PROPERTIES ON SUBCLASS OF SAKAGUCHI TYPE FUNCTIONS USING A MITTAG-LEFFLER TYPE POISSON DISTRIBUTION SERIES

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Received April 6, 2023. Published online November 21, 2023.
Communicated by Grigore Sălăgean

Abstract. Few subclasses of Sakaguchi type functions are introduced in this article, based on the notion of Mittag-Leffler type Poisson distribution series. The class $\mathfrak{p}\text{-}\Phi\mathcal{S}^*(t,\mu,\nu,J,K)$ is defined, and the necessary and sufficient condition, convex combination, growth distortion bounds, and partial sums are discussed.

Keywords: Mittag-Leffler type Poisson distribution; analytic function; conic-type region; geometric properties

MSC 2020: 30C45, 30C50

1. Introduction and preliminaries

Let $\mathbb{U}(r) := \{z \in \mathbb{C}\colon\ |z| < r\}$ be the disk in the complex plane $\mathbb{C}$ centered at the origin, with radius $r > 0$, and denote by $\mathbb{U} := \mathbb{U}(1)$ the unit disk. We denote by $\mathcal{A}$ the class of analytic functions in the unit disk $\mathbb{U}$ normalized by $f(0) = f'(0) - 1 = 0$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. The Taylor series expansion of f is given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}. \tag{1.1}$$

Kanas et al. (see [3]–[6]) were the first to define the conic domain Ω_k $(k \geqslant 0)$ as:

$$\Omega_k = \bigl\{u + \mathrm{i}v\colon\ u > k\sqrt{(u-1)^2 + v^2}\bigr\}. \tag{1.2}$$

DOI: 10.21136/MB.2023.0061-23

Moreover, for fixed k, Ω_k represents the conic region bounded successively by the imaginary axis ($k = 0$). For $k = 1$, it is a parabola, and for $0 < k < 1$, it is the right-hand branch of a hyperbola, and for $k > 1$, it represents an ellipse (see [12]).

The Poisson distribution and the well-known Mittag-Leffler function were discovered and connected systematically by Porwal and Dixit, see [13]. They called it the Mittag-Leffler type Poisson distribution and prevailed moments. The Mittag-Leffler type Poisson distribution is given by (see [13])

$$Y(\Psi, \mu, \nu)(z) = z + \sum_{n=2}^{\infty} \frac{\Psi^{n-1}}{\Gamma(\mu(n-1)+\nu) E_{\mu;\nu}(\Psi)} z^n,$$

where $Y(\Psi, \mu, \nu)(z)$ is a normalized function of class $\mathcal{S}$, since

$$Y(\Psi, \mu, \nu)(0) = 0 = Y'(\Psi, \mu, \nu)(0) - 1.$$

The probability mass function for the Mittag-Leffler type Poisson distribution series is given by

$$P(\Psi, \mu, \nu; n)(z) = \frac{\Psi^n}{E_{\mu;\nu}(\Psi)\Gamma(\mu n + \nu)}, \quad n = 0, 1, 2, \ldots,$$

where

$$E_{\mu;\nu}(\Psi) = \sum_{k=0}^{\infty} \frac{\Psi^k}{\Gamma(\nu + \mu k)}, \quad \mu, \nu, \Psi \in \mathbb{C},$$

and was introduced by Wiman (see [14], [15]), Agrawal (see [1]), and by many others (see for example [8]–[11]). Recently, using the Mittag-Leffler type Poisson distribution series, Alessa et al. (see [2]) defined the convolution operator as

$$\Phi(\Psi, \mu, \nu) f(z) = Y(\Psi, \mu, \nu) * f(z) = z + \sum_{n=2}^{\infty} \varphi_\Psi^n(\mu, \nu) a_n z^n,$$

where

$$\varphi_\Psi^n(\mu, \nu) = \frac{\Psi^{n-1}}{\Gamma(\mu(n-1)+\nu) E_{\mu;\nu}(\Psi)}.$$

They defined and studied a new subclass of analytic functions using this convolution operator. Following are some convolution operators for Mittag-Leffler type Poisson distributions that are useful for defining the subclass of Sakaguchi type functions involving the conic domains.

Definition 1. For K, J, $-1 \leqslant K < J \leqslant 1$, a function $f \in \mathcal{A}$ is in class $\mathfrak{p}$-$\Phi\mathcal{S}^*(t,\mu,\nu,J,K)$ if

$$\Re\Big(\frac{(K-1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J-1)}{(K+1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J+1)}-1\Big) \geqslant \mathfrak{p}\Big|\frac{(K-1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J-1)}{(K+1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J+1)}-1\Big|,$$

where

$$\eta_{(t)}(f;\Psi,\mu,\nu) = \frac{(1-t)z(\Phi(\Psi,\mu,\nu)f(z))'}{\Phi(\Psi,\mu,\nu)f(z)-\Phi(\Psi,\mu,\nu)f(tz)}$$

with $|t| \leqslant 1$, $t \neq 1$.

Remark 1.

(i) For $t = 0$ in class $\mathfrak{p}$-$\Phi\mathcal{S}^*(t,\mu,\nu,J,K)$, we get the class k-$\Omega\mathcal{S}^*(\alpha,\beta,A,B)$ and the geometric properties were discussed by Khan et al. in [7].

(ii) For $t = -1$ we obtain the class $\mathfrak{p}$-$\Phi\mathcal{S}^*(-1,\mu,\nu,J,K)$ if

$$\Re\Big(\frac{(K-1)\eta_{(-1)}(f;\Psi,\mu,\nu)-(J-1)}{(K+1)\eta_{(-1)}(f;\Psi,\mu,\nu)-(J+1)}-1\Big) \geqslant \mathfrak{p}\Big|\frac{(K-1)\eta_{(-1)}(f;\Psi,\mu,\nu)-(J-1)}{(K+1)\eta_{(-1)}(f;\Psi,\mu,\nu)-(J+1)}-1\Big|,$$

where

$$\eta_{(-1)}(f;\Psi,\mu,\nu) = \frac{2z(\Phi(\Psi,\mu,\nu)f(z))'}{\Phi(\Psi,\mu,\nu)f(z)-\Phi(\Psi,\mu,\nu)f(-z)}.$$

(iii) When $J = 1$, $K = -1$, the class $\mathfrak{p}$-$\Phi\mathcal{S}^*(t,\mu,\nu,1,-1)$ is obtained with the following condition:

$$\Re(\eta_{(t)}(f;\Psi,\mu,\nu)-1) \geqslant \mathfrak{p}|\eta_{(t)}(f;\Psi,\mu,\nu)-1|.$$

(iv) When $J = 1-2\alpha$, $K = -1$ with $0 \leqslant \alpha < 1$, we obtain a class $\mathfrak{p}$-$\Phi\mathcal{S}^*(t,\mu,\nu,1-2\alpha,-1)$ with

$$\Re\Big(\frac{\eta_{(t)}(f;\Psi,\mu,\nu)-1}{1-\alpha}\Big) \geqslant \mathfrak{p}\Big|\frac{\eta_{(t)}(f;\Psi,\mu,\nu)-1}{1-\alpha}\Big|.$$

(v) When $J = 1-2\alpha$, $K = -1$ with $0 \leqslant \alpha < 1$ and $\mathfrak{p} = 0$, a class 0-$\Phi\mathcal{S}^*(t,\mu,\nu,1-2\alpha,-1)$ can be obtained if

$$\Re\Big(\frac{\eta_{(t)}(f;\Psi,\mu,\nu)-1}{1-\alpha}\Big) \geqslant 0.$$

2. Main results

Theorem 1. *Let $f \in \mathfrak{p}\text{-}\Phi\mathcal{S}^*(t,\mu,\nu,J,K)$ and is of the form* (1.1). *Then*

$$\begin{aligned} \sum_{n=2}^{\infty} \{2(\mathfrak{p}+1)&|(1+t+\ldots+t^{n-1})-n| \\ &+|n(1+K)-(1+t+\ldots+t^{n-1})(1+J)|\}\varphi_{\Psi}^{n}(\mu,\nu)|a_n| \\ &< |K-J|. \end{aligned} \tag{2.1}$$

The result is sharp for the function given in (2.7).

P r o o f. Suppose that inequality (2.1) holds true, then it is enough to show that

$$\mathfrak{p}\Big|\frac{(K-1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J-1)}{(K+1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J+1)}-1\Big|-\Re\Big(\frac{(K-1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J-1)}{(K+1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J+1)}-1\Big)<1.$$

For this, consider

$$\mathfrak{p}\Big|\frac{(K-1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J-1)}{(K+1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J+1)}-1\Big|-\Re\Big(\frac{(K-1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J-1)}{(K+1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J+1)}-1\Big).$$

As we have set,

$$\eta_{(t)}(f;\Psi,\mu,\nu)=\frac{(1-t)z(\Phi(\Psi,\mu,\nu)f(z))'}{\Phi(\Psi,\mu,\nu)f(z)-\Phi(\Psi,\mu,\nu)f(tz)}$$

therefore after some straightforward simplifications, we have

$$\begin{aligned} &\mathfrak{p}\Big|\frac{(K-1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J-1)}{(K+1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J+1)}-1\Big|-\Re\Big(\frac{(K-1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J-1)}{(K+1)\eta_{(t)}(f;\Psi,\mu,\nu)-(J+1)}-1\Big) \\ &\leqslant (\mathfrak{p}+1)\Big\{\Big|\frac{(K-1)\Upsilon_1-(J-1)\Upsilon_2}{(K+1)\Upsilon_1-(J+1)\Upsilon_2}-1\Big|\Big\}=2(\mathfrak{p}+1)\Big\{\Big|\frac{\Upsilon_2-\Upsilon_1}{(K+1)\Upsilon_1-(J+1)\Upsilon_2}\Big|\Big\} \\ &=2(\mathfrak{p}+1) \\ &\quad\times\Big|\frac{\sum_{n=2}^{\infty}(1+tn-n-t^n)\varphi_{\Psi}^{n}(\mu,\nu)a_n z^n}{(1-t)z(K-J)+\sum_{n=2}^{\infty}[n(1-t)(1+K)-(1-t^n)(1+J)]\varphi_{\Psi}^{n}(\mu,\nu)a_n z^n}\Big| \\ &\leqslant 2(\mathfrak{p}+1) \\ &\quad\times\frac{\sum_{n=2}^{\infty}|1+tn-n-t^n|\varphi_{\Psi}^{n}(\mu,\nu)|a_n|}{|(1-t)(K-J)|-\sum_{n=2}^{\infty}|n(1-t)(1+K)-(1-t^n)(1+J)|\varphi_{\Psi}^{n}(\mu,\nu)|a_n|}, \end{aligned}$$

where

$$\Upsilon_1=(1-t)z(\Phi(f;\Psi,\mu,\nu)f(z))',\quad \Upsilon_2=\Phi(f;\Psi,\mu,\nu)f(z)-\Phi(f;\Psi,\mu,\nu)f(tz).$$

By using (2.1), the above inequality is bounded above by 1, and hence, the proof is completed. □

Corollary 1. *Let* $f \in \mathfrak{p}\text{-}\Phi\mathcal{S}^*(-1, \mu, \nu, J, K)$ *and is of the form* (1.1)*. Then*

$$\sum_{n=2}^{\infty} \Upsilon_3 |a_n| < 2|K - J|, \tag{2.2}$$

where

$$\Upsilon_3 = \{2(\mathfrak{p}+1)|1 - (-1)^n - 2n| + |2n(1+K) - (1 - (-1)^n)(1+J)|\}\varphi_{\Psi}^{n}(\mu, \nu).$$

The result is sharp for the function given in (2.8)*.*

Corollary 2. *Let* $f \in \mathfrak{p}\text{-}\Phi\mathcal{S}^*(t, \mu, \nu, 1, -1)$ *and is of the form* (1.1)*. Then*

$$\sum_{n=2}^{\infty} \Upsilon_4 |a_n| < 1, \tag{2.3}$$

where

$$\Upsilon_4 = \{(\mathfrak{p}+1)|(1 + t + \ldots + t^{n-1}) - n| + |1 + t + \ldots + t^{n-1}|\}\varphi_{\Psi}^{n}(\mu, \nu).$$

The result is sharp for the function given in (2.9)*.*

Corollary 3. *Let* $f \in \mathfrak{p}\text{-}\Phi\mathcal{S}^*(t, \mu, \nu, 1 - 2\alpha, -1)$ *and is of the form* (1.1)*. Then*

$$\sum_{n=2}^{\infty} \Upsilon_5 |a_n| < |\alpha - 1|, \tag{2.4}$$

where

$$\Upsilon_5 = \{(\mathfrak{p}+1)|(1 + t + \ldots + t^{n-1}) - n| + |(\alpha - 1)(1 + t + \ldots + t^{n-1})|\}\varphi_{\Psi}^{n}(\mu, \nu).$$

The result is sharp for the function given in (2.10)*.*

Corollary 4. *Let* $f \in 0\text{-}\Phi\mathcal{S}^*(t, \mu, \nu, 1 - 2\alpha, -1)$ *and is of the form* (1.1)*. Then*

$$\sum_{n=2}^{\infty} \Upsilon_6 |a_n| < |\alpha - 1|, \tag{2.5}$$

where

$$\Upsilon_6 = \{|(1 + t + \ldots + t^{n-1}) - n| + |(\alpha - 1)(1 + t + \ldots + t^{n-1})|\}\varphi_{\Psi}^{n}(\mu, \nu).$$

The result is sharp for the function given in (2.11)*.*

E x a m p l e 1. For the function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{|K-J|}{\Upsilon_7}x_nz^n,\quad z\in\mathbb{U}, \tag{2.6}$$

where

$$\Upsilon_7=\{2(\mathfrak{p}+1)|(1+t+\ldots+t^{n-1})-n|+|n(1+K)-(1+t+\ldots+t^{n-1})(1+J)|\}\varphi_{\Psi}^{n}(\mu,\nu),$$

such that

$$\sum_{n=2}^{\infty}|x_n|=1,$$

we have

$$\sum_{n=2}^{\infty}\Upsilon_7|a_n|=\Big\{\sum_{n=2}^{\infty}\Upsilon_7\Big\}\Big\{\frac{|K-J|}{\Upsilon_7}|x_n|\Big\}=|K-J|\sum_{n=2}^{\infty}|x_n|=|K-J|.$$

Hence, $f\in\mathfrak{p}\text{-}\Phi\mathcal{S}^*(t,\mu,\nu,J,K)$ and the result is sharp.

R e m a r k 2. By fixing some suitable parameter values in function (2.6), we can get the examples of the following classes: $\mathfrak{p}\text{-}\Phi\mathcal{S}^*(-1,\mu,\nu,J,K)$, $\mathfrak{p}\text{-}\Phi\mathcal{S}^*(t,\mu,\nu,1,-1)$, $\mathfrak{p}\text{-}\Phi\mathcal{S}^*(t,\mu,\nu,1-2\alpha,-1)$, $0\text{-}\Phi\mathcal{S}^*(t,\mu,\nu,1-2\alpha,-1)$.

Theorem 2. *Let the function f of the form* (1.1) *be in the class $\mathfrak{p}\text{-}\Phi\mathcal{S}^*(t,\mu,\nu,J,K)$. Then*

$$|a_n|\leqslant\frac{|K-J|}{\Upsilon_7}.$$

The result is sharp for the function $f_1(z)$ given by

$$f_1(z)=z+\sum_{n=2}^{\infty}\frac{|K-J|}{\Upsilon_7}z^n. \tag{2.7}$$

Corollary 5. *Let the function f of the form* (1.1) *be in the class $\mathfrak{p}\text{-}\Phi\mathcal{S}^*(-1,\mu,\nu,J,K)$. Then*

$$|a_n|\leqslant\frac{2|K-J|}{\Upsilon_3}.$$

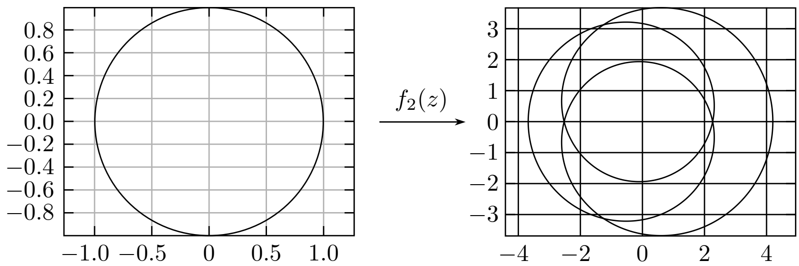
The result is sharp for the function $f_2(z)$ given by

$$f_2(z)=z+\sum_{n=2}^{\infty}\frac{2|K-J|}{\Upsilon_3}z^n. \tag{2.8}$$

By fixing the parameter values, $J = \frac{1}{2}$, $K = \frac{1}{3}$, $\mathfrak{p} = 0.1$, $\mu = 1$, $\nu = 10$, $\Psi = 1$ in (2.8), we get that

$$f_2(z) = z + 0.2608z^2 + 2.9423z^3 + \dots$$

belongs to $\mathfrak{p}$-$\Phi\mathcal{S}^*(-1, \mu, \nu, J, K)$.



(a) Unit disc in z-plane. (b) w-plane.

Figure 1. Pictorial representation of $f_2(z) = z + 0.2608z^2 + 2.9423z^3 + \dots$

Corollary 6. *Let the function f of the form* (1.1) *be in the class* $\mathfrak{p}$-$\Phi\mathcal{S}^*(t, \mu, \nu, 1, -1)$*. Then*

$$|a_n| \leqslant \frac{1}{\Upsilon_4}.$$

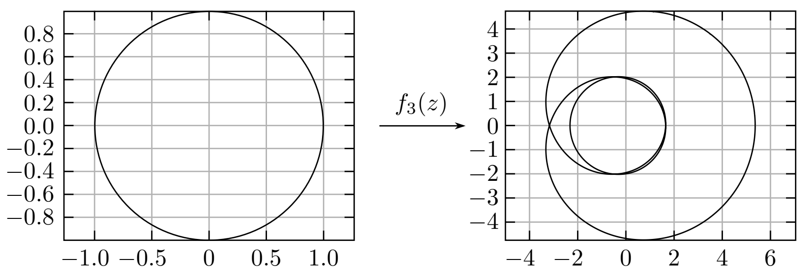
The result is sharp for the function $f_3(z)$ given by

$$f_3(z) = z + \sum_{n=2}^{\infty} \frac{1}{\Upsilon_4} z^n. \tag{2.9}$$

By fixing the parameter values, $t = -\frac{1}{2}$, $\mathfrak{p} = 0.1$, $\mu = 1$, $\nu = 1.8$, $\Psi = 1$ in (2.9), we get that

$$f_3(z) = z + 1.5230z^2 + 2.8423z^3 + \dots$$

belongs to $\mathfrak{p}$-$\Phi\mathcal{S}^*(t, \mu, \nu, 1, -1)$.



(a) Unit disc in z-plane. (b) w-plane.

Figure 2. Pictorial representation of $f_3(z) = z + 1.5230z^2 + 2.8423z^3 + \dots$

Corollary 7. *Let the function f of the form* (1.1) *be in the class* $\mathfrak{p}$-$\Phi\mathcal{S}^*(t, \mu, \nu, 1-2\alpha, -1)$. *Then*

$$|a_n| \leqslant \frac{|\alpha - 1|}{\Upsilon_5}.$$

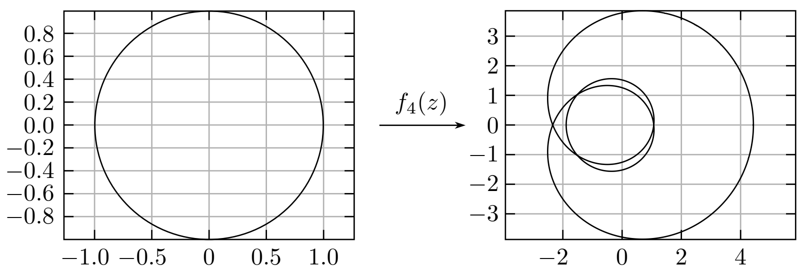
The result is sharp for the function $f_4(z)$ given by

$$f_4(z) = z + \sum_{n=2}^{\infty} \frac{|\alpha - 1|}{\Upsilon_5} z^n. \tag{2.10}$$

By fixing the parameter values, $t = \frac{1}{2}$, $\alpha = \frac{1}{2}$, $\mathfrak{p} = 0.2$, $\mu = 1$, $\nu = 2$, $\Psi = 1$ in (2.10), we get that

$$f_4(z) = z + 1.2725z^2 + 2.1701z^3 + \dots$$

belongs to $\mathfrak{p}$-$\Phi\mathcal{S}^*(t, \mu, \nu, 1-2\alpha, -1)$.



(a) Unit disc in z-plane. (b) w-plane.

Figure 3. Pictorial representation of $f_4(z) = z + 1.2725z^2 + 2.1701z^3 + \dots$

Corollary 8. *Let the function f of the form* (1.1) *be in the class* 0-$\Phi\mathcal{S}^*(t, \mu, \nu, 1-2\alpha, -1)$. *Then*

$$|a_n| \leqslant \frac{|\alpha - 1|}{\Upsilon_6}.$$

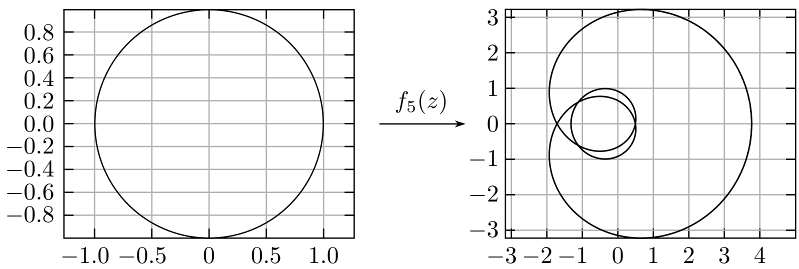
The result is sharp for the function $f_5(z)$ given by

$$f_5(z) = z + \sum_{n=2}^{\infty} \frac{|\alpha - 1|}{\Upsilon_6} z^n. \tag{2.11}$$

By fixing the parameter values, $t = \frac{1}{3}$, $\alpha = \frac{1}{4}$, $\mu = 1$, $\nu = 1$, $\Psi = 1$ in (2.11), we get that

$$f_5(z) = z + 1.2232z^2 + 1.5450z^3 + \ldots$$

belongs to $0\text{-}\Phi\mathcal{S}^*(t, \mu, \nu, 1 - 2\alpha, -1)$.



(a) Unit disc in z-plane. (b) w-plane.

Figure 4. Pictorial representation of $f_5(z) = z + 1.2232z^2 + 1.5450z^3 + \ldots$

Theorem 3. *The class* $\mathfrak{p}\text{-}\Phi\mathcal{S}^*(t, \mu, \nu, J, K)$ *is closed under convex combination.*

Proof. Let $f_m(z) \in \mathfrak{p}\text{-}\Phi\mathcal{S}^*(t, \mu, \nu, J, K)$ such that

$$f_m(z) = z + \sum_{n=2}^{\infty} a_{n,m} z^n, \quad m \in \{1, 2\}.$$

It is enough to show that

$$\mathfrak{t} f_1(z) + (1 - \mathfrak{t}) f_2(z) \in \mathfrak{p}\text{-}\Phi\mathcal{S}^*(t, \mu, \nu, J, K), \quad \mathfrak{t} \in [0, 1].$$

As

$$\mathfrak{t} f_1(z) + (1 - \mathfrak{t}) f_2(z) = z + \sum_{n=2}^{\infty} [\mathfrak{t} a_{n,1} + (1 - \mathfrak{t}) a_{n,2}] z^n,$$

$$\begin{aligned}
\sum_{n=2}^{\infty} \Upsilon_7 |\mathfrak{t} a_{n,1} + (1 - \mathfrak{t}) a_{n,2}| &\leqslant \sum_{n=2}^{\infty} \Upsilon_7 [\mathfrak{t} |a_{n,1}| + (1 - \mathfrak{t}) |a_{n,2}|] \\
&\leqslant \mathfrak{t} \sum_{n=2}^{\infty} \Upsilon_7 |a_{n,1}| + (1 - \mathfrak{t}) \sum_{n=2}^{\infty} \Upsilon_7 |a_{n,2}| \\
&< \mathfrak{t} |K - J| + (1 - \mathfrak{t}) |K - J| = |K - J|.
\end{aligned}$$

Hence,

$$\mathfrak{t} f_1(z) + (1 - \mathfrak{t}) f_2(z) \in \mathfrak{p}\text{-}\Phi\mathcal{S}^*(t, \mu, \nu, J, K),$$

which completes the proof. □

Corollary 9. *The classes* $\mathfrak{p}$-$\Phi\mathcal{S}^*(-1,\mu,\nu,J,K)$, $\mathfrak{p}$-$\Phi\mathcal{S}^*(t,\mu,\nu,1,-1)$, $\mathfrak{p}$-$\Phi\mathcal{S}^*(t,\mu,\nu,1-2\alpha,-1)$, 0-$\Phi\mathcal{S}^*(t,\mu,\nu,1-2\alpha,-1)$ *are closed under convex combination.*

Theorem 4. *Let* $f \in \mathfrak{p}$-$\Phi\mathcal{S}^*(t,\mu,\nu,J,K)$. *Then for* $|z| = r$

$$|f(z)| \leqslant r + \frac{|K-J|}{\{2(\mathfrak{p}+1)|t-1| + |2(1+K)-(1+t)(1+J)|\}\varphi_{\Psi}^2(\mu,\nu)} r^2,$$
$$|f(z)| \geqslant r - \frac{|K-J|}{\{2(\mathfrak{p}+1)|t-1| + |2(1+K)-(1+t)(1+J)|\}\varphi_{\Psi}^2(\mu,\nu)} r^2.$$

The result is sharp for the function given in (2.7) *for* $n = 2$.

Proof. Let $f \in \mathfrak{p}$-$\Phi\mathcal{S}^*(t,\mu,\nu,J,K)$. Using Theorem 1, we can deduce the following inequality:

$$\begin{aligned}|f(z)| &\leqslant |z| + \sum_{n=2}^{\infty} |a_n||z^n| \leqslant |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \\ &\leqslant r + \frac{|K-J|}{\{2(\mathfrak{p}+1)|t-1| + |2(1+K)-(1+t)(1+J)|\}\varphi_{\Psi}^2(\mu,\nu)} r^2.\end{aligned}$$

Similarly,

$$\begin{aligned}|f(z)| &\geqslant |z| - \sum_{n=2}^{\infty} |a_n||z^n| \geqslant |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \\ &\leqslant r - \frac{|K-J|}{\{2(\mathfrak{p}+1)|t-1| + |2(1+K)-(1+t)(1+J)|\}\varphi_{\Psi}^2(\mu,\nu)} r^2.\end{aligned}$$

□

Corollary 10. *Let* $f \in \mathfrak{p}$-$\Phi\mathcal{S}^*(-1,\mu,\nu,J,K)$. *Then for* $|z| = r$

$$|f(z)| \leqslant r + \frac{|K-J|}{\{4(\mathfrak{p}+1) + |2(1+K)|\}\varphi_{\Psi}^2(\mu,\nu)} r^2,$$
$$|f(z)| \geqslant r - \frac{|K-J|}{\{4(\mathfrak{p}+1) + |2(1+K)|\}\varphi_{\Psi}^2(\mu,\nu)} r^2.$$

The result is sharp for the function given in (2.8) *for* $n = 2$.

Corollary 11. *Let* $f \in \mathfrak{p}$-$\Phi\mathcal{S}^*(t,\mu,\nu,1,-1)$. *Then for* $|z| = r$

$$|f(z)| \leqslant r + \frac{1}{\{(\mathfrak{p}+1)|t-1| + |1+t|\}\varphi_{\Psi}^2(\mu,\nu)} r^2,$$
$$|f(z)| \geqslant r - \frac{1}{\{(\mathfrak{p}+1)|t-1| + |1+t|\}\varphi_{\Psi}^2(\mu,\nu)} r^2.$$

The result is sharp for the function given in (2.9) *for* $n = 2$.

Corollary 12. *Let* $f \in \mathfrak{p}\text{-}\Phi\mathcal{S}^*(t,\mu,\nu,1-2\alpha,-1)$*. Then for* $|z| = r$

$$|f(z)| \leqslant r + \frac{|\alpha-1|}{\{(\mathfrak{p}+1)|t-1| + |(\alpha-1)(1+t)|\}\varphi_\Psi^2(\mu,\nu)} r^2,$$
$$|f(z)| \geqslant r - \frac{|\alpha-1|}{\{(\mathfrak{p}+1)|t-1| + |(\alpha-1)(1+t)|\}\varphi_\Psi^2(\mu,\nu)} r^2.$$

The result is sharp for the function given in (2.10) *for* $n = 2$.

Corollary 13. *Let* $f \in 0\text{-}\Phi\mathcal{S}^*(t,\mu,\nu,1-2\alpha,-1)$*. Then for* $|z| = r$

$$|f(z)| \leqslant r + \frac{|\alpha-1|}{\{|t-1| + |(\alpha-1)(1+t)|\}\varphi_\Psi^2(\mu,\nu)} r^2,$$
$$|f(z)| \geqslant r - \frac{|\alpha-1|}{\{|t-1| + |(\alpha-1)(1+t)|\}\varphi_\Psi^2(\mu,\nu)} r^2.$$

The result is sharp for the function given in (2.11) *for* $n = 2$.

Theorem 5. *Let* $f \in \mathfrak{p}\text{-}\Phi\mathcal{S}^*(t,\mu,\nu,J,K)$*. Then for* $|z| = r$

$$|f'(z)| \leqslant 1 + \frac{2|K-J|}{\{2(\mathfrak{p}+1)|t-1| + |2(1+K)-(1+t)(1+J)|\}\varphi_\Psi^2(\mu,\nu)} r,$$
$$|f'(z)| \geqslant 1 - \frac{2|K-J|}{\{2(\mathfrak{p}+1)|t-1| + |2(1+K)-(1+t)(1+J)|\}\varphi_\Psi^2(\mu,\nu)} r.$$

The result is sharp for the function given in (2.7) *for* $n = 2$.

P r o o f. The proof is quite similar to Theorem 5. □

Corollary 14. *Let* $f \in \mathfrak{p}\text{-}\Phi\mathcal{S}^*(-1,\mu,\nu,J,K)$*. Then for* $|z| = r$

$$|f'(z)| \leqslant 1 + \frac{2|K-J|}{\{4(\mathfrak{p}+1) + |2(1+K)|\}\varphi_\Psi^2(\mu,\nu)} r,$$
$$|f'(z)| \geqslant 1 - \frac{2|K-J|}{\{4(\mathfrak{p}+1) + |2(1+K)|\}\varphi_\Psi^2(\mu,\nu)} r.$$

The result is sharp for the function given in (2.8) *for* $n = 2$.

Corollary 15. *Let* $f \in \mathfrak{p}\text{-}\Phi\mathcal{S}^*(t,\mu,\nu,1,-1)$*. Then for* $|z| = r$

$$|f'(z)| \leqslant 1 + \frac{2}{\{(\mathfrak{p}+1)|t-1| + |1+t|\}\varphi_\Psi^2(\mu,\nu)} r,$$
$$|f'(z)| \geqslant 1 - \frac{2}{\{(\mathfrak{p}+1)|t-1| + |1+t|\}\varphi_\Psi^2(\mu,\nu)} r.$$

The result is sharp for the function given in (2.9) *for* $n = 2$.

Corollary 16. *Let $f \in \mathfrak{p}\text{-}\Phi\mathcal{S}^*(t,\mu,\nu,1-2\alpha,-1)$. Then for $|z|=r$*

$$|f'(z)| \leqslant 1+\frac{2|\alpha-1|}{\{(\mathfrak{p}+1)|t-1|+|(\alpha-1)(1+t)|\}\varphi_\Psi^2(\mu,\nu)}r,$$
$$|f'(z)| \geqslant 1-\frac{2|\alpha-1|}{\{(\mathfrak{p}+1)|t-1|+|(\alpha-1)(1+t)|\}\varphi_\Psi^2(\mu,\nu)}r.$$

The result is sharp for the function given in (2.10) *for* $n=2$.

Corollary 17. *Let $f \in 0\text{-}\Phi\mathcal{S}^*(t,\mu,\nu,1-2\alpha,-1)$. Then for $|z|=r$*

$$|f'(z)| \leqslant 1+\frac{2|\alpha-1|}{\{|t-1|+|(\alpha-1)(1+t)|\}\varphi_\Psi^2(\mu,\nu)}r,$$
$$|f'(z)| \geqslant 1-\frac{2|\alpha-1|}{\{|t-1|+|(\alpha-1)(1+t)|\}\varphi_\Psi^2(\mu,\nu)}r.$$

The result is sharp for the function given in (2.11) *for* $n=2$.

3. Partial sums

In this section, we examine the ratio of the function of form (1.1) to its sequence of partial sums

$$f_q(z)=z+\sum_{n=2}^{q}a_nz^n$$

when the coefficients of f are sufficiently small to satisfy condition (2.1). We determine sharp lower bounds for

$$\Re\Big(\frac{f(z)}{f_q(z)}\Big),\quad \Re\Big(\frac{f_q(z)}{f(z)}\Big),\quad \Re\Big(\frac{f'(z)}{f'_q(z)}\Big),\quad \Re\Big(\frac{f'_q(z)}{f'(z)}\Big).$$

Theorem 6. *If f of the form* (1.1) *satisfies condition* (2.1)*. Then*

$$\Re\Big(\frac{f(z)}{f_q(z)}\Big) \geqslant 1-\frac{1}{\vartheta_{q+1}} \quad \forall\, z\in\mathbb{U}, \tag{3.1}$$

$$\Re\Big(\frac{f_q(z)}{f(z)}\Big) \geqslant \frac{\vartheta_{q+1}}{1+\vartheta_{q+1}} \quad \forall\, z\in\mathbb{U}, \tag{3.2}$$

where

$$\vartheta_q=\frac{\Upsilon_7}{|K-J|}. \tag{3.3}$$

The result is sharp for the function given in (2.7).

P r o o f. This can easily be verified:

$$\vartheta_{n+1} > \vartheta_n > 1 \quad \text{for } n > 2.$$

Therefore, in order to prove inequality (3.1), we set

$$\begin{aligned}\vartheta_{q+1}\Big(\frac{f(z)}{f_q(z)} - \Big(1 - \frac{1}{\vartheta_{q+1}}\Big)\Big) &= \frac{\vartheta_{q+1}(f(z) - f_q(z)) + f_q(z)}{f_q(z)} \\ &= \frac{1 + \sum_{n=2}^{q} a_n z^{n-1} + \vartheta_{q+1}\sum_{n=q+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{q} a_n z^{n-1}} \\ &= \frac{1 + \mathfrak{u}_1(z)}{1 + \mathfrak{u}_2(z)}.\end{aligned}$$

We now set

$$\frac{1 + \mathfrak{u}_1(z)}{1 + \mathfrak{u}_2(z)} = \frac{1 + \mathfrak{d}_1(z)}{1 - \mathfrak{d}_2(z)}.$$

A suitable simplification reveals that

$$\mathfrak{d}(z) = \frac{\mathfrak{u}_1(z) - \mathfrak{u}_2(z)}{2 + \mathfrak{u}_1(z) + \mathfrak{u}_2(z)}.$$

Thus, it can be seen that

$$\mathfrak{d}(z) = \frac{\vartheta_{q+1}\sum_{n=q+1}^{\infty} a_n z^{n-1}}{2 + 2\sum_{n=2}^{q} a_n z^{n-1} + \vartheta_{q+1}\sum_{n=q+1}^{\infty} a_n z^{n-1}}.$$

Following the trigonometric inequalities with $|z| < 1$, we obtain the following inequality:

$$|\mathfrak{d}(z)| \leqslant \frac{\vartheta_{q+1}\sum_{n=q+1}^{\infty} |a_n|}{2 - 2\sum_{n=2}^{q} |a_n| - \vartheta_{q+1}\sum_{n=q+1}^{\infty} |a_n|}.$$

This can now be seen as

$$|\mathfrak{d}(z)| \leqslant 1,$$

if and only if

$$2\vartheta_{q+1}\sum_{n=q+1}^{\infty} |a_n| \leqslant 2 - 2\sum_{n=2}^{q} |a_n|.$$

Hence, it implies

$$\sum_{n=2}^{q} |a_n| + \vartheta_{q+1}\sum_{n=q+1}^{\infty} |a_n| \leqslant 1. \tag{3.4}$$

It suffices to show that the left-hand side of (3.1) is bounded above by the following sum to prove the inequality in (3.4):

$$\sum_{n=2}^{\infty} \vartheta_n |a_n|.$$

This is equivalent to

$$\sum_{n=2}^{q}(\vartheta_n - 1)|a_n| + \sum_{n=q+1}^{\infty}(\vartheta_n - \vartheta_{q+1})|a_n| \geqslant 0. \tag{3.5}$$

Due to (3.5), the inequality in (3.1) can now be proven.

The next step is to prove inequality (3.2) by setting

$$\begin{aligned}(1+\vartheta_{q+1})\Big(\frac{f_q(z)}{f(z)} - \frac{\vartheta_{q+1}}{1+\vartheta_{q+1}}\Big) &= \frac{1+\sum_{n=2}^{q} a_n z^{n-1} - \vartheta_{q+1}\sum_{n=q+1}^{\infty} a_n z^{n-1}}{1+\sum_{n=2}^{\infty} a_n z^{n-1}}\\ &= \frac{1+\mathfrak{d}_1(z)}{1-\mathfrak{d}_2(z)},\end{aligned}$$

where

$$|\mathfrak{d}(z)| \leqslant \frac{(1+\vartheta_{q+1})\sum_{n=q+1}^{\infty}|a_n|}{2-2\sum_{n=2}^{q}|a_n| - (\vartheta_{q+1}-1)\sum_{n=q+1}^{\infty}|a_n|} \leqslant 1. \tag{3.6}$$

This last inequality in (3.6) is equivalent to

$$\sum_{n=2}^{q}|a_n| + \vartheta_{q+1}\sum_{n=q+1}^{\infty}|a_n| \leqslant 1. \tag{3.7}$$

In addition, we find that the inequality in (3.7) is bounded above by the following sum:

$$\sum_{n=2}^{\infty}\vartheta_n|a_n|.$$

Assertion (3.2) has been proved. □

Corollary 18. *If f of the form* (1.1) *satisfies conditions* (2.2)–(2.5), *respectively, then*

$$\Re\Big(\frac{f(z)}{f_q(z)}\Big) \geqslant 1 - \frac{1}{\vartheta_{q+1}(i)} \quad \forall\, z \in \mathbb{U},\ i = 1,2,3,4, \tag{3.8}$$

$$\Re\Big(\frac{f_q(z)}{f(z)}\Big) \geqslant \frac{\vartheta_{q+1}(i)}{1+\vartheta_{q+1}(i)} \quad \forall\, z \in \mathbb{U},\ i = 1,2,3,4, \tag{3.9}$$

where

$$\vartheta_q(1) = \frac{\Upsilon_3}{2|K-J|}, \tag{3.10}$$

$$\vartheta_q(2) = \Upsilon_4, \tag{3.11}$$

$$\vartheta_q(3) = \frac{\Upsilon_5}{|\alpha-1|}, \tag{3.12}$$

$$\vartheta_q(4) = \frac{\Upsilon_6}{|\alpha-1|}. \tag{3.13}$$

Equations (3.8) *and* (3.9) *are the partial sums of the classes* $\mathfrak{p}$-$\Phi\mathcal{S}^*(-1,\mu,\nu,J,K)$, $\mathfrak{p}$-$\Phi\mathcal{S}^*(t,\mu,\nu,1,-1)$, $\mathfrak{p}$-$\Phi\mathcal{S}^*(t,\mu,\nu,1-2\alpha,-1)$, 0-$\Phi\mathcal{S}^*(t,\mu,\nu,1-2\alpha,-1)$, *respectively. The result is sharp for the functions given in* (2.8)–(2.11), *respectively.*

Following are ratios involving derivatives.

Theorem 7. *If f of the form* (1.1) *satisfies condition* (2.1), *then*

$$\Re\Big(\frac{f'(z)}{f_q'(z)}\Big) \geqslant 1-\frac{q+1}{\vartheta_{q+1}} \quad \forall\, z\in\mathbb{U},$$
$$\Re\Big(\frac{f_q'(z)}{f'(z)}\Big) \geqslant \frac{\vartheta_{q+1}}{\vartheta_{q+1}+q+1} \quad \forall\, z\in\mathbb{U},$$

where ϑ_q is given by (3.3). *The result is sharp for the function given in* (2.7).

P r o o f. The proof of Theorem 7 is similar to that of Theorem 6. The analogous details are omitted here. □

Corollary 19. *If f of the form* (1.1) *satisfies conditions* (2.2)–(2.5), *respectively, then*

$$\Re\Big(\frac{f'(z)}{f_q'(z)}\Big) \geqslant 1-\frac{q+1}{\vartheta_{q+1}(i)} \quad \forall\, z\in\mathbb{U},\ i=1,2,3,4, \tag{3.14}$$
$$\Re\Big(\frac{f_q'(z)}{f'(z)}\Big) \geqslant \frac{\vartheta_{q+1}(i)}{\vartheta_{q+1}(i)+q+1} \quad \forall\, z\in\mathbb{U},\ i=1,2,3,4, \tag{3.15}$$

where $\vartheta_q(1)$, $\vartheta_q(2)$, $\vartheta_q(3)$, $\vartheta_q(4)$ are given by (3.10)–(3.13) *accordingly. The equations* (3.14) *and* (3.15) *are the partial sums of the classes* $\mathfrak{p}$-$\Phi\mathcal{S}^*(-1,\mu,\nu,J,K)$, $\mathfrak{p}$-$\Phi\mathcal{S}^*(t,\mu,\nu,1,-1)$, $\mathfrak{p}$-$\Phi\mathcal{S}^*(t,\mu,\nu,1-2\alpha,-1)$, 0-$\Phi\mathcal{S}^*(t,\mu,\nu,1-2\alpha,-1)$, *respectively. The result is sharp for the functions given in* (2.8)–(2.11), *respectively.*

4. Concluding remarks and observations

In our present work, by making use of the idea of Mittag-Leffler type Poisson distribution, we have defined and studied certain new subclasses of Sakaguchi type functions. Further, we have discussed some important geometric properties, like necessary and sufficient condition, convex combination, growth and distortion bounds, and partial sums for this newly defined class.

A c k n o w l e d g m e n t s The authors wish to thank the referees and the editor for their valuable comments.

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