ON MEAN VALUE PROPERTIES INVOLVING A LOGARITHM-TYPE WEIGHT

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In memoriam Ivan Netuka

Abstract. Two new assertions characterizing analytically disks in the Euclidean plane $\mathbb{R}^2$ are proved. Weighted mean value property of positive solutions to the Helmholtz and modified Helmholtz equations are used for this purpose; the weight has a logarithmic singularity. The obtained results are compared with those without weight that were found earlier.

Keywords: harmonic function; Helmholtz equation; modified Helmholtz equation; mean value property; logarithmic weight; characterization of balls

MSC 2020: 31A10, 35B05, 35J05

1. Introduction

This note is a tribute to Ivan Netuka—a colleague with whom the author shared interest in potential theory and boundary value problems. Netuka's surveys [12] and [13] (the latter one joint with J. Veselý) motivated him to begin (in his late 70s!!) studies in the vast field of mean value properties. These surveys were very helpful in discovering a gap in this area; it concerned the mean value properties over spheres and balls and the corresponding converse theorems for solutions of the m-dimensional Helmholtz and modified Helmholtz equations (these solutions are also known as meta- and panharmonic functions, respectively). Thus, the author's initial papers in this area (see [7] and [9]) deal with filling in this gap; see also the article [10], where investigations of mean value properties of panharmonic functions are summarized.

Subsequently, the results reviewed in Sections 7 and 8 of the survey [13] attracted the author's attention. In these sections, the so-called inverse mean value properties (the term, presumably, coined by Hansen and Netuka in [3] and [4]) of harmonic func-

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tions are considered. They characterize open sets by quadrature identities valid for some particular class of functions. The best known result of this kind is the following.

Theorem 1.1 (Kuran [5]). *Let D be a domain (= connected open set) of finite (Lebesgue) measure in the Euclidean space $\mathbb{R}^m$ where $m \geqslant 2$. Suppose that there exists a point P_0 in D such that, for every function h harmonic in D and integrable over D, the volume mean of h over D equals $h(P_0)$. Then D is an open ball (disk when $m = 2$) centred at P_0.*

However, it was unknown for a long time whether an assertion similar to this theorem is true if solutions of other partial differential equations are used instead of harmonic functions; of course, the identity for the arithmetic mean over balls must be adjusted to these solutions. Thus, it was interesting to investigate this question.

Indeed, it occurs that a characterization of m-dimensional balls by solutions to the modified Helmholtz equation

$$\nabla^2 u - \mu^2 u = 0, \quad \mu \in \mathbb{R} \setminus \{0\}, \tag{1.1}$$

is possible as was shown in [6]. (Here and below, $\nabla = (\partial_1, \dots, \partial_m)$, $\partial_i = \partial/\partial x_i$, denotes the gradient operator.) In what follows, instead of the cumbersome 'solution of the modified Helmholtz equation' the term 'μ-panharmonic function' is used; this convenient abbreviation was introduced by Duffin (see [1]). Since this characterization is related to a result obtained in this note, we reproduce it, but prior to that we introduce some notation and terminology used below.

The open ball $B_r(x) = \{y \colon |y - x| < r\}$ centred at $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ and having radius r is called admissible with respect to a domain D provided $\overline{B_r(x)} \subset D$. By D_r the union of D and $\bigcup_{x \in \partial D} B_r(x)$ (the r-neighbourhood of a domain $D \subset \mathbb{R}^m$) is denoted. If D has a finite Lebesgue measure and a function f is integrable over D, then

$$M(f, D) = \frac{1}{|D|} \int_D f(x) \, \mathrm{d}x$$

is its volume mean value over D; here $|D|$ denotes the domain's volume. The function

$$a(t) = \Gamma\Big(\frac{m}{2} + 1\Big) \frac{I_{m/2}(t)}{(\frac{1}{2}t)^{m/2}}, \tag{1.2}$$

where I_ν stands for the modified Bessel function of the first kind of order ν, arises in the m-dimensional mean value formula for balls

$$a(\mu r) u(x) = M(u, B_r(x)), \quad x \in D. \tag{1.3}$$

This identity holds if u is μ-panharmonic in a domain $D \subset \mathbb{R}^m$ and $B_r(x)$ is admissible; see [7], pages 676, 677.

Now we are in a position to formulate the following theorem.

Theorem 1.2 (Kuznetsov [6]). *Let $D \subset \mathbb{R}^m$, $m \geqslant 2$, be a bounded domain, and let $r > 0$ be such that $|B_r(0)| = |D|$. Suppose that there exists $x_0 \in D$ such that for some $\mu > 0$ the identity $a(\mu r) u(x_0) = M(u, D)$ holds for every positive function u which is μ-panharmonic in D_r, then $D = B_r(x_0) \setminus A$, where A is a closed set of zero Lebesgue measure such that $x_0 \notin A$.*

Unfortunately, the formulations of this theorem in [6] and [10], § 4.1 miss out the fact that its assumptions imply that the domain D coincides with a ball only up to a closed set of zero measure.

For solutions of the Helmholtz equation (it differs from (1.1) by the sign on the left-hand side), also referred to as μ-metaharmonic functions, the following analogue of the mean value identity (1.3) is valid for every admissible ball $B_r(x)$:

$$a_m(\mu r) u(x) = M(u, B_r(x)), \quad a_m(t) = \Gamma\left(\frac{m}{2} + 1\right) \frac{J_{m/2}(t)}{(\frac{1}{2} t)^{m/2}}. \tag{1.4}$$

As usual, J_ν denotes the Bessel function of order ν; its nth positive zero is denoted by $j_{\nu,n}$ (this standard notation is used below). Since $J_{m/2}(t)$ is monotonic only on a bounded interval adjacent to zero, an upper bound on the size of D is imposed in the following analogue of Theorem 1.2.

Theorem 1.3 (Kuznetsov [8]). *Let $D \subset \mathbb{R}^m$, $m \geqslant 2$, be a bounded domain, and let $r > 0$ be such that $|B_r(0)| = |D|$. Suppose that for some $\mu > 0$ and a point $x_0 \in D$ the identity $u(x_0) a_m(\mu r) = M(u, D)$ holds for every u, which is μ-metaharmonic in D_r. If also*

$$D_r \subset B_{r_0}(x_0), \quad \text{where } \mu r_0 = j_{m/2,1}, \tag{1.5}$$

then $D = B_r(x_0) \setminus A$, where A is a closed set of zero Lebesgue measure and such that $x_0 \notin A$.

Similar to Theorem 1.2, the formulation of this theorem in [8] misses out the fact that its assumptions imply that D coincides with a ball only up to a closed set of zero measure.

In the recent paper (see [11]), some weighted mean value properties of harmonic functions were considered. (It must be pointed out that characterizations of balls found in [11] require the same corrections as those discussed after Theorems 1.2 and 1.3.) For meta- and panharmonic functions in plane domains analogous properties involving a logarithm-type weight immediately follow from the representation valid for any $w \in C^2(D)$ and every admissible disk (see [2], § 2.1),

$$w(x) = \frac{1}{2\pi r} \int_{\partial B_r(x)} w(y)\, \mathrm{d}S_y - \frac{1}{2\pi} \int_{B_r(x)} \nabla^2 w(y) \log \frac{r}{|x-y|}\, \mathrm{d}y,$$

and the expressions for the arithmetic mean over a circumference (the first term on the right); see [7], Theorem 2.1.

Let us formulate these identities because the aim of this note is to obtain the corresponding inverse properties and to compare them with Theorems 1.2 and 1.3. We consider μ-panharmonic functions first.

Theorem 1.4. *Let D be a domain in $\mathbb{R}^2$. If u is μ-panharmonic in D, then*

$$\widehat{a}(\mu r) u(x) = \frac{1}{\pi r^2} \int_{B_r(x)} u(y) \log \frac{r}{|x-y|}\, \mathrm{d}y, \quad \widehat{a}(t) = \frac{2[I_0(t) - 1]}{t^2} \tag{1.6}$$

for every admissible disk $B_r(x)$.

The behaviour of logarithmic-type weight in the identity (1.6) is quite simple: it is a positive function of y within $B_r(x)$, growing from zero attained at $y \in \partial B_r(x)$ to infinity as $|x-y| \to 0$, and is negative when $y \notin \overline{B_r(x)}$.

Let us compare $\widehat{a}(t)$ with $a(t) = 2t^{-1} I_1(t)$, expressing (1.2) for $m = 2$. By the definition of I_0, we have that

$$\widehat{a}(0) = \lim_{t \to +0} \widehat{a}(t) = \frac{1}{2},$$

and $\widehat{a}(t)$ increases monotonically from this value to infinity similar to $a(t)$. Moreover,

$$a(t) - \widehat{a}(t) = \frac{2[t I_1(t) - I_0(t) + 1]}{t^2}$$

is positive for all $t \in [0, \infty)$. Indeed,

$$[t I_1(t) - I_0(t) + 1]' = I_1(t) + t I_1'(t) - I_1(t) = \frac{t[I_2(t) + I_0(t)]}{2},$$

which through straightforward calculations implies that $a(t) - \widehat{a}(t)$ increases on $(0, \infty)$ and

$$\lim_{t \to +0} [a(t) - \widehat{a}(t)] = \lim_{t \to +0} \frac{[t I_1(t) - I_0(t) + 1]'}{t} = \frac{1}{2}.$$

For μ-metaharmonic functions, the analogue of Theorem 1.4 is the following.

Theorem 1.5. *Let D be a domain in $\mathbb{R}^2$. If u is μ-metaharmonic in D, then*

$$\widetilde{a}(\mu r) u(x) = \frac{1}{\pi r^2} \int_{B_r(x)} u(y) \log \frac{r}{|x-y|}\, \mathrm{d}y, \quad \widetilde{a}(t) = \frac{2[1 - J_0(t)]}{t^2} \tag{1.7}$$

for every admissible disk $B_r(x)$.

The behaviour of $\widetilde{a}$ is as follows: $\widetilde{a}(0) = \lim\limits_{t\to+0} \widetilde{a}(t) = \frac{1}{2}$, whereas $\widetilde{a}(t)$ approaches zero as $t \to \infty$ decreasing nonmonotonically, but remaining positive. The latter property of $\widetilde{a}(t)$ distinguishes it from $a_2(t) = 2t^{-1}J_1(t)$—the coefficient in the area mean value identity (1.4). The latter coefficient has infinitely many zeros.

2. On analytic characterization of disks in $\mathbb{R}^2$

The following analogue of Theorem 1.2 is based on the weighted mean value property of μ-panharmonic functions.

Theorem 2.1. *Let $D \subset \mathbb{R}^2$ be a bounded domain, and let $r > 0$ be such that $|D| \geqslant \pi r^2$. Suppose that there exists $x_0 \in D$ such that for some $\mu > 0$ the identity*

$$\widehat{a}(\mu r)u(x_0) = \frac{1}{|D|}\int_D u(y)\log\frac{r}{|x_0 - y|}\,\mathrm{d}y \tag{2.1}$$

holds for every function $u > 0$, which is μ-panharmonic in D_r, then $D = B_r(x_0) \setminus A$, where A is a closed set of zero Lebesgue measure and such that $x_0 \notin A$.

Prior to proving this theorem, we notice that the radially symmetric function

$$\widehat{U}(x) = I_0(\mu|x|), \quad x \in \mathbb{R}^2, \tag{2.2}$$

monotonically increases from $\widehat{U}(0) = 1$ to infinity as $|x|$ goes to infinity. Moreover, it is μ-panharmonic in $\mathbb{R}^2$; this follows by comparing the modified Bessel equation for I_0 (see [14], page 223) with (1.1) in polar coordinates.

Proof of Theorem 2.1. Without loss of generality, we suppose that the domain D is located so that x_0 coincides with the origin. Let us consider the bounded open set $G_i = D \setminus \overline{B_r(0)}$ and the set $G_e = B_r(0) \setminus D$, and assume that there is no set A' of zero measure such that $D = B_r(0) \setminus A'$. Combining this assumption and the inequality $|D| \geqslant \pi r^2$ (see the theorem's formulation), one concludes that G_i is not empty. Since a contradiction is obtained from this fact below, it is shown that $D = B_r(0) \setminus A'$, where A' has zero measure. Since D is open, the set $A = A' \cup \partial B_r(0)$ is closed and has zero measure; moreover, $D = B_r(0) \setminus A$. Taking into account that $\widehat{U}(0) = 1$, we write (2.1) for $\widehat{U}$ as

$$|D|\widehat{a}(\mu r) = \int_D \widehat{U}(y)\log\frac{r}{|y|}\,\mathrm{d}y. \tag{2.3}$$

Since the identity (1.4) holds for $\widehat{U}$ over $B_r(0)$, we write it in the same way,

$$\pi r^2\widehat{a}(\mu r) = \int_{B_r(0)} \widehat{U}(y)\log\frac{r}{|y|}\,\mathrm{d}y. \tag{2.4}$$

Subtracting (2.4) from (2.3), we obtain

$$[|D| - \pi r^2]\widehat{a}(\mu r) = \int_{G_i} \widehat{U}(y) \log \frac{r}{|y|}\,\mathrm{d}y - \int_{G_e} \widehat{U}(y) \log \frac{r}{|y|}\,\mathrm{d}y. \tag{2.5}$$

Here the difference on the right-hand side is negative. Indeed, $\widehat{U} > 0$ everywhere, whereas $\log(r/|y|) < 0$ on $G_i \neq \emptyset$, because $|y| > r$ there. Hence, the first term is negative. The second integral is nonnegative because $\log(r/|y|) > 0$ provided $y \in B_r(0) \setminus \{0\}$. The obtained contradiction proves the theorem. $\square$

Let us compare Theorems 1.2 and 2.1. It occurs that the proof of Theorem 1.2 for $m = 2$ involves the same equalities (2.3)–(2.5), but without the factor $\log(r/|y|)$ in the integrands. (Notice that U stands in [6] for the function denoted here by $\widehat{U}$.) Therefore, the fact that $\widehat{U}(y)$ monotonically increases with $|y|$ is essential for that proof implying that this function is greater (less) than $[\widehat{U}(y)]_{|y|=r}$ on G_i (G_e, respectively). In its turn, this yields that the right-hand side is positive in the analogue of (2.5), because $|G_i| = |G_e|$ in view of the assumption that $|D| = |B_r(0)|$, which, therefore, is crucial for obtaining a contradiction in the proof of Theorem 1.2.

On the other hand, the weight function $\log(r/|y|)$ is crucial for demonstrating that the difference is negative on the right-hand side of (2.5), whereas only the fact that $\widehat{U} > 0$ is of importance about this function. Therefore, the weaker assumption $|D| - \pi r^2 \geqslant 0$ is sufficient for obtaining a contradiction in the proof of Theorem 2.1.

Now, we turn to the inverse of the mean value property (1.7); it is worth recalling that the latter identity is analogous to (1.4), whose inverse is formulated in Theorem 1.3.

Theorem 2.2. *Let $D \subset \mathbb{R}^2$ be a bounded domain, and let $r > 0$ be such that $|D| \geqslant \pi r^2$. Suppose that there exists $x_0 \in D$ such that for some $\mu > 0$ the identity*

$$\widetilde{a}(\mu r) u(x_0) = \frac{1}{|D|} \int_D u(y) \log \frac{r}{|x_0 - y|}\,\mathrm{d}y$$

holds for every positive function u, which is μ-metaharmonic in D_r. If also

$$D_r \subset B_{r_0}(x_0), \quad \text{where } \mu r_0 = j_{0,1}, \tag{2.6}$$

then $D = B_r(x_0) \setminus A$, where A is a closed set of zero Lebesgue measure and such that $x_0 \notin A$.

Since $\widetilde{a} > 0$, the proof of Theorem 2.1 is applicable to this case, but with the following essential distinction: indeed, it involves the radially symmetric function $\widetilde{U}(x) = J_0(\mu|x|)$, $x \in \mathbb{R}^2$, instead of $\widehat{U}$; see (2.2). For this reason, the condition (2.6) is imposed to restrict the domain's size because $\widetilde{U}$ is not positive everywhere and this condition describes the domain surrounding the origin, where $\widetilde{U}(x) > 0$. It must be emphasized that (2.6) is more restrictive than (1.5) used in Theorem 1.3. However, the weaker assumption $|D| \geqslant \pi r^2$ replaces the strict equality.

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