_*(\varrho) = p^+_{B_{2\varrho}(x_c)}$.

Proof. Let $x_c \in B_{R_1/2}(x_0)$, $0 < \varrho \leqslant \frac{1}{4}R$ and $R \leqslant \frac{1}{4}R_1$. Thus we have

$$B_\varrho(x_c) \subset B_{R/4}(x_c) \subset B_{4R}(x_c) \subset B_{2R_1}(x_0).$$

Using (3.11), (3.13) and (3.22), we get

(3.27)

$$\begin{aligned}\int_{B_\varrho} |\nabla u|^{p_*(\varrho)}\,\mathrm{d}x \\ &\leqslant 2^{p^+-1}\Big(\int_{B_\varrho}(1+|\nabla u-\nabla v|^{p_*(R)})\,\mathrm{d}x + \int_{B_\varrho}(1+|\nabla v|^{p_*(R)})\,\mathrm{d}x\Big) \\ &\leqslant C\Big(\varrho^n + R^{\beta/2}\int_{B_{2R}}(R^{-n\delta}|\nabla u|^{p(x)}+|f|^{1+\delta})\,\mathrm{d}x + \Big(\frac{\varrho}{R}\Big)^n\int_{B_R}(1+|\nabla u|^{p_*(R)})\,\mathrm{d}x\Big) \\ &\leqslant C\Big(R^n + \Big(R^{\beta/2-n\delta}+\Big(\frac{\varrho}{R}\Big)^n\Big)\int_{B_{2R}}(1+|\nabla u|^{p_*(R)})\,\mathrm{d}x + R^{\beta/2}\int_{B_{2R}}|f|^{1+\delta}\,\mathrm{d}x\Big).\end{aligned}$$

Since (3.26) is satisfied, it follows that $(1+\delta)2n/\beta < m^-$. Therefore, by the Hölder inequality we have

$$\int_{B_{2R}} |f|^{1+\delta}\,\mathrm{d}x \leqslant C(n, \|b\|_{p'(\cdot)m(\cdot)}, \|d\|_{m(\cdot)})R^{n-\beta/2}.$$

From this inequality and (3.27), we arrive at

$$\int_{B_\varrho} |\nabla u|^{p_*(\varrho)}\,\mathrm{d}x \leqslant C\Big(R^{\beta/2-n\delta}+\Big(\frac{\varrho}{R}\Big)^n\Big)\int_{B_{2R}}(1+|\nabla u|^{p_*(R)})\,\mathrm{d}x + CR^n. \tag{3.28}$$

From (3.26) it follows that $\frac{1}{2}\beta - n\delta > 0$. We proved (3.28) under the conditions $\varrho \leqslant \frac{1}{4}R$ and $R \leqslant \frac{1}{4}R_1$. Suppose that $0 < \varrho \leqslant \frac{1}{8}R$ and $R \leqslant \frac{1}{2}R_1$. Setting $R' = \frac{1}{2}R$, then $0 < \varrho \leqslant \frac{1}{4}R'$ and $R' \leqslant \frac{1}{4}R_1$, and so by (3.28) we have

$$\int_{B_\varrho} |\nabla u|^{p_*(\varrho)}\,\mathrm{d}x \leqslant C\Big(R^{\beta/2-n\delta}+\Big(\frac{\varrho}{R}\Big)^n\Big)\int_{B_R}(1+|\nabla u|^{p_*(R)})\,\mathrm{d}x + CR^n. \tag{3.29}$$

Now we define the function $\varphi\colon\ (0, R_1] \to \mathbb{R}$ as

$$\varphi(\varrho) := \int_{B_\varrho}(1+|\nabla u|^{p_*(\varrho)})\,\mathrm{d}x.$$

This is a positive function and by (3.9) and Lemma 3.1 it is also bounded. Moreover, we observe that, since the function $p_*(t)$ is non-decreasing, it readily follows that $\varphi(s) \leqslant 2\varphi(t)$ whenever $s \leqslant t$ and from (3.29) we have

$$\varphi(\varrho) \leqslant C_0\Big(R^{\beta/2-n\delta}+\Big(\frac{\varrho}{R}\Big)^n\Big)\varphi(R) + C_0R^n$$

with $0 < C_0 < \infty$, whenever $0 < \varrho \leqslant \frac{1}{8}R$. Therefore, by Lemma 3.2 of [1], for any τ with $0 < \tau < n$ there exist C and ε_0 depending on n, $\lambda(M)$, $\Lambda(M)$, M, $p(\cdot)$,

$\|b\|_{p'(\cdot)m(\cdot)}$, $\|d\|_{m(\cdot)}$, τ and $\operatorname{dist}(\Omega', \partial\Omega)$ such that $\varepsilon_0 \leqslant \frac{1}{2}R_1$ and if $R^{\beta/2-n\delta} \leqslant \varepsilon_0$, then

$$\varphi(\varrho) \leqslant C\Big(\frac{\varrho}{R}\Big)^{n-\tau}(\varphi(R) + R^{n-\tau}) \tag{3.30}$$

whenever $\varrho \leqslant \frac{1}{16}R$.

Since $0 < \frac{1}{2}\beta - n\delta < 1$, it holds $\varepsilon_1 := \varepsilon_0^{1/(\beta/2-n\delta)} < \varepsilon_0 \leqslant \frac{1}{2}R_1$. Set $R_\tau := \frac{1}{16}\varepsilon_1$. It follows that $R_\tau < \frac{1}{32}R_1$ and if $\varrho \leqslant R_\tau$, i.e., $\varrho \leqslant \frac{1}{16}\varepsilon_1$, then from (3.30) it holds

$$\varphi(\varrho) \leqslant C\Big(\frac{\varrho}{\varepsilon_1}\Big)^{n-\tau}(\varphi(\varepsilon_1) + \varepsilon_1^{n-\tau}).$$

In order to estimate $\varphi(\varepsilon_1)$, we use Lemma 3.1 and the inequality

$$p_*(\varepsilon_1)\Big(1 + \frac{\delta}{2}\Big) \leqslant p(x)(1+\delta) \quad \forall x \in B_{2\varepsilon_1}(x_c).$$

Then we conclude that

$$\begin{aligned}\varphi(\varepsilon_1) &\leqslant 2|B_{\varepsilon_1}| + \int_{B_{\varepsilon_1}} |\nabla u|^{(1+\delta)p(x)}\,\mathrm{d}x \leqslant 2 + C\Big(|B_{2\varepsilon_1}|^{-\delta} + \int_{B_{2\varepsilon_1}} |f|^{1+\delta}\,\mathrm{d}x\Big)\\ &\leqslant C(n, \lambda(M), \Lambda(M), M, p(\cdot), m(\cdot), \|b\|_{p'(\cdot)m(\cdot)}, \|d\|_{m(\cdot)}, \alpha_1, \beta_1, \beta_2, \tau, \operatorname{dist}(\Omega', \partial\Omega)),\end{aligned}$$

which completes the proof of Proposition 3.4. □

P r o o f of Theorem 1.2. Choose numbers $\tau \in (0, n), \theta > 0$ and $\delta \in (0, \delta_0]$ satisfying (3.26) such that

$$\theta\alpha_2 - \tau > 0, \tag{3.31}$$

$$\frac{\beta}{2} - n(\theta + \delta) - \tau > 0, \tag{3.32}$$

$$\frac{\beta}{2} - n\Big(\theta + \frac{1+\delta}{m^-}\Big) > 0. \tag{3.33}$$

Suppose that $B_{2R_1}(x_0)$ is as above but R_1 is a number satisfying (3.9) with this δ and R_τ is as in Proposition 3.4.

Let $x_c \in B_{R_1/4}(x_0)$ and $\varrho < (\frac{1}{4}R\tau)^{1+\theta}$. Set $R = (2\varrho)^{1/(1+\theta)}$. Then $2\varrho < R < \frac{1}{2}R\tau$. Let v be the unique solution of the problem (3.10). It easily follows that

$$\int_{B_\varrho} |\nabla u - (\nabla u)_\varrho|^{p_*}\,\mathrm{d}x \leqslant C(n, p^+)\Big(\int_{B_\varrho} |\nabla u - \nabla v|^{p_*}\,\mathrm{d}x + \int_{B_\varrho} |\nabla v - (\nabla v)_\varrho|^{p_*}\,\mathrm{d}x\Big). \tag{3.34}$$

Now we estimate the integrals on the right-hand side of (3.34). By (3.13) and Proposition 3.4 we obtain

$$\begin{aligned}\fint_{B_R} |\nabla v - (\nabla v)_R|^{p_*}\,\mathrm{d}x &\leqslant C(n, p^+)|B_R|^{-1}\int_{B_R} |\nabla v|^{p_*}\,\mathrm{d}x\\ &\leqslant C|B_R|^{-1}\int_{B_R} (1 + |\nabla u|)^{p_*}\,\mathrm{d}x \leqslant CR^{-\tau}.\end{aligned}$$

Therefore, using the inequalities $(\varrho/R)^{\alpha_2}R^{-\tau} < \varrho^{(\theta\alpha_2-\tau)/(1+\theta)} < 1$ and

$$\begin{aligned}\int_{B_\varrho} &|\nabla v-(\nabla v)_\varrho|^{p_*}\,\mathrm{d}x\\ &\leqslant \begin{cases} C\varrho^n\Big(\Big(\dfrac{\varrho}{R}\Big)^{\alpha_2}R^{-\tau}\Big)^{p_*/2}\Big(\Big(\Big(\dfrac{\varrho}{R}\Big)^{\alpha_2}R^{-\tau}\Big)^{1-p_*/2}+1\Big) & \text{if } 1<p_*<2,\\ C\varrho^n\Big(\dfrac{\varrho}{R}\Big)^{\alpha_2}R^{-\tau} & \text{if } p_*\geqslant 2,\end{cases}\end{aligned}$$

which follow from (3.31) and (3.12), respectively, we have

$$\int_{B_\varrho}|\nabla v-(\nabla v)_\varrho|^{p_*}\,\mathrm{d}x \leqslant C\varrho^{n+(\theta\alpha_2-\tau)/(2(1+\theta))}. \tag{3.35}$$

On the other hand, by Lemma 3.3 and Proposition 3.4, we get

$$\begin{aligned}\int_{B_\varrho}|\nabla u-\nabla v|^{p_*}\,\mathrm{d}x &\leqslant CR^{\beta/2}\int_{B_{2R}}(R^{-n\delta}|\nabla u|^{p(x)}+|f|^{1+\delta})\,\mathrm{d}x\\ &\leqslant C\varrho^n(\varrho^{(\beta/2-n\theta-n\delta-\tau)/(1+\theta)}+\varrho^{(\beta/2-n\theta-n(1+\delta)/m^-)/(1+\theta)}).\end{aligned} \tag{3.36}$$

Setting

$$\varepsilon=\min\Big\{\frac{\theta\alpha_2-\tau}{2(1+\theta)},\frac{\frac12\beta-n(\theta+\delta)-\tau}{1+\theta},\frac{\frac12\beta-n(\theta+(1+\delta)/m^-)}{1+\theta}\Big\},$$

by (3.31)–(3.33) it follows that $\varepsilon>0$. Thus, substituting (3.35), (3.36) into (3.34), we have

$$\begin{aligned}\int_{B_\varrho}|\nabla u-(\nabla u)_\varrho|^{p^-}\,\mathrm{d}x &\leqslant |B_\varrho|^{1-p^-/p_*}\Big(\int_{B_\varrho}|\nabla u-(\nabla u)_\varrho|^{p_*}\,\mathrm{d}x\Big)^{p^-/p_*}\\ &\leqslant C\varrho^{n-np^-/p_*}\varrho^{np^-/p_*+\varepsilon p^-/p_*}=C\varrho^{n+\varepsilon p^-/p_*}\leqslant C\varrho^{n+\varepsilon p^-/p^+},\end{aligned}$$

which implies from Campanato's theorem (Theorem 2.9 of [22]) $u\in C^{1,\alpha}(B_{R_1/8}(x_0))$ with $\alpha=\varepsilon/p^+$. The proof of Theorem 1.2 is complete. □

Remark 3.5. Note that, for example, the numbers

$$\theta=\frac{\beta\,(\beta m^- - 2n)}{2nm^-(n+\alpha_2)},\quad \tau=\frac{\beta\alpha_2(\beta m^- - 2n)}{4nm^-(n+\alpha_2)}$$

and $\delta\in(0,\delta_0]$ such that

$$\begin{aligned}\delta<\min\Big\{&\frac{\beta(2n+\alpha_2)}{4n(n+\alpha_2)},\frac{\beta(nm^-\alpha_2+(2n+\alpha_2)(nm^-+2n-m^-\beta))}{4n^2m^-(n+\alpha_2)},\\ &\frac{(m^-\beta-2n)(n+\alpha_2-\beta)}{2n(n+\alpha_2)}\Big\}\end{aligned}$$

satisfy all the conditions as assumed in the proof of Theorem 1.2 under the condition (1.11).

4. Proof of Theorem 1.3

Theorem 1.2 gives the interior $C^{1,\alpha}$ regularity for the bounded weak solutions. Therefore, to prove Theorem 1.3 it is sufficient to prove only Hölder continuity of the gradient in neighbourhood of the boundary.

First we give a result on the higher integrability for the Dirichlet problem (1.1), and (1.2).

Lemma 4.1. *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n$, whose boundary is denoted by $\partial\Omega$. Let the variable exponent p, and the coefficients A and B satisfy the conditions of Lemma* 3.1. *Suppose that $g \in W^{1,\infty}(\Omega)$ with $\|g\|_{W^{1,\infty}(\Omega)} \leqslant G$ and $u \in W^{1,p(\cdot)}(\Omega)$ is a bounded weak solution of the Dirichlet problem* (1.1), (1.2) *and $\sup\limits_{\Omega}|u| \leqslant M$. Then, there exist positive constants R_0, C and $\delta_0 \in (0, m^- - 1]$ depending on n, $p(\cdot)$, $a_0(M)$, $\Lambda(M)$, Ω and G, and C also on $a_1(M)$, $b_0(M)$ and M, and δ_0 also on M and $m(\cdot)$ such that $|\nabla u| \in L^{(1+\delta_0)p(\cdot)}(\Omega)$ and for every $z \in \overline{\Omega}$, $R \in (0, R_0)$ and $\delta \in (0, \delta_0]$, it holds*

$$\fint_{\Omega_R(z)} |\nabla u|^{(1+\delta)p(x)}\,\mathrm{d}x \leqslant C\biggl(\biggl(\fint_{\Omega_{2R}(z)} |\nabla u|^{p(x)}\,\mathrm{d}x\biggr)^{1+\delta} + \fint_{\Omega_{2R}(z)} |f|^{1+\delta}\,\mathrm{d}x\biggr), \tag{4.1}$$

where $\Omega_R(z) = \Omega \cap B_R(z)$ and f is as in Lemma 3.1.

P r o o f. From the Stein extension theorem (Theorem 5.24 of [3]) it is possible to extend g to a $W^{1,\infty}(\mathbb{R}^n)$ function with $\|g\|_{W^{1,\infty}(\mathbb{R}^n)} \leqslant G$. We define additionally $u = g$ on $\mathbb{R}^n \setminus \overline{\Omega}$ and extend a_2 and b_1 as in Lemma 2.9 and f as in Lemma 3.1 to be zero outside Ω. By [13], Proposition 4.1.7 the variable exponent p can be extended to $\mathbb{R}^n$ without changing the fundamental properties. Let $x_0 \in \partial\Omega$ and let us consider the ball $B_{2R_1}(x_0)$. We know that $u \in C(\overline{\Omega})$. Thus, we can choose a number R_1 such that

$$|(u-g)(x)| \leqslant \frac{a_0(M)}{3\Lambda(M)} \quad \forall\, x \in B_{2R_1}(x_0). \tag{4.2}$$

Now we prove that there exist positive constants $R_0 \leqslant R_1$, $\varepsilon \in (0,1)$ and C_0 such that

$$\fint_{B_R} |\nabla u - \nabla g|^{p(x)}\,\mathrm{d}x \leqslant C_0\biggl(\fint_{B_{2R}} |\nabla u - \nabla g|^{p(x)/(1+\varepsilon)}\,\mathrm{d}x\biggr)^{1+\varepsilon} + C_0 \fint_{B_{2R}} f(x)\,\mathrm{d}x \tag{4.3}$$

for all balls $B_R := B_R(z) \subset B_{2R} \subset B_{2R_1}(x_0)$ with $R \leqslant R_0$. Let us consider the following three cases.

Case (1): In the case when $B_{3R/2} \subset B_{2R_1}(x_0) \cap \Omega$, take $\zeta \in C_0^1(B_{3R/2})$ such that $\zeta = 1$ on B_R and $0 \leqslant \zeta \leqslant 1$, $|\nabla\zeta| \leqslant 4/R$ on $B_{3R/2}$. Taking $\varphi = \zeta^{p^+}(u - (u)_{3R/2})$ as a test function in (1.9) and arguing as in the proof of Lemma 3.1 by using (4.2), we conclude that for any $\varepsilon > 0$ with $\varepsilon < \min\{p^- - 1, 1/(n-1)\}$ there is $R_0 > 0$ with $R_0 \leqslant R_1$ such that

$$\fint_{B_R} |\nabla u|^{p(x)}\,\mathrm{d}x \leqslant C\biggl(\fint_{B_{3R/2}} |\nabla u|^{p(x)/(1+\varepsilon)}\,\mathrm{d}x\biggr)^{1+\varepsilon} + C\fint_{B_{3R/2}} f(x)\,\mathrm{d}x$$

for every $R \in (0, R_0]$ with $B_{3R/2} \subset \Omega$. Therefore, using the above inequality and the obvious inequality

$$\fint_{B_R} |\nabla u - \nabla g|^{p(x)}\,\mathrm{d}x \leqslant 2^{p^+}\biggl(\fint_{B_R} |\nabla u|^{p(x)}\,\mathrm{d}x + \fint_{B_R} |\nabla g|^{p(x)}\,\mathrm{d}x\biggr),$$

we obtain the desired inequality (4.3).

Case (2): In the case when $B_{3R/2} \subset B_{2R_1}(x_0) \setminus \Omega$, the left-hand side of (4.3) equals 0, and so (4.3) is obviously true.

Case (3): In the case when $B_{3R/2} \cap \partial\Omega \neq \emptyset$, let $\zeta \in C_0^1(B_{2R})$ be as in the proof of Lemma 3.1 and take $\varphi = \zeta^{p^+}(u-g)$ as a test function in (1.9). Proceeding as in the proof of Lemma 3.1, we get

$$\begin{aligned}(4.4)\qquad \int_\Omega A(x,u,\nabla u)\nabla\varphi\,\mathrm{d}x \geqslant{}& \frac{2a_0(M)}{3}\int_{\Omega_{2R}} \zeta^{p^+}|\nabla u|^{p(x)}\,\mathrm{d}x \\ &- C\int_{B_{2R}} \biggl(f(x) + \Bigl|\frac{u-g}{R}\Bigr|^{p(x)}\biggr)\,\mathrm{d}x,\end{aligned}$$

where $C = C\left(p^+, p^-, a_0(M), a_1(M), b_0(M), G\right)$.

We estimate the integral $\int_{B_{2R}} |(u-g)/R|^{p(x)}\,\mathrm{d}x$. Putting $D := \{x \in B_{2R}\colon u - g = 0\}$, it follows from the condition on the domain Ω that there is a constant $C > 0$ such that $|D| \geqslant C|B_{2R}|$. Thus, by [13], Lemma 8.2.3 we have

$$(4.5)\qquad C_1\|u-g\|_{p(\cdot),B_{2R}} \leqslant \|u-g-(u-g)_{2R}\|_{p(\cdot),B_{2R}} \leqslant C_2\|u-g\|_{p(\cdot),B_{2R}}.$$

Without loss of generality we may assume that $C_1 < 1$ and $C_2 \geqslant 1$. If we choose $R_0 > 0$ so small that $R_0 \leqslant R_1$ and $C_2\|u-g\|_{p(\cdot),B_{2R}} \leqslant 1$ for any $R \in (0, R_0]$, it follows that

$$(4.6)\qquad \|u-g\|_{p(\cdot),B_{2R}} \leqslant 1, \quad \|u-g-(u-g)_{2R}\|_{p(\cdot),B_{2R}} \leqslant 1.$$

By using (4.5) and (4.6) we obtain

$$\begin{aligned}(4.7)\qquad \int_{B_{2R}} \Bigl|\frac{u-g}{R}\Bigr|^{p(x)}\,\mathrm{d}x \leqslant{}& CR^{-(p^+_{B_{2R}}-p^-_{B_{2R}})(p^+_{B_{2R}}+p^-_{B_{2R}})/(p^+_{B_{2R}})} \\ &\times\biggl(\int_{B_{2R}} \Bigl|\frac{u-g-(u-g)_{2R}}{R}\Bigr|^{p(x)}\,\mathrm{d}x\biggr)^{p^-_{B_{2R}}/p^+_{B_{2R}}}\end{aligned}$$

Since p is log-Hölder continuous in $\mathbb{R}^n$, it follows by (2.3) that there is a constant C such that $R^{-(p^+_{B_{2R}}-p^-_{B_{2R}})} \leqslant C$ for any $R>0$, from which by using (4.6) and (4.7) it follows

$$\int_{B_{2R}} \Big|\frac{u-g}{R}\Big|^{p(x)}\,\mathrm{d}x \leqslant C\Big(\int_{B_{2R}} \Big|\frac{u-g-(u-g)_{2R}}{R}\Big|^{p(x)}\,\mathrm{d}x + 1\Big). \tag{4.8}$$

Choosing $R_0>0$ so small that for any $R\in(0,R_0]$ it holds $\|\nabla u-\nabla g\|_{p(\cdot)/(1+\varepsilon),B_{2R}} \leqslant 1$ with $\varepsilon<\min\{p^- -1, 1/(n-1)\}$ and using [13], Proposition 8.2.11, from (4.8) we get

$$\int_{B_{2R}} \Big|\frac{u-g}{R}\Big|^{p(x)}\,\mathrm{d}x \leqslant C\Big(|B_{2R}|\Big(\fint_{B_{2R}} |\nabla u-\nabla g|^{p(x)/(1+\varepsilon)}\,\mathrm{d}x\Big)^{1+\varepsilon} + 1\Big).$$

Therefore, from (4.4) we arrive at

$$\begin{aligned}\int_{\Omega} A(x,u,\nabla u)\nabla\varphi\,\mathrm{d}x \geqslant{}& \frac{2a_0(M)}{3}\int_{\Omega_{2R}} \zeta^{p^+}|\nabla u|^{p(x)}\,\mathrm{d}x\\ &- C|B_{2R}|\Big(\fint_{B_{2R}} |\nabla u-\nabla g|^{p(x)/(1+\varepsilon)}\,\mathrm{d}x\Big)^{1+\varepsilon}\\ &- C\int_{B_{2R}} f(x)\,\mathrm{d}x.\end{aligned} \tag{4.9}$$

On the other hand, by (1.8) and (4.2), we have

$$\int_{\Omega} B(x,u,\nabla u)\varphi\,\mathrm{d}x \leqslant \frac{a_0(M)}{3}\int_{\Omega_{2R}} \zeta^{p^+}|\nabla u|^{p(x)}\,\mathrm{d}x + \frac{a_0(M)}{3}\int_{B_{2R}} f(x)\,\mathrm{d}x. \tag{4.10}$$

Substituting (4.9) and (4.10) into (1.9), we obtain

$$\int_{\Omega_R} |\nabla u|^{p(x)}\,\mathrm{d}x \leqslant C|B_{2R}|\Big(\fint_{B_{2R}} |\nabla u-\nabla g|^{p(x)/(1+\varepsilon)}\,\mathrm{d}x\Big)^{1+\varepsilon} + C\int_{B_{2R}} f(x)\,\mathrm{d}x$$

and so

$$\begin{aligned}\int_{B_R} |\nabla u-\nabla g|^{p(x)}\,\mathrm{d}x &= \int_{\Omega_R} |\nabla u-\nabla g|^{p(x)}\,\mathrm{d}x \leqslant 2^{p^+}\int_{\Omega_R} |\nabla u|^{p(x)}\,\mathrm{d}x + C\\ &\leqslant C_0|B_{2R}|\Big(\fint_{B_{2R}} |\nabla u-\nabla g|^{p(x)/(1+\varepsilon)}\,\mathrm{d}x\Big)^{1+\varepsilon} + C_0\int_{B_{2R}} f(x)\,\mathrm{d}x\end{aligned}$$

from which (4.3) follows.

By the Gehring lemma (see [14], Theorem 3.7), it follows from (4.3) that there is a $\delta_0\in(0,m^- -1]$ such that

$$\fint_{B_R} |\nabla u-\nabla g|^{(1+\delta)p(x)}\,\mathrm{d}x \leqslant C\Big(\Big(\fint_{B_{2R}} |\nabla u-\nabla g|^{p(x)}\,\mathrm{d}x\Big)^{1+\delta} + \fint_{B_{2R}} |f|^{1+\delta}\,\mathrm{d}x\Big) \tag{4.11}$$

holds for $B_R \subset B_{2R} \subset B_{2R_1}(x_0)$ with $R \leqslant R_0$ and $\delta \in (0, \delta_0]$. Since Ω is a bounded Lipschitz domain, there exists a constant $\sigma \in (0,1)$ such that

$$\sigma|B_R(z)| \leqslant |\Omega_R(z)| \leqslant |B_R(z)| \quad \forall\, z \in \overline{\Omega} \text{ and } R \in (0, R_0]$$

and so from (4.11) we get

$$\fint_{\Omega_R} |\nabla u - \nabla g|^{(1+\delta)p(x)}\,\mathrm{d}x \leqslant C\biggl(\biggl(\fint_{\Omega_{2R}} |\nabla u - \nabla g|^{p(x)}\,\mathrm{d}x\biggr)^{1+\delta} + \fint_{\Omega_{2R}} |f|^{1+\delta}\,\mathrm{d}x\biggr).$$

Consequently (4.1) is proved. By the compactness of $\overline{\Omega}$, we can see that $|\nabla u| \in L^{(1+\delta_0)p(\cdot)}(\Omega)$. Lemma 4.1 is proved. □

Now we suppose that Assumptions (H1)–(H3) and (1.3) are satisfied. Since $g \in C^{1,\alpha_0}(\partial\Omega)$ and Ω is a C^{1,α_0} domain, from the extension theory (Lemma 6.38 of [21]) we may assume that $g \in C^{1,\alpha_0}(\overline{\Omega})$ and $\|g\|_{C^{1,\alpha_0}(\overline{\Omega})} \leqslant C\|g\|_{C^{1,\alpha_0}(\partial\Omega)}$, where $C = C(\Omega)$.

We use the notation $\overline{\Omega}_R = \overline{\Omega}_R(z) = \overline{B_R}(z) \cap \overline{\Omega}$. Let R_0 and δ_0 be as in Lemma 4.1. As proved above, we know that $u \in C^{0,\alpha_1}(\overline{\Omega})$, $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ and $|\nabla u| \in L^{(1+\delta_0)p(\cdot)}(\Omega)$. As in Section 3, we assume that $|B_{R_0}| \leqslant 1$ and $R_0 \leqslant 1$. Let $x_0 \in \partial\Omega$ and $\delta \in (0, \delta_0]$. Let $R_1 > 0$ be so small that $R_1 \leqslant R_0$, $\int_{\Omega_{2R_1}(x_0)} |\nabla u|^{p(x)}\,\mathrm{d}x \leqslant 1$ and

$$p^+_{\Omega_{2R_1}(x_0)}\Bigl(1 + \frac{\delta}{2}\Bigr) \leqslant p^-_{\Omega_{2R_1}(x_0)}(1+\delta). \tag{4.12}$$

Let $\Omega_R = \Omega_R(x_c) \subset \Omega_{2R} \subset \Omega_{2R_1}(x_0)$. Put $p_*(R) = p^+_{\Omega_{2R}}$ and let $x_* \in \overline{\Omega}_{2R}$ be such that $p(x_*) = p_*(R) = p_*$. Define $\overline{A}(\eta) = A(x_*, u(x_*), \eta)$ and consider the problem

$$\begin{cases} \mathrm{div}\overline{A}(\nabla v) = 0 & \text{in } \Omega_R, \\ v = u & \text{on } \partial\Omega_R. \end{cases} \tag{4.13}$$

The following result is taken from [27], see also [16], Lemma 4.2.

Lemma 4.2. *There is a unique solution v of the problem* (4.13) *such that $v \in C^{1,\alpha_2}(\overline{\Omega}_{R/2}) \cap W^{1,p_*}(\Omega_R)$ and*

$$\sup_{\Omega_{R/2}} |\nabla v|^{p_*} \leqslant C\Bigl(R^{-n}\int_{\Omega_R} |\nabla v|^{p_*}\,\mathrm{d}x + G^{p_*}\Bigr), \tag{4.14}$$

$$\operatorname*{osc}_{\Omega_\varrho} \nabla v \leqslant C\Bigl(\frac{\varrho}{R}\Bigr)^{\alpha_2}\Bigl(\sup_{\Omega_{R/2}} |\nabla v| + GR^{\alpha_0}\Bigr) \quad \text{for } 0 < \varrho < \frac{\mathrm{R}}{2}, \tag{4.15}$$

$$\int_{\Omega_R} |\nabla v|^{p_*}\,\mathrm{d}x \leqslant C\int_{\Omega_R} (|\nabla u|^{p_*} + 1)\,\mathrm{d}x, \tag{4.16}$$

$$\sup_{\Omega_R} |u - v| \leqslant \operatorname*{osc}_{\Omega_R} u,$$

where $\alpha_2 \in (0,1)$ and C depends on n, p_*, $\lambda(M)$, $\Lambda(M)$ and α_0.

Analogously to Lemma 3.3 and Proposition 3.4, we have following Lemmas 4.3 and 4.4, the proof of which is similar to that of Lemma 3.3 and Proposition 3.4, and is omitted here.

Lemma 4.3. *Let v be the unique solution of the problem* (4.13) *and let* $0 < \delta \leqslant \delta_0$, *where δ_0 is as in Lemma* 4.1. *Let R and R_1 be as above. Then we have*

$$\int_{\Omega_R} |\nabla u - \nabla v|^{p_*}\,\mathrm{d}x \leqslant CR^{\beta/2} \int_{\Omega_{2R}} (R^{-n\delta}|\nabla u|^{p(x)} + |f|^{1+\delta})\,\mathrm{d}x,$$

where C depends on n, $p(\cdot)$, $\lambda(M)$, $\Lambda(M)$, M, G and Ω, and β and f are as in (1.12) *and* (3.1), *respectively.*

Lemma 4.4. *Let $\Omega_{2R_1}(x_0)$ be as above and, moreover, let R_1 be a number satisfying* (4.12) *with $\delta \in (0, \delta_0]$ as in* (3.26). *Then, given $\tau \in (0, n)$, there exist positive constants $R_\tau < \frac{1}{16}R_1$ and C_τ depending on τ, n, $p(\cdot)$, $m(\cdot)$, $\lambda(M)$, $\Lambda(M)$, α_1, β_1, β_2, $\|b\|_{p'(\cdot)m(\cdot)}$, $\|d\|_{m(\cdot)}$ and G, such that*

$$\int_{\Omega_\varrho(x_c)} |\nabla u|^{p_*(\varrho)}\,\mathrm{d}x \leqslant C_\tau \varrho^{n-\tau} \quad \forall\, x_c \in \overline{\Omega}_{R_1/2}(x_0),\ \forall\, \varrho \in (0, R_\tau),$$

where $p_*(\varrho) = p^+_{\Omega_{2\varrho}(x_c)}$.

P r o o f of Theorem 1.3. Let $\tau \in (0, n)$, $\theta > 0$ and $\delta \in (0, \delta_0]$ be numbers satisfying (3.26) and (3.31)–(3.33). Suppose that $\Omega_{2R_1}(x_0)$ is as above but R_1 is a positive number satisfying (4.12) with the above δ and R_τ is as in Lemma 4.4. Let $x_c \in \overline{\Omega}_{R_1/4}(x_0)$ and $\varrho < (\frac{1}{4}R_\tau)^{1+\theta}$. Set $R = (2\varrho)^{1/(1+\theta)}$. Let v be the unique solution of the problem (4.13). Noting that it follows from (4.14)–(4.16) that

$$⨍_{\Omega_\varrho} |\nabla v - (\nabla v)_\varrho|^{p_*}\,\mathrm{d}x \leqslant C\Big(\frac{\varrho}{R}\Big)^{\alpha_2} ⨍_{\Omega_R} (1 + |\nabla u|^{p_*})\,\mathrm{d}x \leqslant C\varrho^{(\theta\alpha_2 - \tau)/(1+\theta)}$$

and using same argument as was done in the proof of Theorem 1.2, we conclude that

$$\int_{\Omega_\varrho} |\nabla u - (\nabla u)_\varrho|^{p^-}\,\mathrm{d}x \leqslant C\varrho^{n+\varepsilon p^-/p^+},$$

where, for example,

$$\varepsilon = \min\Big\{\frac{\theta\alpha_2 - \tau}{1+\theta}, \frac{\frac{1}{2}\beta - n(\theta+\delta) - \tau}{1+\theta}, \frac{\frac{1}{2}\beta - n(\theta + (1+\delta)/m^-)}{1+\theta}\Big\}$$

and C depends on n, $p(\cdot)$, $m(\cdot)$, $\lambda(M), \Lambda(M)$, M, α_1, β_1, β_2, $\|b\|_{p'(\cdot)m(\cdot)}$, $\|d\|_{m(\cdot)}$, G and Ω. This implies that the conclusions of Theorem 1.3 hold. Theorem 1.3 is proved. □

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