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NON-HOMOGENEOUS DIRECTIONAL EQUATIONS: SLICE SOLUTIONS BELONGING TO FUNCTIONS OF BOUNDED L-INDEX IN THE UNIT BALL

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Abstract. For a given direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ we study non-homogeneous directional linear higher-order equations whose all coefficients belong to a class of joint continuous functions which are holomorphic on intersection of all directional slices with a unit ball. Conditions are established providing boundedness of L-index in the direction with a positive continuous function L satisfying some behavior conditions in the unit ball. The provided conditions concern every solution belonging to the same class of functions as the coefficients of the equation. Our considerations use some estimates involving a directional logarithmic derivative and distribution of zeros on all directional slices in the unit ball.

Keywords: bounded index; bounded L-index in direction; slice function; holomorphic function; directional differential equation; bounded l-index; directional derivative; unit ball

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1. Introduction

American mathematician Lepson in [20] and his scholar MacDonnell in [21] were the first to introduce the notion of bounded index for entire functions of one variable. An entire function $f\colon \mathbb{C} \to \mathbb{C}$ is called a function of bounded index if there exists $m \in \mathbb{Z}_+$ such that for every $p \in \mathbb{Z}_+$ and all $z \in \mathbb{C}$ one has

$$\frac{|f^{(p)}(z)|}{p!} \leqslant \max_{0\leqslant s\leqslant m} \frac{|f^{(s)}(z)|}{s!}. \tag{1.1}$$

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The least such integer m satisfying (1.1) is called the index of the function f. These functions have fascinating properties: some local regular behavior (see [14]), a uniform distribution of zeros and a bounded logarithmic derivative outside an exceptional set (see [15]), their antiderivative has bounded value distribution (see [16]), etc. Moreover, every entire solution of a linear higher-order homogeneous differential equation with constant coefficients possesses property (1.1), i.e., it is of bounded index (see [24]). Later a similar fact was established when the coefficients $a_j = a_j(z)$ of the equation $f^{(n)}(t) + \sum\limits_{j=0}^{n-1} a_j f^{(j)}(t) = 0$ are polynomials (see [25]). It is obviuos that every polynomial has bounded index. However, Shah in [26] and Hayman in [16] proved that each entire function having bounded index is a function of exponential type.

Therefore, Kuzyk and Sheremeta proposed an extension of this notion introducing a function of bounded l-index (see [18]). They replaced $p!$ and $s!$ by $p!\, l^p(z)$ and $s!\, l^s(z)$ in the previous definition, respectively, where $l\colon \mathbb{C} \to \mathbb{R}_+$ belongs to continuous functions. Such a class of functions having bounded l-index is very wide. If the multiplicities of zeros of an entire function are uniformly bounded, then there exists a positive continuous function l providing boundedness of the l-index for the entire function (see [12], [28]). This notion was also generalized for functions of several complex variables by two approaches. The first approach uses all possible partial derivatives in the definition (bounded index in joint variables, see [6], [22], [23]) and the second approach uses directional derivatives in the definition (bounded index in a direction, see [5], [7], [17]). Although the first approach seems more natural, the second approach allows to deduce more multi-dimensional analogs of known one-dimensional propositions describing properties of functions having bounded index. In particular, the bounded index in direction helps to obtain an analog of logarithmic criterion even for bounded index in joint variables (see [1]). This criterion is a central tool in applications of the theory of a bounded index to differential equations.

Recently, we started to study a sufficiently general class of holomorphic functions. Our researches were inspired by the following Favorov's problem (see [8]):

Question 1.1 ([8]). Are deducible known propositions on properties of analytic functions having bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ for joint continuous and holomorphic on the slices $\{z^0 + t\mathbf{b}\colon\ t \in \mathbb{C}\}$ functions?

Here we continue our investigations initialized in [2], [3], [4]. A concept of L-index boundedness in direction for slice analytic functions of several complex variables was introduced and many criteria of L-index boundedness in direction were obtained. We present some applications of these criteria to consider slice analytic solutions of directional differential equations. Since functions of bounded index have many applications in analytic theory of differential equations (see [6], [10], [23]), we study the local behavior of slice holomorphic functions of bounded L-index in direction which satisfies some linear higher-order directional differential equations.

In [11], a question was considered on the additional conditions, providing index boundedness of every slice entire solution for linear higher-order directional differential equation with slice entire coefficients. Here we consider the question for slice holomorphic functions in the unit ball.

Let us introduce some notations from [2]. Let $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_+^* = [0, \infty)$, $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{1} = (1, \ldots, 1)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction, $\mathbb{B}^n = \{z \in \mathbb{C}^n \colon |z| < 1\}$ be a unit ball, $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ be a unit disc, $L \colon \mathbb{B}^n \to \mathbb{R}_+$ be a continuous function. For a given $z \in \mathbb{B}^n$, we denote $S_z = \{t \in \mathbb{C} \colon z + t\mathbf{b} \in \mathbb{B}^n\}$. Clearly, $\mathbb{D} = \mathbb{B}^1$. The slice functions on S_z for fixed $z^* \in \mathbb{B}^n$ will be denoted as $g_{z^*}(t) = F(z^* + t\mathbf{b})$ and $l_{z^*}(t) = L(z^* + t\mathbf{b})$ for $t \in S_z$.

Let $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ be a class of functions which are holomorphic on every slice $\{z^* + t\mathbf{b} \colon t \in S_{z^*}\}$ for each $z^* \in \mathbb{B}^n$ and let $\mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ be a class of functions from $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ which are joint continuous. The notation $\partial_{\mathbf{b}} F(z)$ stands for the derivative of the function $g_z(t)$ at the point 0, i.e., for every $p \in \mathbb{N}$, $\partial_{\mathbf{b}}^p F(z) = g_z^{(p)}(0)$, where $g_z(t) = F(z + t\mathbf{b})$ is an analytic function of complex variable $t \in S_z$ for given $z \in \mathbb{B}^n$. Besides, we denote by $\langle a, c \rangle = \sum_{j=1}^{n} a_j \overline{c_j}$ the Hermitian inner product in $\mathbb{C}^n$, where $a, c \in \mathbb{C}^n$.

The hypothesis on joint continuity together with the hypothesis on holomorphy in one direction do not imply holomorphy in whole n-dimensional unit ball. Some examples to demonstrate it were presented in [2].

A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is said to be of *bounded L-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ (we denote this class by $B_{\mathbf{b}}(L, \mathbb{B}^n)$) (see [2]) if there exists $m_0 \in \mathbb{Z}_+$ such that for all $m \in \mathbb{Z}_+$ and each $z \in \mathbb{B}^n$, inequality

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m!\, L^m(z)} \leqslant \max_{0 \leqslant k \leqslant m_0} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k!\, L^k(z)} \tag{1.2}$$

holds. The least such integer number m_0, obeying (1.2), is called the L-index in the direction $\mathbf{b}$ of the function F and is denoted by $N_{\mathbf{b}}(F, L, \mathbb{B}^n)$. For $n = 1$, $\mathbf{b} = 1$, $L(z) = l(z)$, and $z \in \mathbb{C}$ instead $z \in \mathbb{B}$, inequality (1.2) defines a function of bounded l-index with the l-index $N(F, l) \equiv N_1(F, l, \mathbb{C})$ (see [19], [28], [29]), and if in addition $l(z) \equiv 1$, then we obtain a definition of index boundedness with index $N(F) \equiv N_1(F, 1, \mathbb{C})$ (see [20], [21]). It is also worth mentioning paper [31], which introduces the concept of generalized index. It is quite close to the bounded l-index.

To obtain constructive and sufficiently general results, one must impose additional restrictions on the function L, since continuity alone is not enough for this.

For $z\in\mathbb{B}^n$ we denote

$$\lambda_{\mathbf{b}}(\eta)=\sup\Big\{\sup_{t,\tau\in S_z}\Big\{\frac{L(z+t\mathbf{b})}{L(z+\tau\mathbf{b})}\colon\ |t-\tau|\leqslant\frac{\eta}{\min\{L(z+t\mathbf{b}),L(z+\tau\mathbf{b})\}}\Big\}\colon\ z\in\mathbb{B}^n\Big\}.$$

Let $Q_{\mathbf{b}}(\mathbb{B}^n)$ be the class of positive continuous functions $L\colon\ \mathbb{B}^n\to\mathbb{R}_+$ satisfying for given $\beta>1$ and for every $\eta\in[0,\beta]$ the condition

$$\lambda_{\mathbf{b}}(\eta)<\infty$$

and for all $z\in\mathbb{B}^n$ the inequality

$$L(z)>\frac{\beta|\mathbf{b}|}{1-|z|}. \tag{1.3}$$

The last condition (1.3) is, in a sense, final.

2. Auxiliary propositions

Throughout this section, we will assume that $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$, $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$, where $\mathbf{b}\in\mathbb{C}\setminus\{0\}$, $n\geqslant 2$. Let us formulate several theorems proved in articles [3], [4], which we will use in what follows. The following theorem from [4] describes sufficient conditions for the boundedness of L-index in direction in terms of the local behavior of the maximum of the modulus of the function.

Theorem 2.1 ([4])**.** *If there exist $r_1,r_2\in(0,\beta]$, $r_1<r_2$, and exists $P_1\geqslant 1$ for all $z\in\mathbb{B}^n$ such that*

$$\max\{|F(z+t\mathbf{b})|\colon\ |t|=r_2/L(z)\}\leqslant P_1\max\{|F(z+t\mathbf{b})|\colon\ |t|=r_1/L(z)\},$$

then $F\in B_{\mathbf{b}}(L,\mathbb{B}^n)$.

We also need the following analogue of the theorem obtained by Fricke (see [15]) for entire functions of bounded index of one complex variable.

Theorem 2.2 ([4])**.** *If $F\in B_{\mathbf{b}}(L,\mathbb{B}^n)$, then for all $R\in(0,\beta)$ there exists $P_2(R)\geqslant 1$, exists $\eta(R)\in(0,R)$ and for all $z\in\mathbb{B}^n$ there exists $r=r(z)\in[\eta(R),R]$ such that*

$$\max\{|F(z+t\mathbf{b})|\colon\ |t|=r/L(z)\}\leqslant P_2(R)\min\{|F(z+t\mathbf{b})|:|t|=r/L(z)\}. \tag{2.1}$$

Also, we use an analogue of logarithmic criterion for function from the class $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$. The one-dimensional analogue of the criterion is efficient to investigate boundedness of l-index of infinite products (see [13], [27], [30]). As a necessary conditions the criterion was obtained by Fricke (see [14], [15]) for entire functions of one complex variable having bounded index. We put

$$G_r(F) := G_r^{\mathbf{b}}(F) := \bigcup_{a\in\mathbb{B}^n:\ F(a)=0} \{a+t\mathbf{b}:\ t\in\mathbb{C},\ |t|<r/L(a)\}.$$

For given $z\in\mathbb{B}^n$ and $r>0$ we denote by

$$n_z(r) = n_{\mathbf{b}}(r,z,1/F) := \sum_{|a_k^0|\leqslant r} 1$$

the counting function of zeros a_k^0 of the slice function $g_z(t) = F(z+t\mathbf{b})$ in the disc $\{t\in\mathbb{C}:\ |t|\leqslant r\}$. If for given $z\in\mathbb{B}^n$ and for all $t\in S_z:\ g_z(t) = F(z+t\mathbf{b})\equiv 0$, then we put $n_z(r) = -1$. Denote also $n(r) = \sup_{z\in\mathbb{B}^n} n_z(r/L(z))$.

Theorem 2.3 ([4]). *If $F\in B_{\mathbf{b}}(L,\mathbb{B}^n)$, then*

(1) *for all $r\in(0,\beta]$ there exists $P=P(r)>0$ and for all $z\in\mathbb{B}^n\setminus G_r^{\mathbf{b}}(F)$:*

$$|\partial_{\mathbf{b}}F(z)|\leqslant PL(z)|F(z)|;$$

(2) *for all $r\in(0,\beta]$ there exists $\widetilde{n}(r)\in\mathbb{Z}_+$ and for all $z^0\in\mathbb{B}^n, F(z^0+t\mathbf{b})\not\equiv 0$:*

$$n_{\mathbf{b}}(r/L(z^0),z^0,1/F)\leqslant\widetilde{n}(r).$$

The following statement is actually obtained in the proof of Theorem 2.3 in [4]:

Lemma 2.1. *If $F\in B_{\mathbf{b}}(L,\mathbb{B}^n)$, then for all $z^0\in\mathbb{B}^n\setminus G_r^{\mathbf{b}}(F)$, $r\in(0,\beta)$, and for all $\widetilde{a}^k = z^0 + a_k^0\mathbf{b}\in\mathbb{B}^n$, $F(\widetilde{a}^k)=0$, we have*

$$|z^0-\widetilde{a}_k| > \frac{r|\mathbf{b}|}{2L(z^0)\lambda_{\mathbf{b}}(r)}. \tag{2.2}$$

The following (Theorem 2.4) is an analogue of Hayman's theorem (see [16]) proved for entire functions of single variable.

Theorem 2.4 ([3]). *A function $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is of bounded L-index in the direction $\mathbf{b}$ if and only if there exists $p\in\mathbb{Z}_+$, exists $C>0$ and for all $z\in\mathbb{B}^n$:*

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L^{(p+1)}(z)}\leqslant C\max\Big\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{L^{-k}(z)}:\ 0\leqslant k\leqslant p\Big\}.$$

In the proof of Lemma 2.2 we will use methods from a proof of its counterparts in [7] for another class of functions which are analytic in some domain from n-dimensional complex space.

Lemma 2.2. *Let $F \in B_{\mathbf{b}}(L, \mathbb{B}^n)$. Then for all $r > 0$ and for all $m \in \mathbb{N}$ there exists $P = P(r, m) > 1$ such that for all $z \in \mathbb{B}^n \setminus G_r^{\mathbf{b}}(F)$: $|\partial_{\mathbf{b}}^m F(z)| \leqslant PL^m(z)|F(z)|$.*

Proof. By Theorem 2.2 with $R = r/(2\lambda_{\mathbf{b}}(r))$, there exist $P_2 = P_2(R) \geqslant 1$, $\eta = \eta(R) \in (0, R)$ such that for every $z \in \mathbb{B}^n$ and some $r^* = r^*(z) \in [\eta(R), R]$, inequality (2.1) is true with $r = r^*$. Applying the Cauchy inequality, for $r = r^*$ we have

$$\frac{1}{m!}|\partial_{\mathbf{b}}^m F(z)| \leqslant \Big(\frac{L(z)}{r}\Big)^m \max\Big\{|F(z + t\mathbf{b})|\colon\ |t| = \frac{r}{L(z)}\Big\}.$$

Hence, by inequalities (2.1) and $\eta \leqslant r = r^*$ we get

$$\frac{1}{m!}|\partial_{\mathbf{b}}^m F(z)| \leqslant P_2\Big(\frac{L(z)}{\eta}\Big)^m \min\Big\{|F(z + t\mathbf{b})|\colon\ |t| = \frac{r^*}{L(z)}\Big\}.$$

Inequality (2.2) from Lemma 2.1 implies that $F(z + t\mathbf{b}) \neq 0$ on the set $\{t\colon\ |t| \leqslant r/(2\lambda_2^{\mathbf{b}}(r)L(z))\}$ for every $z \in \mathbb{B}^n \setminus G_r^{\mathbf{b}}(F)$. By the maximum modulus principle in variable $t \in \mathbb{C}$ for the function $1/F(z + t\mathbf{b})$ one has

$$\frac{1}{|F(z)|} \leqslant \frac{1}{\min\{|F(z + t\mathbf{b})| : |t| = r^*/L(z)\}},$$

i.e., $|F(z)| \geqslant \min\{|F(z + t\mathbf{b})|\colon\ |t| = r^*/L(z)\}$. Thus,

$$|\partial_{\mathbf{b}}^m F(z)| \leqslant m!\, P_2\eta^{-m}L^m(z)|F(z)|.$$

Since z is arbitrary, we finally obtain the required inequality with $P = P_2 m!\, \eta^{-m}$. $\square$

3. The main result

Let us consider the directional differential equation

$$h_0(z)\partial_{\mathbf{b}}^p w + h_1(z)\partial_{\mathbf{b}}^{p-1} w + \ldots + h_p(z)w = h(z), \tag{3.1}$$

where h_j, h are functions from the class $\mathcal{H}_{\mathbf{b}}^n$, $j \in \{0, 1, \ldots, p\}$. For entire functions of bounded L-index in direction $p = 2$ and entire functions h_0, h_1, h_2, h, the equation was investigated in [5], [10], and in the case of holomorphic in the unit ball coefficients h_j, h, equation (3.1) was studied in [7].

Here we consider the equation under a much weaker assumption that the coefficients of (3.1) are slice holomorphic functions in the unit ball. Let us remind that a case of entire on directional slices functions was examined in [11]. Denote

$$H(z) = h(z)\prod_{j=0}^{p} h_j(z), \quad n(r,H) = \sup_{z\in\mathbb{B}^n} n_{\mathbf{b}}(r/L(z), z, 1/H),$$

$$r^* = \sup_{s\geqslant 1} \frac{s-1}{8(n(s,H)+1)\lambda_{\mathbf{b}}(s)}, \quad E(F) = \bigcup_{\substack{z\in Z_F:\\ \forall\, t\in\mathbb{C}, F(z+t\mathbf{b})\equiv 0}} \{z+t\mathbf{b}\colon\ t\in\mathbb{C}\},$$

where Z_F is a zero set of the function F. We also denote $G_r = (G_r(h)\setminus E(h)) \cup G_r(g_0) \cup \bigcup\limits_{j=1}^{p}(G_r(h_j)\setminus E(h_j))$.

We prove the following theorem.

Theorem 3.1. *Let $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$ with $\beta > 3$ in condition* (1.3) *and $\{h_0(z),\dots, h_p(z), h(z)\} \subset \mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)\cap B_{\mathbf{b}}(L,\mathbb{B}^n)$. If there exist $r\in(0;\min\{\frac{1}{3}\beta, r^*\})$ and $T>0$ such that $\mathbb{B}^n\setminus G_r\neq\emptyset$ and for all $z\in\mathbb{B}^n\setminus G_r(h_0)$ and for all $j\in\{1,\dots,p\}$ it is*

$$|h_j(z)| \leqslant TL^j(z)|h_0(z)|, \tag{3.2}$$

then every solution $F\in\mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ of equation (3.1) *has bounded L-index in the direction $\mathbf{b}$.*

Proof. Our proof is based on some ideas from [11]. It is easy to see that the condition $F\in\mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ implies $\partial_{\mathbf{b}}^m F\in\mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ for all $m\in\mathbb{N}$. It should also be noted that Theorem 2.3 and the restrictions of Theorem 3.1 provide the validity of the inequalities $n(r,H)<\infty$ and $r^*>0$.

Since by condition $\mathbb{B}^n\setminus G_r\neq\emptyset$, from conditions of Theorem 3.1 and Lemma 2.2 by inequality (3.2) it follows that exists $r\in(0,\min\{\beta; r^*\})$, exists $T^* = \max\{P, T, P\cdot T\} > 0$ for all $z\in\mathbb{B}^n\setminus G_r$:

$$|\partial_{\mathbf{b}}h(z)| \leqslant P|h(z)|L(z) \leqslant T^*|h(z)|L(z), \tag{3.3}$$

$$|h_j(z)| \leqslant T^*|h_0(z)|L^j(z), \quad j\in\{1,2,\dots,p,\},$$

$$\begin{aligned}|\partial_{\mathbf{b}}h_j(z)| &\leqslant PL(z)|h_j(z)| \leqslant P\cdot T|h_0(z)|L^{j+1}(z)\\ &\leqslant T^*|h_0(z)|L^{j+1}(z), \quad j\in\{0,1,2,\dots,p,\}.\end{aligned} \tag{3.4}$$

Let us now apply the derivative $\partial_{\mathbf{b}}$ to both sides of equation (3.1):

$$h_0(z)\partial_{\mathbf{b}}^{p+1}F(z) + \sum_{j=1}^{p} h_j(z)\partial_{\mathbf{b}}^{p+1-j}F(z) + \sum_{j=0}^{p}\partial_{\mathbf{b}}h_j(z)\partial_{\mathbf{b}}^{p-j}F(z) = \partial_{\mathbf{b}}h(z).$$

Then using inequalities (3.3), (3.4) and $|h(z)| \leqslant \sum\limits_{j=0}^{p} |h_j(z)||\partial_{\mathbf{b}}^{p-j} F(z)|$ for all $z \in \mathbb{B}^n \setminus G_r$, we successively obtain

$$|\partial_{\mathbf{b}} h(z)| \leqslant T^* |h(z)| L(z) \leqslant T^* L(z) \sum_{j=0}^{p} |h_j(z)||\partial_{\mathbf{b}}^{p-j} F(z)|$$

and

$$\begin{aligned}
&|h_0(z)||\partial_{\mathbf{b}}^{p+1} F(z)| \\
&\quad \leqslant |\partial_{\mathbf{b}} h(z)| + \sum_{j=1}^{p} |h_j(z)||\partial_{\mathbf{b}}^{p+1-j} F(z)| + \sum_{j=0}^{p} |\partial_{\mathbf{b}} h_j(z)||\partial_{\mathbf{b}}^{p-j} F(z)| \\
&\quad \leqslant T^* L(z) \sum_{j=0}^{p} |h_j(z)||\partial_{\mathbf{b}}^{p-j} F(z)| + \sum_{j=1}^{p} |h_j(z)||\partial_{\mathbf{b}}^{p+1-j} F(z)| \\
&\qquad + \sum_{j=0}^{p} |\partial_{\mathbf{b}} h_j(z)||\partial_{\mathbf{b}}^{p-j} F(z)| \\
&\quad \leqslant T^* |h_0(z)||L^{p+1}(z)| \biggl((T^*+1) \sum_{j=0}^{p} \frac{|\partial_{\mathbf{b}}^{p-j} F(z)|}{L^{p-j}(z)} + \sum_{j=1}^{p} \frac{|\partial_{\mathbf{b}}^{p+1-j} F(z)|}{L^{p+1-j}(z)} \biggr) \\
&\quad \leqslant P_3 |h_0(z)| L^{p+1}(z) \max_{0 \leqslant j \leqslant p} \frac{|\partial_{\mathbf{b}}^{j} F(z)|}{L^{j}(z)},
\end{aligned}$$

where $P_3 = T^*((T^*+1)(p+1)+p) > 0$. Thus, for all $z \in \mathbb{B}^n \setminus G_r$,

$$L^{-(p+1)}(z)|\partial_{\mathbf{b}}^{p+1} F(z)| \leqslant P_3 \max\{L^{-j}(z)|\partial_{\mathbf{b}}^{j} F(z)| \colon\ 0 \leqslant j \leqslant p\}. \tag{3.5}$$

For any point $z' \in A := H(h_0) \setminus \bigcup\limits_{j=1}^{p} (G_r(h_j) \setminus E(h_j))$ there exists a sequence of the points $z^m \in \mathbb{B}^n \setminus G_r$ satisfying (3.5) with $z = z^m$ and such that $z^m \to z'$ as $m \to \infty$. Substituting $z = z^m$ in (3.5) and passing to the limit as $m \to \infty$, by the joint continuity of the function F, we obtain that inequality is valid for all $z \in A \cup (\mathbb{B}^n \setminus G_r)$.

If all zeros of the function H belong to $E(H)$, i.e., $\mathbb{B}^n = A \cup (\mathbb{B}^n \setminus G_r)$, then by Theorem 2.4 the function F, asa function from the class $\mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$, has bounded L-index in the direction $\mathbf{b}$. Otherwise, $n(s,H) \geqslant 1$.

It is easy to see that for $r \in (0, \min\{\frac{1}{3}\beta, r^*\})$ and $r^* = \sup\limits_{s \geqslant 1} (s-1)/(8(n(s,H)+1) \times \lambda_{\mathbf{b}}(s))$, there exists $r' \in [1, \frac{1}{3}\beta)$ such that $r \leqslant (r'-1)/(8(n(r',H)+1)\lambda_{\mathbf{b}}(r'))$.

Let $z^* \in \mathbb{B}^n$ be an arbitrary point and $K^* = \{z^* + t\mathbf{b} \colon\ |t| \leqslant r'/L(z^*)\}$. The slice holomorphic functions $h_0, h_1, \ldots, h_p, h$ have bounded L-index in the direction $\mathbf{b}$, therefore by Theorem 2.3, the set K^* contains at most $n(r',H)$ zeros of these functions or $K^* \subset Z_H$.

Let c_m^* be the points such that $H(z^*+c_m^*\mathbf{b})=0$ (i.e., c_m^* are the zeros of the slice function H) and $z^*+c_m^*\mathbf{b}\in K^*\cap(Z_h\setminus E(h))\cup\bigcup_{j=0}^{p}(Z_{h_j}\setminus E(h_j))$, where $m\in\mathbb{N}$, $m\leqslant n(r',H)$. Remark, that $L(z^*+c_m^*\mathbf{b})\geqslant L(z^*)/\lambda_{\mathbf{b}}(r')$, because $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$. Then, obviously,

$$\begin{aligned}\widetilde{K}_m^* &:= \Big\{z^*+t\mathbf{b}\colon\ |t-c_m^*|\leqslant\frac{r}{L(z^*+c_m^*\mathbf{b})}\Big\}\\ &\subset\Big\{z^*+t\mathbf{b}\colon\ |t-c_m^*|\leqslant\frac{r'-1}{8(n(r',H)+1)\lambda_{\mathbf{b}}(r')L(z^*+c_m^*\mathbf{b})}\Big\}\\ &\subset K_m^*:=\Big\{z^*+t\mathbf{b}\colon\ |t-c_m^*|\leqslant\frac{r'-1}{8(n(r',H)+1)L(z^*)}\Big\}.\end{aligned}$$

Therefore, for each point $z^*+t\mathbf{b}\in K^*\setminus\bigcup_{z^*+c_m^*\mathbf{b}\in K^*}K_m^*$, inequality (3.5) is true and thus, for these points $z^*+t\mathbf{b}$ from inequalities $L(z^*)\geqslant L(z^*+t\mathbf{b})/\lambda_{\mathbf{b}}(r')$ and (3.5) it follows

$$\begin{aligned}\frac{|\partial_{\mathbf{b}}^{p+1}F(z^*+t\mathbf{b})|}{L^{p+1}(z^*)} &\leqslant(\lambda_{\mathbf{b}}(r'))^{p+1}\frac{|\partial_{\mathbf{b}}^{p+1}F(z^*+t\mathbf{b})|}{L^{p+1}(z^*+t\mathbf{b})}\\ &\leqslant P_3(\lambda_{\mathbf{b}}(r'))^{p+1}\max_{0\leqslant j\leqslant p}\Big\{\frac{|\partial_{\mathbf{b}}^{j}F(z^*+t\mathbf{b})|}{L^{j}(z^*+t\mathbf{b})}\Big\}\\ &\leqslant P_3(\lambda_{\mathbf{b}}(r'))^{p+1}\max_{0\leqslant j\leqslant p}\Big\{\frac{|\partial_{\mathbf{b}}^{j}F(z^*+t\mathbf{b})|}{L^{j}(z^*)}(\lambda_{\mathbf{b}}(r'))^{j}\Big\}\\ &\leqslant P_3(\lambda_{\mathbf{b}}(r'))^{2p+1}\max_{0\leqslant j\leqslant p}\Big\{\frac{|\partial_{\mathbf{b}}^{j}F(z^*+t\mathbf{b})|}{L^{j}(z^*)}\Big\}=P_4w_{z^*}(t),\end{aligned}\tag{3.6}$$

where $w_{z^*}(t)=\max\{|\partial_{\mathbf{b}}^{j}F(z^*+t\mathbf{b})|/L^{j}(z^*)\colon\ 0\leqslant j\leqslant p\}$ and $P_4=P_3(\lambda_{\mathbf{b}}(r'))^{2p+1}$.

Let D be a sum of diameters K_m^*. Then $D\leqslant 2(r'-1)n(r',H)/(8(n(r',H)+1)\times L(z^*))<r'-1/(4L(z^*))$. Therefore there exist numbers $r_1\in[\frac{1}{4}r',\frac{1}{2}r']$ and $r_2\in[\frac{1}{4}(3r'+1);r']$ such that for $z^*+t\mathbf{b}\in C_j=\{z^*+t\mathbf{b}\colon\ |t|=r_j/L(z^*)\}$, $j\in\{1,2\}$, one has $z^*+t\mathbf{b}\in K^*\setminus\bigcup_{c_m^*\in K^*}K_m^*$. Let $z^*+t_1\mathbf{b}\in C_1$ and $z^*+t_2\mathbf{b}\in C_2$ be arbitrary points. We connect these points $z^*+t_1\mathbf{b}$, $z^*+t_2\mathbf{b}$ by a smooth curve $\gamma=\{z^*+t(s)\mathbf{b}\colon\ 0\leqslant s\leqslant 1\}$ (i.e., $z^*+t(0)\mathbf{b}=z^*+t_1\mathbf{b}, z^*+t(1)\mathbf{b}=z^*+t_2\mathbf{b}$) such that $w_{z^*}(t)\neq 0$ and $\gamma\subset K^*\setminus\bigcup_{c_m^*\in K^*}K_m^*$.

Let us describe in detail the construction of this curve γ, using the ideas from the proof in [9] with adapting them for the unit ball. First, let $\ell\colon\ t(s)=(t_2-t_1)s+t_1$, $s\in[0,1]$ be the line segment connecting the points t_1 and t_2. Let $t_k^*=(t_2-t_1)s_k+t_1$, $s_k\in(0,1)$ be points such that $w_{z^*}(t_k^*)=0$. The number $m_0=m_0(z^*+t_1\mathbf{b},z^*+t_2\mathbf{b})$

of such points t_k^* is finite. Without loss generality of our reasoning, we can assume that (t_k^*) is the sequence of these points in ascending order of values $|t_1 - t_k^*|$, $k \in \{1, 2, \dots, p\}$. We choose

$$r_0 \in \Big(0, \min\Big\{|t_k^* - t_{k+1}^*|, |t_1^* - t_1|, |t_{m_0}^* - t_2|, \frac{r'}{4\pi L(z^*)}\colon\ 1 \leqslant k \leqslant m_0 - 1\Big\}\Big).$$

We now choose circles γ_k centered at the points t_k^* and with radii $r_k' < r_0/2^k$ such that $w_{z^*}(t) \neq 0$ for all t on these circles. It is possible because $F \not\equiv 0$.

Every such circle γ_k is divided into two semicircles by the line ℓ. The required piecewise-analytic curve ℓ^* consists of segments of line ℓ, which connect the circles in series between themselves or with the points t_1, t_2, and of arcs of semicircles of the constructed circles. If the curve ℓ^* intersects a set K_m^*, then we replace part of the curve $\ell^* \cap K_m^*$ with a semicircle centered at the point c_m^* and radius $r'/(8(n(r', H) + 1)L(z^*))$. For the resulting curve, we keep the notation $\gamma = \ell^*$.

From the construction of the curve γ described above, it can be seen that the following estimate is valid for the length of the curve

$$\begin{aligned} |\gamma| &\leqslant |\mathbf{b}|\Big(\frac{\pi r_1}{L(z^*)} + \frac{r_2 - r_1}{L(z^*)} + \frac{\pi r_0}{L(z^*)} + \frac{\pi n(r', H) r'}{8(n(r', H) + 1)L(z^*)}\Big) \\ &< \frac{|\mathbf{b}|}{L(z^*)}\Big(\frac{\pi r'}{2} + r' + \frac{\pi r'}{8}\Big) < \frac{3|\mathbf{b}| r'}{L(z^*)}. \end{aligned} \tag{3.7}$$

Then on γ, inequality (3.6) is valid, that is,

$$\frac{|\partial_{\mathbf{b}}^{p+1} F(z^* + t(s)\mathbf{b})|}{L^{p+1}(z^*)} \leqslant P_4 w_{z^*}(t(s)) \quad \text{for } 0 \leqslant s \leqslant 1.$$

From the construction of the curve γ described above, it is also clear that the function $z = z^* + t(s)\mathbf{b}\colon\ [0, 1] \to \mathbb{C}$ is piecewise analytic. Hence, for arbitrary $k \in \mathbb{Z}_+$, $j \in \mathbb{Z}_+$, $k \leqslant p$, either

$$(\forall s \in [0, 1])\colon\ \frac{|\partial_{\mathbf{b}}^k F(z^* + t(s)\mathbf{b})|}{L^k(z^*)} \equiv \frac{|\partial_{\mathbf{b}}^j F(z^* + t(s)\mathbf{b})|}{L^j(z^*)}, \tag{3.8}$$

or there exists a finite set of points $s_m \in [0; 1]$ such that

$$\frac{|\partial_{\mathbf{b}}^k F(z^* + t(s)\mathbf{b})|}{L^k(z^*)}\bigg|_{s=s_m} = \frac{|\partial_{\mathbf{b}}^j F(z^* + t(s)\mathbf{b})|}{L^j(z^*)}\bigg|_{s=s_m}. \tag{3.9}$$

Hence, for the function $w_{z^*}(t(s)) = \max\{|\partial_{\mathbf{b}}^j F(z^* + t(s)\mathbf{b})|/L^j(z^*)\colon\ 0 \leqslant j \leqslant p\}$, two cases are possible:

(1) In an interval of analyticity of the curve γ the function $w_{z^*}(t(s))$ identically equals simultaneously to some derivatives, that is, (3.8) holds. It means that $w_{z^*}(t(s)) \equiv |\partial_{\mathbf{b}}^j F(z^* + t(s)\mathbf{b})|/L^j(z^*)$ for some $j \leqslant p$. Clearly, the function $\partial_{\mathbf{b}}^j F(z^* + t(s)\mathbf{b})$ is analytic. Then $|\partial_{\mathbf{b}}^j F(z^* + t(s)\mathbf{b})|$ is a continuously differentiable function on the interval of analyticity except the points where this derivative equals zero: $|\partial_{\mathbf{b}}^j F(z^* + t(s)\mathbf{b})| = 0$. However, there are not the points, because in the opposite case $w_{z^*}(t(s)) = 0$. But it contradicts the construction of the curve γ.

(2) In an interval of analyticity of the curve γ, the function $w_{z^*}(t(s))$ equals simultaneously to some derivatives at a finite number of points s_k, that is, (3.9) holds. Then the points s_k divide the interval of analyticity into a finite number of segments, in which of them $w_{z^*}(t(s))$ equals to one from the partial derivatives, i.e., $w_{z^*}(t(s)) \equiv |\partial_{\mathbf{b}}^j F(z^* + t(s)\mathbf{b})|/L^j(z^*)$ for some $j \leqslant p$. As above, in each of these segments the functions $|\partial_{\mathbf{b}}^j F(z^* + t(s)\mathbf{b})|$, and $w_{z^*}(t(s))$ are continuously differentiable except the points s_k.

Therefore, the function $|g_{z^*}(t(s))|$ is continuous on $[0,1]$ and continuously differentiable except, possibly, a finite set of points. Moreover, for a complex-valued function of real variable the inequality

$$\frac{\mathrm{d}}{\mathrm{d}s}|\varphi(s)| \leqslant \Big|\frac{\mathrm{d}}{\mathrm{d}s}\varphi(s)\Big|$$

holds except the points s where $\varphi(s) = 0$. Therefore in view of (3.6) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}|g_{z^*}(t(s))| &\leqslant \max_{0\leqslant j\leqslant p}\Big\{\frac{\mathrm{d}}{\mathrm{d}s}\frac{|\partial_{\mathbf{b}}^j F(z^* + t(s)\mathbf{b})|}{L^j(z^*)}\Big\}\\ &\leqslant \max_{0\leqslant j\leqslant p}\Big\{\frac{|\partial_{\mathbf{b}}^{j+1} F(z^* + t(s)\mathbf{b})|}{L^{j+1}(z^*)}|t'(s)|L(z^*)\Big\}\\ &\leqslant \max_{0\leqslant j\leqslant p+1}\Big\{\frac{|\partial_{\mathbf{b}}^j F(z^* + t(s)\mathbf{b})|}{L^j(z^*)}\Big\}|t'(s)|L(z^*)\\ &\leqslant P_5|g_{z^*}(t(s))||t'(s)|L(z^*),\end{aligned}$$

where $P_5 = \max\{1, P_4\}$. Hence, using (3.7), we have

$$\begin{aligned}\Big|\ln\frac{|g_{z^*}(t_2)|}{|g_{z^*}(t_1)|}\Big| &= \Big|\int_0^1 \frac{1}{|g_{z^*}(t(s))|}\frac{\mathrm{d}}{\mathrm{d}s}|g_{z^*}(t(s))|\,\mathrm{d}s\Big| \leqslant P_5 L(z^*)\int_0^1 |t'(s)|\,\mathrm{d}s\\ &\leqslant P_5 L(z^*)|\gamma| \leqslant 3|\mathbf{b}|r' P_5,\end{aligned}$$

that is,

$$|g_{z^*}(t_2)| \leqslant |g_{z^*}(t_1)|\exp\{3|\mathbf{b}|r' P_5\}. \tag{3.10}$$

It is possible to choose t_2 such that $|g_{z^*}(t_2)| = |F(z^* + t_2\mathbf{b})| = \max\{|F(z^* + t\mathbf{b})|\colon z^* + t\mathbf{b} \in C_2\}$. Hence and from inequality (3.10) we get

$$\begin{aligned}(3.11)\qquad \max\Big\{|F(z^* + t\mathbf{b})|\colon |t| = \frac{3r' + 1}{4L(z^*)}\Big\} &\leqslant \max\{|F(z^* + t\mathbf{b})|\colon z^* + t\mathbf{b} \in C_2\}\\ &= |F(z^* + t_2\mathbf{b})| = |g_{z^*}(t_2)|\\ &\leqslant |g_{z^*}(t_1)| \exp\{3|\mathbf{b}|r' P_5\}.\end{aligned}$$

Recalling that $z^* + t_1\mathbf{b} \in C_1 = \{z^* + t\mathbf{b}\colon |t| = r_1/L(z^*)\}$ and $r_1 \in [\frac{1}{4}r', \frac{1}{2}r']$, for all $j \in \{1, 2, \ldots, p\}$, by Cauchy's inequality in variable t on the circle $\{t \in \mathbb{C}\colon |t - t_1| = r'/(4L(z^*))\}$ we obtain

$$\begin{aligned}\frac{r'^j}{(4L(z^*))^j} |\partial_{\mathbf{b}}^j F(z^* + t_1\mathbf{b})| &\leqslant j!\, \max\Big\{|F(z^* + t\mathbf{b})|\colon |t - t_1| = \frac{r'}{4L(z^*)}\Big\}\\ &\leqslant p!\, \max\Big\{|F(z^* + t\mathbf{b})|\colon |t| = \frac{3r'}{4L(z^*)}\Big\}.\end{aligned}$$

Thus,

$$(3.12)\qquad |g_{z^*}(t_1)| \leqslant p!\, \max\{1, (4/r')^p\} \max\Big\{|F(z^* + t\mathbf{b})|\colon |t| = \frac{3r'}{4L(z^*)}\Big\}.$$

We put $P_6 = p! \max\{1, (4/r')^p\} \exp\{3|\mathbf{b}|r' P_5\}$. Then inequalities (3.11) and (3.12) imply that

$$\max\Big\{|F(z^* + t\mathbf{b})|\colon |t| = \frac{3r' + 1}{4L(z^*)}\Big\} \leqslant P_6 \max\Big\{|F(z^* + t\mathbf{b})|\colon |t| = \frac{3r'}{4L(z^*)}\Big\}.$$

Finally, by Theorem 2.1, the function $F \in B_{\mathbf{b}}(L, \mathbb{B}^n)$. □

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