THE MINIMAL CLOSED MONOIDS FOR THE GALOIS CONNECTION End-Con

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Abstract. The minimal nontrivial endomorphism monoids $M = \operatorname{EndCon}(A,F)$ of congruence lattices of algebras (A,F) defined on a finite set A are described. They correspond (via the Galois connection End-Con) to the maximal nontrivial congruence lattices $\operatorname{Con}(A,F)$ investigated and characterized by the authors in previous papers. Analogous results are provided for endomorphism monoids of quasiorder lattices $\operatorname{Quord}(A,F)$.

Keywords: endomorphism monoid; congruence lattice; quasiorder lattice; finite algebra

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1. Preliminaries

In this short note we describe the minimal nontrivial endomorphism monoids $M = \operatorname{End}\operatorname{Con}(A,F)$ of congruence lattices of algebras (A,F) defined on a finite set A.

Congruence relations (i.e., compatible equivalence relations) are one of the basic tools for the investigation of universal algebras (A,F). A nice property of equivalence relations (or, more general, of quasiorders, i.e., reflexive and transitive relations) is that their compatibility with the operations F of an algebra depends only on their compatibility with unary polynomial functions $f \in A^A$. Thus, one can focus on unary algebras (A,F) with $F \subseteq A^A$ or even on monounary algebras (A,f) via $\operatorname{Con}(A,F) = \bigcap_{f\in F} \operatorname{Con}(A,f)$ (for the investigation of monounary algebras we refer to the monograph [5], but they are also discussed in numerous recent publications, of which we mention only [1], [2], [9] because these are close to our own research).

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For fixed A, the congruence lattices $\operatorname{Con}(A,F)$ themselves form a lattice (with respect to inclusion), which can be characterized as

$$\mathcal{E}_A := \{\operatorname{Con}(A,F)\colon\ F \subseteq A^A\},$$

and was investigated, e.g., in [7] (see also [9]). Each congruence lattice is a complete sublattice of $\operatorname{Eq}(A)$ (the lattice of all equivalence relations), in particular, it contains the trivial congruences $\Delta_A = \{(x,x)\colon\ x \in A\}$ and $\nabla_A = A \times A$. Due to the Galois connection End-Con (see below), the endomorphism monoids $M = \operatorname{End}\operatorname{Con}(A,F)$ of such congruence lattices also form a lattice

$$\mathcal{M}_A := \{\operatorname{End}\operatorname{Con}(A,F)\colon\ F \subseteq A^A\},$$

which is dual to $\mathcal{E}_A$.

The coatoms of $\mathcal{E}_A$ (i.e., the maximal elements below the top element $\operatorname{Eq}(A) \in \mathcal{E}_A$) were determined in [7], Theorem 4.3 as congruence lattices of the form $\operatorname{Con}(A,f)$ for special functions f of type I, II and III (and their structure was studied in detail in [8]). It is natural to ask for the other (the monoid) side of the Galois connection, that is, to consider the atoms of $\mathcal{M}_A$, in other words, the minimal elements of $\mathcal{M}_A$ above the least element $T \in \mathcal{M}_A$. This is done in the present short note.

We explicitly describe the atoms of $\mathcal{M}_A$, i.e., $\operatorname{End}\operatorname{Con}(A,f)$ for these special functions f (Theorem 2.1 (A)). As shown in [6], the same functions of type I, II and III also give the coatoms in the lattice of quasiorder lattices $\operatorname{Quord}(A,F)$, therefore we also shall characterize the corresponding atoms $\operatorname{End}\operatorname{Quord}(A,f)$ (Theorem 2.1 (B)).

To fix the notions and notation, recall that a binary relation $\theta \subseteq A\times A$ is *compatible* with (or *invariant* for) a function $f \in A^A$; we also say f *preserves* ϱ, denoted by $f \triangleright \varrho$, if

$$\forall\, x,y \in A\colon\ (x,y) \in \theta \Rightarrow (fx,fy) \in \theta.$$

Equivalently, this expresses the fact that f is an *endomorphism* of θ ($f \in \operatorname{End}\theta$) and (provided that θ is an equivalence relation ($\theta \in \operatorname{Eq}(A)$)) that θ is a *congruence* of the algebra (A,f) ($\theta \in \operatorname{Con}(A,f)$).

The relation $\triangleright$ induces a Galois connection, namely End-Con, between unary mappings and equivalence relations, defined by

$$\operatorname{End} Q := \{f \in A^A\colon\ \forall\, \varrho \in Q\colon\ f \triangleright \varrho\} \quad \text{for } Q \subseteq \operatorname{Eq}(A)$$

and

$$\operatorname{Con}(A,F) := \operatorname{Con} F := \{\theta \in \operatorname{Eq}(A)\colon\ \forall\, f \in F\colon\ f \triangleright \varrho\} \quad \text{for } F \subseteq A^A.$$

The Galois closures are just the congruence lattices $\operatorname{Con}(A,F) \in \mathcal{E}_A$ and the monoids $\operatorname{End} Q \in \mathcal{M}_A$. Thus, we are looking for the minimal nontrivial Galois closed endomorphism monoids. For a two-element base set A there exist only two (namely the trivial) equivalence relations. Thus, A^A is the only Galois closed endomorphism monoid. We exclude this trivial case and, from now on, make the assumption $3 \leqslant |A| < \infty$.

The least monoid $T \in \mathcal{M}_A$ consists of all unary functions that preserve all equivalence relations on A, that is, we have $T = \operatorname{End}\operatorname{Eq}(A)$. Therefore, the monoid T and the functions in it are called *trivial*. Since $3 \leqslant |A|$, it is known (e.g., [11], [10]) that $T := \{\mathrm{id}_A\} \cup C_A$, where id_A is the identity mapping and C_A denotes the set of all unary constant functions on A.

We now define the above mentioned functions of type I, II and III, which play the central role for describing the minimal endormorphism monoids of congruence lattices.

Definition 1.1. A unary function $f \in A^A$ is called a function of type I, II or III, respectively, if it is *nontrivial* and satisfies the following conditions:

(I) $f^2 = f$,

(II) f^2 is a constant, say u, and $|\{x \in A\colon\ fx = u\}| \geqslant 3$,

(III) $f^p = \mathrm{id}_A$ for a prime p such that the permutation f has at least two cycles of length p.

Some examples of special functions f of type I and II can be seen in Figure 1.

Every function $f \in A^A$ can be presented as a (directed) graph with vertex set A and edge set $f^\bullet := \{(x, y) \in A^2\colon\ fx = y\}$. Two elements $x, y \in A$ are called *connected* if there exist $m, n \in \mathbb{N}$ such that $f^m x = f^n y$ (see, e.g., [5], page 11). This is an equivalence relation and its equivalence classes are called (*connected*) *components* of f. A *nontrivial component* of a function f of type I is an at least two-element component of the graph of f, i.e., any at least two-element set of the form $f^{-1}(z)$ for a fixed point z of f.

2. The endomorphism monoids of the coatoms of $\mathcal{E}_A$

As already mentioned, the coatoms of $\mathcal{E}_A$ are well-known as $\operatorname{Con}(A, f)$ for functions of type I, II, III (see [7], Theorem 4.3). In addition, we mention that the same functions also determine the coatoms $\operatorname{Quord}(A, f)$ in the lattice of quasiorder lattices of algebras on the base set A (see [6], Theorem 3.1).

We now consider the other side of the Galois connection End-Con and determine $\operatorname{End}\operatorname{Con}(A, f)$ for all coatoms $\operatorname{Con}(A, f)$, i.e., the minimal nontrivial endomorphism monoids in $\mathcal{M}_A$. In fact, this means to determine $\operatorname{End}\operatorname{Con}(A, f)$ for all functions of type I, II and III. Before we give the result in Theorem 2.1, we need two more definitions.

For a function f of type I with exactly one nontrivial component (whose fixed point is denoted by z) let $\hat{f}$ be defined as follows:

$$(*) \qquad \hat{f}x := \begin{cases} z & \text{if } fx = x, \\ x & \text{otherwise.} \end{cases}$$

For a function f with a 2-element image $\operatorname{Im}(f) = \{z, u\}$, let f' be defined as follows:

$$(**) \qquad f'x := \begin{cases} u & \text{if } fx = z, \\ z & \text{if } fx = u. \end{cases}$$

These functions $\hat{f}$ and f' (for f of type I or II) are shown in Figure 1, where $\{a, \ldots, a'\}$ and $\{b, \ldots, b'\}$ schematically represent arbitrary nonempty subsets of A (i.e., also one-element sets are allowed) while $\{b', \ldots, b''\}$ is an arbitrary (possibly empty) subset. Note that $\hat{\hat{f}} = f$ and $f'' = f$. It is straightforward to check that $\operatorname{Con}(A, \hat{f}) = \operatorname{Con}(A, f)$ and $\operatorname{Con}(A, f') = \operatorname{Con}(A, f)$.

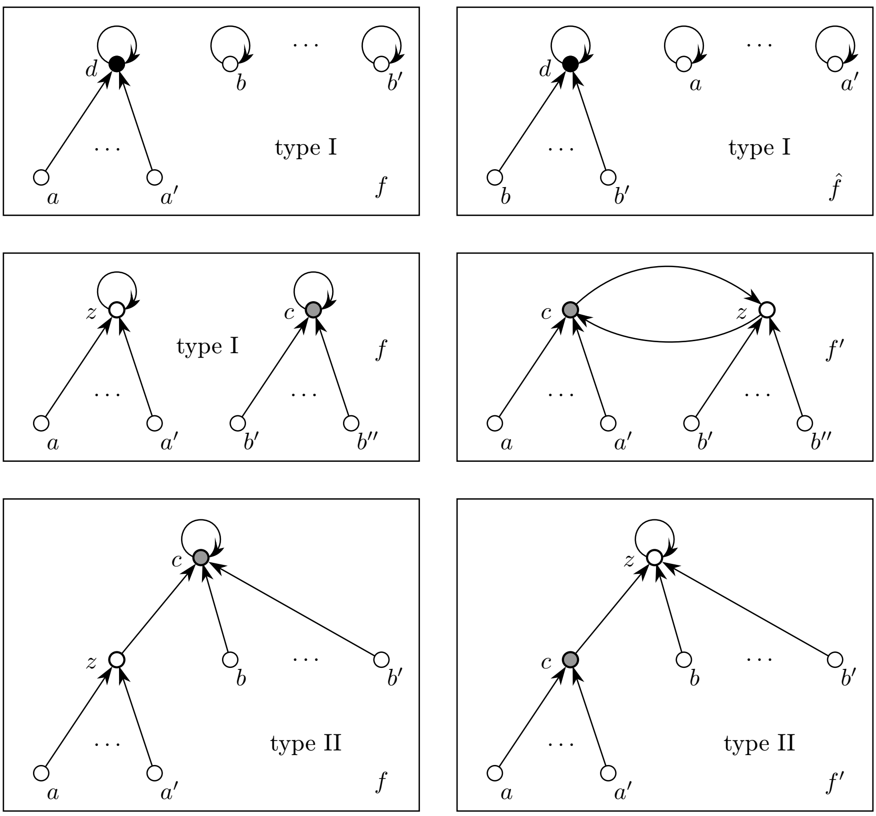

Figure 1. The functions $\hat{f}$ and f'.

Theorem 2.1. *Let* $3 \leqslant |A| < \infty$.

(A) *The following table describes the Galois closure* $\operatorname{End}\operatorname{Con}(A, f)$ *for all functions* f *of type* I, II *or* III. *The number* s *indicates the number of nontrivial functions in the closure.*

	type of f	$\lvert\operatorname{Im}(f)\rvert$	*number of nontrivial components* K *of* f	*other conditions*	*Galois closure* $\operatorname{End}\operatorname{Con}(A, f)$	s
(1)	I	$\geqslant 3$	$\geqslant 2$		$\{f\} \cup T$	1
(2)	I	$\geqslant 3$	1	$\lvert K\rvert \geqslant 3$	$\{f, \hat{f}\} \cup T$	2
(3)	I	$\geqslant 3$	1	$\lvert K\rvert = 2$	$\{f, \hat{f}, (\hat{f})'\} \cup T$	3
(4)	I	2	2		$\{f, f'\} \cup T$	2
(5)	I	2	1	$\lvert A\rvert > 3$	$\{f, f', \hat{f}\} \cup T$	3
(6)	I	2	1	$\lvert A\rvert = 3$	$\{f, f', \hat{f}, (\hat{f})'\} \cup T$	4
(7)	II	$\geqslant 3$			$\{f\} \cup T$	1
(8)	II	2			$\{f, f'\} \cup T$	2
(9)	III			*cycle length* p	$\{f, f^2, \ldots, f^{p-1}\} \cup T$	$p - 1$

(B) *The Galois closures* $\operatorname{End}\operatorname{Quord}(A, f)$ *for the functions of type* I *and* II *are always* $\{f\} \cup T$ *and for functions of type* III *we have* $\operatorname{End}\operatorname{Quord}(A, f) = \operatorname{End}\operatorname{Con}(A, f) = \{f, f^2, \ldots, f^{p-1}\} \cup T$.

Proof. (A): Observe that $g \in \operatorname{End}\operatorname{Con}(A, f)$ (for nontrivial g) is equivalent to $\operatorname{Con}(A, g) = \operatorname{Con}(A, f)$ (because $g \in \operatorname{End}\operatorname{Con}(A, f)$ implies $\operatorname{Con}(A, f) \subseteq \operatorname{Con}(A, g)$, which yields equality since $\operatorname{Con}(A, f)$ is a coatom in $\mathcal{E}_A$ for the functions f of type I, II and III).

Let f be as described in one of the cases (1)–(8) and let $M := \operatorname{End}\operatorname{Con}(A, f)$. The description of the functions $g \in M \setminus T$, i.e., those with $\operatorname{Con}(A, g) = \operatorname{Con}(A, f) \subsetneq \operatorname{Eq}(A)$, follows from results in [3] summarized in [3], Proposition 4.11. Moreover, we know $1 \leqslant |M \setminus T| \leqslant 4$ by [3], Proposition 4.12. The functions which appear in each line of the last but one column of the table in (A), are all different and have the same congruences as f (as mentionend above); hence, they belong to $M \setminus T$. Therefore it is enough to know the cardinality s of $M \setminus T$ (with the notation from [3] this is $|\mathcal{R}(f)|$). If it coincides with the number s in the last column, then we are done.

At first, we note that the cases (3) and (5) are equivalent: in fact, if f is a function of form (3), then $g := \hat{f}$ is of form (5) and we have $\{g, g', \widehat{g}\} = \{\hat{f}, (\hat{f})', f\}$ (since $\hat{\hat{f}} = f$). Thus (3) follows from (5) (and vice versa).

Let f_i denote a function of the form (i) $(i \in \{1, 2, 4, 5, 6, 7, 8\})$. Now we shall discuss each case in more detail.

- ⧐ f_1 satisfies the conditions in [3], Lemma 3.17, therefore $s = |M \setminus T| = 1$ according to [3], Lemma 4.1.
- ⧐ f_2 satisfies the conditions in [3], Lemma 3.15, therefore $s = 2$ according to [3], Lemma 4.7.
- ⧐ f_4 satisfies the conditions in [3], Lemma 3.16 (a), therefore $s = 2$ according to [3], Lemma 4.6.
- ⧐ f_5 satisfies the conditions in [3], Lemma 3.14 (a), therefore $s = 3$ according to [3], Lemma 4.8.
- ⧐ f_6 satisfies the conditions in [3], Lemma 3.12 (b), therefore $s = 4$ according to [3], Lemma 4.9, cf. also [3], Figure 3.12.
- ⧐ f_7 satisfies the conditions in [3], Lemma 3.7, therefore $s = 1$ according to [3], Lemma 4.1.
- ⧐ f_8 satisfies the conditions in [3], Lemma 3.6, therefore $s = 2$ according to [3], Lemma 4.3.

The remaining case (9) for functions f of type III directly follows from [7], Proposition 2.8, where for permutations f of prime power order p^m it was shown that $\{g \in A^A \setminus T\colon\ \mathrm{Con}\,(A, f) \subseteq \mathrm{Con}\,(A, g)\} = \{f, f^2, \dots, f^{p^m-1}\}$. Here we have to take $m = 1$. We remark that the result also could be derived from results in [3], [4].

(B): Since congruences are special invariant quasiorders, for every $f \in A^A$ we have $\mathrm{End\,Quord}\,(A, f) \subseteq \mathrm{End\,Con}\,(A, f)$. Thus, one only has to check which functions from $\mathrm{End\,Con}\,(A, f)$ (as described in (A)) "survive" in the possibly smaller monoid $\mathrm{End\,Quord}\,(A, f)$. As in Part (A), for $g \in A^A$ we have $g \in \mathrm{End\,Quord}\,(A, f)$ if and only if $\mathrm{Quord}\,(A, g) = \mathrm{Quord}\,(A, f)$, cf. [6].

It is straightforward to see that the quasiorder lattices $\mathrm{Quord}\,(A, f)$, $\mathrm{Quord}\,(A, \hat{f})$, $\mathrm{Quord}\,(A, f')$ and $\mathrm{Quord}\,(A, (\hat{f})')$ differ. For this one can choose suitable $(x, y) \in A^2$ such that the principal quasiorders $\alpha_f(x, y) \in \mathrm{Quord}\,(A, f)$ generated by (x, y) differ for the functions f under consideration. For example, we have $(a, z) \in \alpha_{\hat{f}}(a, b) \setminus \alpha_f(a, b)$, $(u, z) \in \alpha_{f'}(a, u) \setminus \alpha_f(a, u)$ and $(a, z) \in \alpha_{(\hat{f})'}(b, a) \setminus \alpha_f(b, a)$ (for notation see Figure 1).

Concerning functions f of type III , the equality $\mathrm{End\,Quord}\,(A, f) = \mathrm{End\,Con}\,(A, f)$ was proved in [7], Lemma 2.7. □

Some of the results of Theorem 2.1 are already contained implicitly in [7], Proposition 4.8.

Dealing with the functions of type I, II and III , we noticed a small but interesting lattice theoretic application concerning the lattice $\mathcal{E}_A$ of congruence lattices, which one can derive directly from results in [12]. We are closing our paper with this application.

Proposition 2.2. *For a finite set A, every distributive sublattice of* $\mathrm{Eq}(A)$ *containing Δ_A and ∇_A is the meet of coatoms of type* I *or* II *in $\mathcal{E}_A$.*

P r o o f. In [12], it is shown that any distributive sublattice of $\mathrm{Eq}(A)$ containing Δ_A and ∇_A (denoted $\mathbf{L}'$ in [12]) can be represented as a congruence lattice $\mathrm{Con}(A, F)$. For this representation the authors use algebras (A, F) with functions $f \in F$ which arise in the following situation: Let $\theta, \theta_1, \theta_2 \in \mathrm{Eq}(A)$ such that $\theta_1 \nsubseteq \theta$ and $\theta \nsubseteq \theta_2$. This implies $|A| \geqslant 3$ and we can fix elements $a, b, c, d \in A$ with $(a, b) \in \theta \setminus \theta_2$ and $(c, d) \in \theta_1 \setminus \theta$. Then f is defined as follows:

$$fx := \begin{cases} c & \text{if } (a, x) \in \theta_2, \\ d & \text{otherwise.} \end{cases}$$

Claim. *There exist functions g, h of type* I *or* II *such that* $\mathrm{Con}(A, f) = \mathrm{Con}(A, g) \cap \mathrm{Con}(A, h)$ *(the cases $g = h$ or $f = g = h$ are not excluded).*

We observe that only the following four cases are possible and prove the claim for each case. Note that $a \neq b$, $c \neq d$, $fa = c$, $fb = d$ and $\mathrm{Im}(f) = \{c, d\}$ by definition of f, in particular, f is nontrivial.

Case 1: $(a, d) \notin \theta_2$ and $(a, c) \in \theta_2$. Then $fd = d$ and $fc = c$, thus both c and d are fixed points and every element is mapped to one of them, i.e., f is of type I (and we can take $g := h := f$).

Case 2: $(a, d) \notin \theta_2$ and $(a, c) \notin \theta_2$. Then we have $fa = c$, $fc = fd = d$ and $fx \in \{c, d\}$. If there exists $u \notin \{c, d\}$ with $fu = d$, then f is of type II and we are done ($g := h := f$). Otherwise, $fx = c$ for every $x \in A \setminus \{c, d\}$. Consider the maps

$$gx := \begin{cases} d & \text{if } x = d, \\ c & \text{otherwise,} \end{cases} \quad \text{and} \quad hx := \begin{cases} d & \text{if } x \in \{c, d\}, \\ x & \text{otherwise.} \end{cases}$$

Both g and h are functions of type I, see Figure 2. Moreover, $f = g \circ h$, therefore $\mathrm{Con}(A, g) \cap \mathrm{Con}(A, h) \subseteq \mathrm{Con}(A, f)$. We show equality. In fact, let $\psi \in \mathrm{Con}(A, f)$.

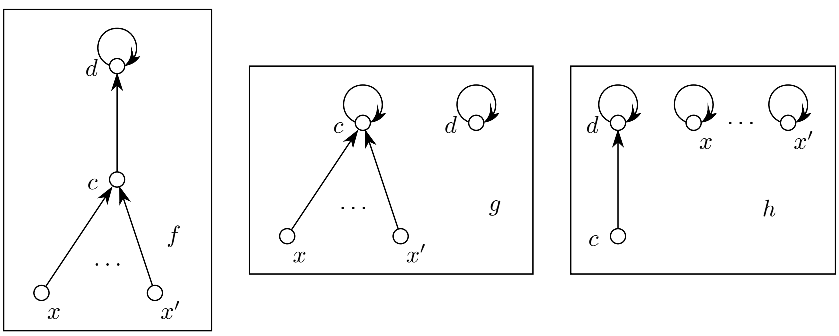

Figure 2. Functions f, g, h with $\mathrm{Con}(A, f) = \mathrm{Con}(A, g) \cap \mathrm{Con}(A, h)$ (Case 2).

If $(c,d) \notin \psi$, then, as $f \triangleright \psi \not\ni (c,d)$, the ψ-equivalence classes of c and d do not contain elements from $A \setminus \{c,d\}$, respectively, and are hence singletons. Thus, we have $\psi = \psi_1 \cup \Delta_A$, where $\psi_1 = \psi \cap (A \setminus \{c,d\})^2$. One then immediately checks that both g and h preserve ψ.

If, on the other hand, $(c,d) \in \psi$, then clearly $g \triangleright \psi$, but we can also show $h \triangleright \psi$. To see this, take $(x,y) \in \psi$ with $x \in A \setminus \{c,d\}$, then $(hx,hy) = (x,y) \in \psi$ if $y \in A \setminus \{c,d\}$, $(hx,hy) = (x,d) \in \psi$ if $y = d$, and $(hx,hy) = (x,d)$ if $y = c$, but $(x,d) \in \psi$ by transitivity, since $(x,c),(c,d) \in \psi$; from this one concludes that $(hx,hy) \in \psi$ for every $(x,y) \in \psi$, i.e., $h \triangleright \psi$, equivalently $\psi \in \operatorname{Con}(A,h)$. Consequently, $\psi \in \operatorname{Con}(A,g) \cap \operatorname{Con}(A,h)$.

Case 3: $(a,d) \in \theta_2$ and $(a,c) \in \theta_2$. Then $fd = fc = c$ and $fx \in \{c,d\}$. If there exists $u \notin \{c,d\}$ with $fu = c$, then f is of type II and we are done. Otherwise, $fx = d$ for all $x \in A \setminus \{c,d\}$ and we can proceed analogously as in Case 2 using the functions of type I

$$gx := \begin{cases} c & \text{if } x = c, \\ d & \text{otherwise}, \end{cases} \quad \text{and} \quad hx := \begin{cases} c & \text{if } x \in \{c,d\}, \\ x & \text{otherwise}, \end{cases}$$

giving $f = g \circ h$ and $\operatorname{Con}(A,f) = \operatorname{Con}(A,g) \cap \operatorname{Con}(A,h)$.

Case 4: $(a,d) \in \theta_2$ and $(a,c) \notin \theta_2$. Then $fd = c$ and $fc = d$ and $fx \in \{c,d\}$ for every $x \in A$. Further, $g := f'$ (as defined in $(**)$ above, see also Figure 1) is a function of type I and we have $\operatorname{Con}(A,f) = \operatorname{Con}(A,g)$ (formally put $h := g$). Thus, the claim is proved.

$\operatorname{Con}(A,F)$ is the intersection of all $\operatorname{Con}(A,f)$ with $f \in F$. Because of the just proved claim, $\operatorname{Con}(A,F)$ is also the intersection of $\operatorname{Con}(A,g)$ with functions $g \in A^A$ of type I or II only, i.e., it is the intersection of coatoms (due to the already mentioned result [7], Theorem 4.3). $\square$

A c k n o w l e d g e m e n t . The authors thank the anonymous reviewer for his/her profound and very helpful report; due to it some gaps in the proofs could be closed and the presentation could be improved considerably.

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