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ON THE LATTICE OF PRONORMAL SUBGROUPS OF DICYCLIC, ALTERNATING AND SYMMETRIC GROUPS

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Abstract. In this paper, the structures of collection of pronormal subgroups of dicyclic, symmetric and alternating groups G are studied in respect of formation of lattices $\mathrm{L}(G)$ and sublattices of $\mathrm{L}(G)$. It is proved that the collections of all pronormal subgroups of A_n and S_n do not form sublattices of respective $\mathrm{L}(\mathrm{A}_n)$ and $\mathrm{L}(\mathrm{S}_n)$, whereas the collection of all pronormal subgroups $\mathrm{LPrN}(\mathrm{Dic}_n)$ of a dicyclic group is a sublattice of $\mathrm{L}(\mathrm{Dic}_n)$. Furthermore, it is shown that $\mathrm{L}(\mathrm{Dic}_n)$ and $\mathrm{LPrN}(\mathrm{Dic}_n)$ are lower semimodular lattices.

Keywords: alternating group; dicyclic group; pronormal subgroup; lattice of subgroups; lower semimodular lattice

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1. Introduction and notation

It is known that the set of all subgroups of a given finite group G forms a lattice denoted by $\mathrm{L}(G)$ with $H\wedge K = H\cap K$ and $H\vee K = \langle H\cup K\rangle = \langle H, K\rangle$, see Grätzer [4], Schmidt [13], Suzuki [15]. For the group theoretic concepts and notations, we refer to Passi [7], Schmidt [13], Suzuki [15].

The following nomenclature is being used throughout this article in which G denotes a finite group.

- ▷ $\mathrm{LN}(G)$ - collection of all normal subgroups of G, which is a sublattice of $\mathrm{L}(G)$.
- ▷ $\mathrm{LPrN}(G)$ - collection of all pronormal subgroups of G.
- ▷ $\mathrm{LSPrN}(G)$ - collection of all strongly pronormal subgroups of G.
- ▷ $|G|$ - order of G - cardinality of G.
- ▷ $|\mathrm{L}(G)|$ - number of subgroups of G - cardinality of $\mathrm{L}(G)$.

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▷ $o(a)$ - order of an element a of G.
▷ e - neutral (identity) element in G.
▷ $\langle H, K\rangle$ - subgroup generated by H and K.
▷ $\mathrm{d}(n)$ - number of divisors of number n.
▷ Dic_n - dicyclic group of order $4n$.
▷ (k, l) - greatest common divisor of k, l.
▷ $[k, l]$ - least common multiplier of k, l.

Note that S_n denotes the symmetric group on n symbols and A_n denotes the alternating group on n symbols, which is a normal subgroup of S_n.

The following definition of a pronormal subgroup of a finite group is due to Hall [5], see Vdovin [16].

Definition 1.1 ([16])**.** Let G be a group and H be a subgroup of G. Then H is said to be *pronormal* if H and given conjugates of H in G, say H^g, are also conjugates in the subgroup generated by H and H^g, namely $\langle H, H^g\rangle$.

We recall the examples of pronormal subgroups in various groups as follows; see Mann [8], Peng [11], Rose [12], Vdovin [16].

▷ Every Hall subgroup (a subgroup whose order is coprime to its index) of a finite solvable group is pronormal.
▷ Every normal subgroup of a group is pronormal.
▷ Every maximal subgroup of a group is pronormal.
▷ Every Sylow p-subgroup of a finite group is pronormal.
▷ Every Carter subgroup (a nilpotent self-normalizing subgroup) of a finite solvable group is pronormal.

The collection of pronormal subgroups of dihedral groups and the collection of all Hall subgroups of alternating groups, symmetric groups and dihedral groups are studied in Mitkari and Kharat [9], [10].

2. Structure of pronormal subgroups of dicyclic groups

In this section, some properties of the collection of pronormal subgroups of Dic_n are investigated.

Definition 2.1. The *dicyclic group* (also called binary dihedral group) with parameter n of order $4n$ is defined as for $n \geqslant 3$, $\mathrm{Dic}_n = \langle a, b\colon a^{2n} = e, b^2 = a^n, bab^{-1} = a^{-1}\rangle$.

The complete listing of subgroups of Dic_n is in the following theorem, for more details see Tărnăuceanu [6].

Theorem 2.2. *Every subgroup of* Dic_n *is cyclic or dicyclic. A complete listing of the subgroups is as follows:*

(1) $\langle a^d\rangle$, *where* $d|2n$, *with index* $2d$,

(2) $\langle a^k, a^i b\rangle$, *where* $k|n$ *and* $0 \leqslant i \leqslant k-1$, *with index* k.

Every subgroup of Dic_n *occurs exactly once in this listing.*

Remark 2.3.

(1) A subgroup of Dic_n is said to be of *Type (1)* if it is a cyclic subgroup as stated in (1) of Theorem 2.2.

(2) A subgroup of Dic_n is said to be of *Type (2)* if it is a dicyclic subgroup as stated in (2) of Theorem 2.2.

Lemma 2.4. *In* Dic_n, *we have the following:*

(1) $(a^k b)^{-1} = b^{-1}(a^k)^{-1} = b^3(a^{-k}) = ba^{n-k} = a^{k-n}b$,

(2) $ba^k = a^{2n-k}b = a^{-k}b$, $ba^k b^{-1} = a^{-k}$,

(3) *elements of group* $H = \langle a^m, a^i b\rangle$ *are either of form* a^{km} *or of form* $a^{km+i}b$ *for some* k.

Lemma 2.5. *Every subgroup of* Dic_n *of Type* (1) *is normal.*

Proof. We have $a^k b\langle a^d\rangle(a^k b)^{-1} = a^k b\langle a^d\rangle a^{k-n}b = \langle a^d\rangle$ for any $d|2n$. Therefore $\langle a^d\rangle$ is normal. $\square$

We obtain the conjugate subgroup of a Type (2) subgroup determined by an element of group Dic_n in the following lemma.

Lemma 2.6. *Let* $H = \langle a^m, a^i b\rangle$ *be a subgroup of* Dic_n. *Then* $H^{a^k} = \langle a^m, a^{2k+i}b\rangle$, $H^{a^k b} = \langle a^m, a^{2k-i}b\rangle$ *and* $\langle H, H^{a^k}\rangle = \langle a^m, a^{2k}, a^i b\rangle$.

Proof. As $ba^{-k} = a^k b$, $a^k(a^i b)a^{-k} = a^{2k+i}b$, and we have that $H^{a^k} = \langle a^m, a^{2k+i}b\rangle$. Next, $(a^k b)a^m(a^k b)^{-1} = a^k(ba^m b^{-1})a^{-k} = a^k a^{-m}a^{-k}$ and $(a^k b)a^i b \times (a^k b)^{-1} = (a^k b)a^i bb^{-1}a^{-k} = a^k ba^{i-k} = a^k a^{k-i}b = a^{2k-i}b$, so that $H^{a^k b} = \langle a^m, a^{2k-i}b\rangle$. Finally, $\langle H, H^{a^k}\rangle = \langle a^m, a^i b, a^{2k+i}b\rangle = \langle a^m, a^{2k}, a^i b\rangle$, because $a^{2k+i}b(a^i b)^{-1} = a^{2k}$. $\square$

We prove that the conjugate of a given subgroup of Dic_n is determined by a power of the generator a of Dic_n.

Lemma 2.7. *For any subgroup* H *of* Dic_n *and for any* $k \in \mathbb{Z}$, *there is a* $j \in \mathbb{Z}$ *such that* $H^{a^k b} = H^{a^j}$.

Proof. Let $H = \langle a^m, a^i b\rangle$. By Lemma 2.6 we have $H^{a^{k-i}} = \langle a^m, a^{2(k-i)+i}b\rangle = \langle a^m, a^{2k-i}b\rangle = H^{a^k b}$. $\square$

In what follows, we characterize the pronormal subgroups of Dic_n.

Theorem 2.8. *A subgroup of* Dic_n *is pronormal unless it is of the form* $\langle a^m, a^i b\rangle$, *where* $4|m|n$ *and* $0 \leqslant i \leqslant m-1$.

Proof. Let H be a subgroup of Dic_n. If H is of Type (1), then it is pronormal since it is normal by Lemma 2.5. We therefore assume the possibilities only when H is a subgroup of Type (2).

Claim 1. *If* $H = \langle a^m, a^i b\rangle$ *is a subgroup of Type* (2) *and* m *is not divisible by* 4, *then* H *is pronormal.*

In the view of Lemma 2.7, it is sufficient to consider H^x for an element $x = a^k$ of Dic_n. As $H^x = \langle a^m, a^{2k+i}b\rangle$, we claim that $\langle H^x, H\rangle = \langle a^g, a^i b\rangle$, where $g = (m, 2k)$. Indeed, note that $\langle H^x, H\rangle = \langle a^m, a^{2k}, a^i b\rangle$, and as $g|m$ and $g|2k$, we have $\langle H^x, H\rangle \subseteq \langle a^g, a^i b\rangle$. Moreover, if $z \in \langle a^g, a^i b\rangle$, then either $z = a^{lg}$ or $z = a^{gq+i}b$ for some $l, q \in \mathbb{N}$. If $z = a^{lg}$, then $z \in \langle a^m, a^{2k}, a^i b\rangle = \langle H^x, H\rangle$. If $z = a^{gq+i}b$, for some $t_1, t_2 \in \mathbb{Z}$ we have $g = mt_1 + 2kt_2$, hence $z = a^{(mt_1+2kt_2)q+i}b \in \langle a^m, a^{2k}, a^i b\rangle = \langle H^x, H\rangle$, and this proves that $\langle H^x, H\rangle = \langle a^g, a^i b\rangle$.

We claim that H and H^x are conjugates in $\langle H^x, H\rangle = \langle a^g, a^i b\rangle$, i.e., there exists $y \in \langle H^x, H\rangle$ such that $H^x = H^y$ holds. We have $g = (m, 2k)$, so let $m = gm'$ and $2k = gk'$ for some $m', k' \in \mathbb{Z}$. Note that if m is even, then $2|g$ and since $4 \nmid m$, we have $(m', 2) = 1$. Also, if m is odd, then $2 \nmid m'$ and so $(m', 2) = 1$. In both the cases we have $(m', 2) = 1$ and therefore there exist $d_1, d_2 \in \mathbb{Z}$ such that $1 = m'd_1 + 2d_2$. Now, $gk' = m'gd_1k' + 2gd_2k' = md_1k' + 2gd_2k' = ms_1 + 2gs_2$, where $s_1 = d_1k'$ and $s_2 = d_2k'$, i.e., $2k = ms_1 + 2gs_2$. Put $y = a^{gs_2}$. Then $H^y = \langle a^m, a^{2gs_2+i}b\rangle$ and so it contains an element $a^{2k+i}b$ of H^x and consequently, $H^x \subseteq H^y$. Therefore, $H^x = H^y$ since H^x and H^y have the same number of elements.

Claim 2. *If* $H = \langle a^m, a^i b\rangle$ *is a subgroup of Type* (2) *and* $m \geqslant 1$ *is divisible by* 4, *then* H *is not pronormal.*

In order to show that H is not pronormal in Dic_n, it is sufficient to find an element $g \in \mathrm{Dic}_n$ such that H and H^g are not conjugates in $\langle H, H^g\rangle$. We have $\langle H, H^a\rangle = \langle a^m, a^i b, a^2\rangle$. As m is even, we have $\langle H, H^a\rangle = \langle a^2, b\rangle$ if i is even and $\langle H, H^a\rangle = \langle a^2, ab\rangle$ if i is odd. As such, we have the following two cases.

Case 1: Suppose that i is odd. In this case, $\langle H, H^a\rangle = \langle a^2, ab\rangle$, and if H and H^a are conjugates in $\langle H, H^a\rangle$, then there must exist an element $x \in \langle H, H^a\rangle$ such that $H^a = H^x$ and such x is of the form a^{2p} for some p or $a^{2p+1}b$ for some p.

Subcase 1.1: If $x = a^{2p}$, then $H^x = \langle a^m, a^{4p+i}b\rangle = H^a = \langle a^m, a^{2+i}b\rangle$ and we must have $a^{mq}a^{4p+i}b = a^{2+i}b$ for some $q \in \mathbb{Z}$. But then, $a^{mq+4p} = a^2$, i.e., $a^{mq+4p-2} = e$. Now, $o(a) = 2n$ and $4|n$, therefore we have $2n|4p+mq-2$ and so $4|4p+mq-2$. Also, $4|m$ and so we must have $4|-2$, which is not true and therefore no such x exists.

Subcase 1.2: If $x = a^{2p+1}b$, then $H^x = \langle a^m, a^{4p+2-i}b\rangle = H^a = \langle a^m, a^{2+i}b\rangle$, and so we must have $a^{mq}a^{4p+2-i}b = a^{2+i}b$ for some q. As such, $a^{mq+4p+2-i} = a^{2+i}$, i.e., $a^{mq+4p-2i} = e$. Now, $o(a) = 2n$ and $4|n$, so we have $2n|4p+mq-2i$ and $4|4p+mq-2i$. Also, $4|m$ and so we must have $4|2i$, which is not possible as i is odd, and so, no such x exists. Therefore, in this Case 1, H and H^a are not conjugates in $\langle H, H^a\rangle$.

Case 2: Suppose that i is even. In this case, $\langle H, H^a\rangle = \langle a^2, b\rangle$, and if H and H^a are conjugates in $\langle H, H^a\rangle$, then there must exist an element $x \in \langle H, H^a\rangle$ such that $H^a = H^x$ and such x is of the form a^{2p} for some p or $a^{2p}b$ for some p.

Subcase 2.1: If $x = a^{2p}$, then $H^x = \langle a^m, a^{4p+i}b\rangle = H^a = \langle a^m, a^{2+i}b\rangle$, and so we have $a^{mq}a^{4p+i}b = a^{2+i}b$ for some $q \in \mathbb{Z}$. As such, $a^{mq+4p} = a^2$, i.e., $a^{mq+4p-2} = e$. Now, $o(a) = n$ and $4|n$ and so we have $2n|4p+mq-2$ and $4|4p+mq-2$. Also, $4|m$ and so we must have $4|-2$, which is not true and so no such x exists.

Subcase 2.2: If $x = a^{2p}b$, then $H^x = \langle a^m, a^{4p-i}b\rangle = H^a = \langle a^m, a^{2+i}b\rangle$ and so we have $a^{mq}a^{4p-i}b = a^{2+i}b$ for some q. Accordingly, $a^{mq+4p-i} = a^{2+i}$, i.e., $a^{mq+4p-2i-2} = e$. Now, $o(a) = 2n$ and $4|n$, and so we have $2n|4p+mq-2i-2$ and $4|4p+mq-2i-2$. Also, $4|m$ and so we must have $4|2$, which is not true and so no such x exists. Therefore, in Case 2 also, H and H^a are not conjugates in $\langle H, H^a\rangle$. Consequently, in either of these cases, the subgroup H is not pronormal. □

It is known that the number of subgroups of $\mathrm{Dic}_n = |\mathrm{L}(\mathrm{Dic}_n)|$ = number of divisors of $2n$ + sum of divisors of n, $n \geqslant 3$. We have the following formula for the number of pronormal subgroups of Dic_n, i.e., $|\mathrm{LPrN}(\mathrm{Dic}_n)|$.

Corollary 2.9. *For any* n, $|\mathrm{LPrN}(\mathrm{Dic}_n)| = \mathrm{d}(2n) + \sum\limits_{4\nmid d'|n} d'$.

P r o o f. From Theorem 2.8, for every choice of a divisor m of n which is not divisible by 4 there is a dicyclic pronormal subgroup $\langle a^m, a^i b\rangle$ for every i. Moreover, every divisor m of $2n$ will determine a cyclic pronormal subgroup $\langle a^m\rangle$ of Dic_n and these are the only pronormal subgroups of Dic_n. □

We prove that the set of all pronormal subgroups of Dic_n forms a sublattice of the subgroup lattice of Dic_n for any n.

Theorem 2.10. $\mathrm{LPrN}(\mathrm{Dic}_n)$ *is a sublattice of* $\mathrm{L}(\mathrm{Dic}_n)$.

P r o o f. We show that the intersection of two pronormal subgroups of Dic_n is again pronormal. Let H and K be two pronormal subgroups of Dic_n. If one of these subgroups is cyclic of the form $\langle a^k\rangle$, then by Lemma 2.5, we are through. So, let $H = \langle a^m, a^i b\rangle$ and $K = \langle a^r, a^j b\rangle$ for some $m, r \geqslant 1$, $m|n$, $r|n$, $0 \leqslant i \leqslant m-1$, $0 \leqslant j \leqslant r-1$, moreover $4 \nmid m, 4 \nmid r$ by Theorem 2.8. Suppose that for some k, $a^k b \in H \cap K$. Then there is l such that $H \cap K = \langle a^{[m,r]}, a^l b\rangle$. As $4 \nmid m$ and $4 \nmid r$, we also have that $4 \nmid [m,r]$ and $H \cap K = \langle a^{[m,r]}, a^l b\rangle$ is pronormal by Theorem 2.8. If $H \cap K$ is cyclic, it is pronormal by Lemma 2.5. Therefore the intersection of any two pronormal subgroups is a pronormal subgroup.

Next, we prove that the subgroup generated by the union of two pronormal subgroups is pronormal. Let H and K be two pronormal subgroups of Dic_n.

Case I: Suppose that both H and K are a subgroups of Type (2), say $H = \langle a^m, a^i b\rangle$ and $K = \langle a^r, a^j b\rangle$ for some $m, r \geqslant 1$, $m|n$, $r|n$, $0 \leqslant i \leqslant m-1$, $0 \leqslant j \leqslant r-1$, moreover $4 \nmid m$, $4 \nmid r$ by Theorem 2.8.

We contend that $\langle H \cup K\rangle = \langle a^g, a^i b\rangle$, where $g = (m, r, i-j)$. Indeed, for $S = \langle a^g, a^i b\rangle$ and $x \in S$, we have $x = a^{gk_1+i}b$, for some $k_1 \in \mathbb{Z}$. However, since $g = (m, r, i-j)$, there exist $p_1, p_2, p_3 \in \mathbb{Z}$ such that $g = mp_1 + rp_2 + (i-j)p_3$ and so $x = a^{(mp_1+rp_2+(i-j)p_3)k_1+i}b$, which is a finite product of elements of H and K, and so $x \in \langle H \cup K\rangle$, therefore $S \subseteq \langle H \cup K\rangle$. Now to show that $S \supseteq \langle H \cup K\rangle$, it is sufficient to show that $a^j b \in S$. We have $a^i b \in S$, $a^{j-i} \in S$ and so $a^j b \in S$. Consequently, $\langle a^m, a^i b, a^r, a^j b\rangle \subseteq S$, i.e., $S \supseteq \langle H \cup K\rangle$.

Now, since H and K are pronormal, we have $4 \nmid m$ and $4 \nmid r$, and so $4 \nmid g$, which implies that $\langle H \cup K\rangle$ is pronormal.

Case II: Suppose that both H and K are a cyclic subgroups of Type (1), then obviously $\langle H \cup K\rangle$ is also cyclic of Type (1) which is normal by Lemma 2.5 and so pronormal.

Case III: Suppose that one of H and K is a cyclic subgroups of Type (1) and the other one is of Type (2), say $H = \langle a^r\rangle$ and $K = \langle a^m, a^i b\rangle$. Then $\langle H \cup K\rangle = \langle a^g, a^i b\rangle$, where $g = (m, r)$. Now, $4 \nmid m$, so $4 \nmid g$, which implies that $\langle H \cup K\rangle$ is pronormal.

We conclude that given pronormal subgroups H and K of Dic_n, we have that both $H \vee K = \langle H \cup K\rangle$ and $H \wedge K = H \cap K$ are pronormal. Therefore $\mathrm{LPrN}(\mathrm{Dic}_n)$ is a sublattice of $\mathrm{L}(\mathrm{Dic}_n)$. □

Now, we establish lattice theoretic property, namely lower semimodularity in $\mathrm{L}(\mathrm{Dic}_n)$ and $\mathrm{LPrN}(\mathrm{Dic}_n)$.

Definition 2.11 ([14]). A lattice L is said to be *lower semimodular* (LSM) if it satisfies the following condition:

▷ If $T \prec T \vee S$, then $T \wedge S \prec S$ for $T, S \in L$.

Theorem 2.12. $\mathrm{L}(\mathrm{Dic}_n)$ *is lower semimodular.*

Proof. Let $T, S \in \mathrm{L}(\mathrm{Dic}_n)$ be such that $T \prec T \vee S$. We claim that $T \wedge S \prec S$.

Case 1: Let T and S be two cyclic subgroups of Type (1), say $T = \langle a^t\rangle$ and $S = \langle a^s\rangle$. Clearly, $T \vee S = \langle a^g\rangle$, where $g = (t, s)$. Note that $\langle a^t\rangle \prec \langle a^g\rangle$ if and only if $gp = t$ for a prime $p|n$.

Note that $T \wedge S = \langle a^l \rangle$, where $l = [s,t]$ and as $g|s$, we have $s = gq$ for a positive integer q, then $l = [gq, gp] = g[q,p] = gqp$. (Note that $p \nmid q$ as $(t,s) \neq t$). Consequently, $S \wedge T = \langle a^l \rangle = \langle a^{sp} \rangle \prec \langle a^s \rangle = S$.

Case 2: Let T be a cyclic subgroup of Type (1) and S be a subgroup of Type (2), say $T = \langle a^t \rangle$ and $S = \langle a^s, a^i b \rangle$.

Clearly, $T \vee S = \langle a^g, a^i b \rangle$, where $g = (t,s)$. We have that $\langle a^t \rangle \prec \langle a^g, a^i b \rangle$ is true if and only if $g = t$.

Note that $T \wedge S = \langle a^l \rangle$, where $l = [s,t]$, as $g|s$, we have $s = gq = tq$ for a positive integer q, then $l = [gq, t] = g[q, 1] = gq = tq = s$. Consequently, $S \wedge T = \langle a^l \rangle = \langle a^s \rangle \prec \langle a^s, a^i b \rangle = S$.

Case 3: Let T be a subgroup of Type (2) and S be a cyclic subgroup of Type (1), say $T = \langle a^t, a^i b \rangle$ and $S = \langle a^s \rangle$.

Clearly, $T \vee S = \langle a^g, a^i b \rangle$, where $g = (t,s)$. We have that $\langle a^t, a^i b \rangle \prec \langle a^g, a^i b \rangle$ is true if and only if $gp = t$ for a prime $p|n$.

Note that $T \wedge S = \langle a^l \rangle$, where $l = [s,t]$, as $g|s$, we have $s = gq$ for a positive integer q, then $l = [gq, gp] = g[q,p] = gqp$. (Note that, $p \nmid q$ as $(t,s) \neq t$). Consequently, $S \wedge T = \langle a^l \rangle = \langle a^{sp} \rangle \prec \langle a^s \rangle = S$.

Case 4: Let T and S be subgroups of Type (2), say $T = \langle a^t, a^i b \rangle$ and $S = \langle a^s, a^j b \rangle$, where $t, s|n$ and $0 \leqslant i \leqslant t-1$, $0 \leqslant j \leqslant s-1$. It is easy to see that $T \vee S = \langle a^g, a^i b \rangle$, where $g = (t, s, i-j)$. We have that $\langle a^t, a^i b \rangle \prec \langle a^g, a^i b \rangle$ is true if and only if $gp = t$ for some a $p|n$.

Subcase 4.1: Suppose that the equation $tx_1 + sx_2 = i - j$ has a solution, namely (x_1, x_2). Number $i - j$ is a multiple of (t,s) in this case. Substituting the values of t and s we get $gpx_1 + gqx_2 = g\alpha$, where $s = gq$ and $i - j = g\alpha$. If $p|q$, then $p|\alpha$ since the equation has a solution. Consequently, we get $g = t$, which is a contradiction to the assumption $T \prec T \vee S$. And so we must have $p \nmid q$. Note that $S \wedge T = \langle a^l, a^k b \rangle$, where $l = [s,t]$, then $l = [gq, gp] = g[q,p] = gqp$. Consequently, $S \wedge T = \langle a^l, a^k b \rangle = \langle a^{sp}, a^k b \rangle \prec \langle a^s, a^j b \rangle = S$.

Subcase 4.2: Suppose that equation $tx_1 + sx_2 = i - j$ has no solution. If for some $0 < l < 2n, a^l b \in H \cap K$, then there are p, q such that $a^{pt+i} b = a^l b = a^{qs+j} b$. This means that $tp + i \equiv sq + j \pmod{2n}$, it means that for some $kt(-p) + sq + 2kn = i - j$. As $t|n$, $s|n$, there are u, v such that $tu = n$, $sv = n$. Then $2kn = t(ku) + s(kv)$ and $t(-p + ku) + s(q + kv) = i - j$, which means that the equation $tx_1 + sx_2 = i - j$ has a solution, a contradiction. Consequently, in this case, we have $S \wedge T = \langle a^l \rangle$, where $l = [s,t]$. If $p \nmid q$, then $(p,q) = 1$, therefore $g = (t,s)$ and so the equation $tx_1 + sx_2 = i - j$ will have a solution, which is a contradiction. Therefore $p|q$ is true, so $t|s$ and so $[s,t] = l = s$. This concludes, $S \wedge T = \langle a^l \rangle = \langle a^s \rangle \prec \langle a^s, a^j b \rangle = S$. Hence $\mathrm{L}(\mathrm{Dic}_n)$ is lower semimodular. □

Corollary 2.13. $\mathrm{LPrN}(\mathrm{Dic}_n)$ *is lower semimodular.*

P r o o f. In order to show that $\mathrm{LPrN}(\mathrm{Dic}_n)$ is lower semimodular, it is sufficient to show that $\mathrm{LPrN}(\mathrm{Dic}_n)$ is a cover preserving the sublattice of $\mathrm{L}(\mathrm{Dic}_n)$. Already, we have $\mathrm{LPrN}(\mathrm{Dic}_n)$ as a sublattice of $\mathrm{L}(\mathrm{Dic}_n)$ by Theorem 2.8. We show that for given pronormal subgroups T and S of $\mathrm{L}(\mathrm{Dic}_n)$ such that $T \prec S$ in $\mathrm{LPrN}(\mathrm{Dic}_n)$, we have to have $T \prec S$ in $\mathrm{L}(\mathrm{Dic}_n)$.

Case I: Suppose that both T and S are cyclic subgroups of Type (1) with $T \prec S$ in $\mathrm{LPrN}(\mathrm{Dic}_n)$. As every cyclic subgroup of Type (1) is normal and so pronormal, we have $T \prec S$ in $\mathrm{L}(\mathrm{Dic}_n)$.

Case II: Suppose that T is a cyclic subgroup of Type (1) and S is a subgroup of Type (2) such that $T \prec S$ in $\mathrm{LPrN}(\mathrm{Dic}_n)$.

Let $T = \langle a^t \rangle$ and $S = \langle a^s, a^i b \rangle$ such that $T \prec S$. S is pronormal, so $4 \nmid s$ by Theorem 2.8 and also $s | t$, say $sq = t$. If $T \not\prec S$ in $\mathrm{L}(\mathrm{Dic}_n)$, then we must have $\langle a^t \rangle \subseteq \langle a^s \rangle \subseteq S$ and if $q \neq 1$, then $\langle a^t \rangle \subsetneq \langle a^s \rangle$. Subgroups of Type (1) are pronormal in Dic_n and therefore $\langle a^s \rangle \in \mathrm{LPrN}(\mathrm{Dic}_n)$, a contradiction to the assumption that $T \prec S$ in $\mathrm{LPrN}(\mathrm{Dic}_n)$. Hence, we must have $t = s$ and so $T \prec S$ in $\mathrm{L}(\mathrm{Dic}_n)$ as well.

Case III: Suppose that T and S are subgroups of Type (2) such that $T \prec S$ in $\mathrm{LPrN}(\mathrm{Dic}_n)$.

Let $T = \langle a^t, a^j b \rangle$ and $S = \langle a^s, a^i b \rangle$ such that $T \prec S$. Subgroups S, T are pronormal and therefore $4 \nmid t$ and $4 \nmid s$. If $T \not\prec S$ in $\mathrm{L}(\mathrm{Dic}_n)$, then there exists a dicyclic subgroup, say $X = \langle a^x, a^k b \rangle$ containing T and contained in S, which implies that $\langle a^t \rangle \subseteq \langle a^x \rangle$, but as X is not pronormal, we must have $4 | x$ and as $x | t$, we have $4 | t$, a contradiction to the fact that T is a dicyclic pronormal subgroup of Dic_n. Hence, no such subgroup exists. Consequently, $\mathrm{LPrN}(\mathrm{Dic}_n)$ is a cover preserving sublattice of $\mathrm{L}(\mathrm{Dic}_n)$ and hence lower semimodular. □

3. Essential elements in $\mathrm{LPrN}(\mathrm{Dic}_n)$

Definition 3.1 ([2]). An element $e \in L$ is called *essential* if $e \wedge a \neq 0$ holds for each element $a \in L, a \neq 0$.

In this section we determine a number of essential elements of $\mathrm{LPrN}(\mathrm{Dic}_n)$ and $\mathrm{L}(\mathrm{Dic}_n)$.

Theorem 3.2. *Let* $\mathrm{LEssPrN}(\mathrm{Dic}_n)$ *be the collection of essential elements of the lattice* $\mathrm{LPrN}(\mathrm{Dic}_n)$. *Then* $|\mathrm{LEssPrN}(\mathrm{Dic}_n)| = \mathrm{d}(a) + \sum\limits_{4 \nmid d' | a} d'$, *where* $a = 2n/(2p_1p_2 \dots p_z)$ *and* $p_1, p_2, \dots, p_m$ *are mutually different odd primes. Moreover,* $\mathrm{LEssPrN}(\mathrm{Dic}_n)$ *is a filter (dual ideal) generated by* $\langle a^{2n/(2p_1p_2\dots p_z)} \rangle$.

P r o o f. Firstly, we determine essential elements of $\mathrm{L}(\mathrm{Dic}_n)$. Note that if a subgroup E of Dic_n is an essential element in $\mathrm{L}(\mathrm{Dic}_n)$, then by definition $E \wedge A \neq \{e\}$ for any subgroup $A \neq \{e\}$, i.e., a subgroup that intersects every subgroup nontrivially. In particular, E intersects every atom of $\mathrm{L}(\mathrm{Dic}_n)$ and therefore E contains every atom. Note that a subgroup of Dic_n is an atom of $\mathrm{L}(\mathrm{Dic}_n)$ if and only if it is a cyclic subgroup of a prime order of Type (1) and all these subgroups are pronormal in Dic_n. As $\mathrm{LPrN}(\mathrm{Dic}_n)$ is a sublattice of $\mathrm{L}(\mathrm{Dic}_n)$, lattices $\mathrm{LPrN}(\mathrm{Dic}_n)$ and $\mathrm{L}(\mathrm{Dic}_n)$ have the same atoms, so these subgroups are also all atoms of $\mathrm{LPrN}(\mathrm{Dic}_n)$. But then, the join of atoms in $\mathrm{L}(\mathrm{Dic}_n)$ is nothing but

$$\bigvee_{m=1}^{z} \langle a^{2^{\alpha} \prod_{m=1}^{z} p_m^{\alpha_m - 1}} \rangle \vee \langle a^{2^{\alpha-1} \prod_{m=1}^{z} p_m^{\alpha_m}} \rangle = \langle a^{2^{\alpha-1} \prod_{m=1}^{z} p_m^{\alpha_m - 1}} \rangle.$$

Consequently, the only essential elements of $\mathrm{L}(\mathrm{Dic}_n)$ and of $\mathrm{LPrN}(\mathrm{Dic}_n)$ are the subgroups of respective lattices which contain the subgroup $\langle a^{2^{\alpha-1} \prod_{m=1}^{z} p_m^{\alpha_m - 1}} \rangle$. Therefore, $\mathrm{Less}(\mathrm{Dic}_n)$ is a filter in $\mathrm{L}(\mathrm{Dic}_n)$ generated by $\langle a^{2n/(2p_1p_2\ldots p_z)} \rangle$. Consequently, $|\mathrm{Less}(\mathrm{Dic}_n)| = \mathrm{d}(2n/(2p_1p_2\ldots p_z)) + \sum\limits_{d^* \mid 2n/(2p_1p_2\ldots p_z)} d^*$.

Secondly, we determine essential elements of $\mathrm{LPrN}(\mathrm{Dic}_n)$. In order to find a number of essential elements in $\mathrm{LPrN}(\mathrm{Dic}_n)$ it is sufficient to find a number of subgroups which contain $K = \langle a^{2n/(2p_1p_2\ldots p_z)} \rangle$. Note that a number of cyclic subgroups containing K is $\mathrm{d}(2n/(2p_1p_2\ldots p_z))$. In view of Theorem 2.8 we have a number of dicyclic pronormal subgroups containing K is $\sum\limits_{4 \nmid d^* \mid 2n/(2p_1p_2\ldots p_z)} d^*$. Therefore

$$|\mathrm{LEssPrN}(\mathrm{Dic}_n)| = \mathrm{d}\Big(\frac{2n}{2p_1p_2\ldots p_z}\Big) + \sum_{4 \nmid d^* \mid 2n/(2p_1p_2\ldots p_z)} d^*.$$

Consequently, $\mathrm{LEssPrN}(\mathrm{Dic}_n)$ is a filter generated by $K = \langle a^{2^{\alpha-1} \prod_{m=1}^{z} p_m^{\alpha_m - 1}} \rangle = \langle a^{2n/(2p_1p_2\ldots p_z)} \rangle$. $\square$

4. Structure of pronormal subgroups of symmetric and alternating groups

In this section, the collection of pronormal subgroups of S_n and A_n, namely, $\mathrm{LPrN}(\mathrm{S}_n)$ and $\mathrm{LPrN}(\mathrm{A}_n)$, respectively, are studied in respect of formation of sublattices of $\mathrm{L}(\mathrm{S}_n)$ and $\mathrm{L}(\mathrm{A}_n)$.

In what follows, a subgroup H of a group G is *strongly pronormal* if for all subgroups K of H and $g \in G$, the subgroup K^g is a conjugate to a subgroup of H (not necessarily to K) by an element of $\langle H, K^g \rangle$.

Proposition 4.1 ([16]). *Let $m, n \in N$ and $1 < m \leqslant n$. Then the following statements hold:*

- *A subgroup S_m of S_n is pronormal if and only if $m > \frac{1}{2}n$.*
- *A subgroup S_m of S_n is strongly pronormal if and only if $m > n-2$. For $\frac{1}{2}n < m < n-1$, in particular, a subgroup S_m of S_n is pronormal but it is not strongly pronormal.*

Note that $\mathrm{LPrN}(\mathrm{S}_4)$ is not a sublattice of $\mathrm{L}(\mathrm{S}_4)$, where $\mathrm{S}_4 = \langle (1234), (12) \rangle$. Also $M_1 = \langle (123), (12) \rangle$ and $M_2 = \langle (234), (23) \rangle$ are subgroups isomorphic to S_3 and being maximal subgroups, both M_1 and M_2 are pronormal. However, $M_1 \wedge M_2 = \langle (23) \rangle$, which is not pronormal, therefore $\mathrm{LPrN}(\mathrm{S}_4)$ is not a sublattice of $\mathrm{L}(\mathrm{S}_4)$. In fact, we have the following result about $\mathrm{LPrN}(\mathrm{S}_n)$ for $n \geqslant 4$.

Theorem 4.2. $\mathrm{LPrN}(\mathrm{S}_n)$ *is not a sublattice of* $\mathrm{L}(\mathrm{S}_n)$ *for* $n \geqslant 4$.

P r o o f. *Case I:* Suppose that n is even. Consider subgroups $M_1 = \langle (123 \ldots \frac{1}{2} \times (n+2)), (12) \rangle$ and $M_2 = \langle (23 \ldots \frac{1}{2}(n+4)), (23) \rangle$. Note that both M_1 and M_2 are pronormal being isomorphic to $\mathrm{S}_{(n+2)/2}$ by Proposition 4.1. Moreover, $M_1 \wedge M_2 = \langle (23 \ldots \frac{1}{2}(n+2)), (23) \rangle \cong S_{n/2}$, which is not pronormal in S_n by Proposition 4.1. As such, we conclude that whenever n is even we get two pronormal subgroups whose meet is not pronormal, which proves that in this case $\mathrm{LPrN}(\mathrm{S}_n)$ is not a sublattice of $\mathrm{L}(\mathrm{S}_n)$.

Case II: Suppose that n is odd. Consider subgroups $M_1 = \langle (123 \ldots \frac{1}{2}(n+1)), (12) \rangle$ and $M_2 = \langle (23 \ldots \frac{1}{2}(n+3)), (23) \rangle$. Note that both M_1 and M_2 are pronormal being isomorphic to $\mathrm{S}_{(n+1)/2}$ by Proposition 4.1. Moreover, $M_1 \wedge M_2 = \langle (23 \ldots \frac{1}{2}(n+1)), (23) \rangle \cong S_{(n-1)/2}$, which is not pronormal in S_n by Proposition 4.1. As such, we conclude that whenever n is odd, we get two pronormal subgroups whose meet is not pronormal, which proves that in this case $\mathrm{LPrN}(\mathrm{S}_n)$ is not a sublattice of $\mathrm{L}(\mathrm{S}_n)$.

Consequently, a collection of pronormal subgroups of S_n, $\mathrm{LPrN}(\mathrm{S}_n)$, is not a sublattice of $\mathrm{L}(\mathrm{S}_n)$ for $n \geqslant 4$. □

Corollary 4.3. $\mathrm{LSPrN}(\mathrm{S}_n)$ *is not a sublattice of* $\mathrm{L}(\mathrm{S}_n)$ *for* $n \geqslant 4$.

P r o o f. Clearly, for $n = 3$ every subgroup is pronormal as every subgroup is maximal. For $n \geqslant 4$, consider the subgroups, say M_1 and M_2, which are maximal subgroups of S_n, each one is isomorphic to S_{n-1} and so strongly pronormal. Note that $M_1 \wedge M_2 \cong S_{n-2}$, which is not strongly pronormal by Proposition 4.1, i.e., an intersection of two strongly pronormal subgroups of S_n is not strongly pronormal in general. □

We use the following facts, see respectively Benesh [1] and Giovanni [3].

1. In the alternating group A_5, all non-cyclic subgroups are pronormal. Moreover, every subgroup of order 2 of A_5 is not pronormal.

2. Every subgroup K of A_n which is isomorphic to A_{n-1} is a maximal subgroup of A_n and that means that K is also a pronormal subgroup of A_n. We shall use this fact for $K = A_{n-1}$, which can be naturally considered as a subgroup of A_n. Let $\{a_1, a_2, \ldots, a_k\} \subseteq \{1, 2, \ldots n\}$, $S(\{a_1, \ldots a_k\})$ be a symmetric group of the set $\{a_1, \ldots, a_k\}$, which can be naturally considered as a subgroup of S_n. For $1 \leqslant i \leqslant n$, let $X_i = \{1, \ldots, n\} - \{i, n\}$. For $n \geqslant 5$ every subgroup H of A_n of form $\langle S(X_i), S(\{i, n\}\rangle \cap A_n$ and of the form $\langle S(\{1, 2\}), S(\{3, \ldots, n\})\rangle \cap A_n$ is a maximal subgroup of A_n (isomorphic to S_{n-2}) and that means H is also a pronormal subgroup of A_n.

Theorem 4.4. $\mathrm{LPrN}(A_n)$ *is not a sublattice of* $\mathrm{L}(A_n)$ *for* $n \geqslant 5$.

Proof. *Case 1:* Let $n = 5$. Let $K = A_4$ and $H_s = \langle S(\{1, 2\}), S(\{3, 4, 5\})\rangle \cap A_5 = \langle (1, 2), (3, 4, 5), (3, 4)\rangle \cap A_5$. We know that subgroups K and H_s are pronormal subgroups of A_5. Moreover, $K \cap H_s = \langle (1, 2)(3, 4)\rangle$, which is not a pronormal subgroup of A_5, because it has 2 elements. Therefore $\mathrm{LPrN}(A_5)$ is not a sublattice of $\mathrm{L}(A_5)$.

Case 2: Let $n = 6$. Let $K = A_5$, $H_5 = \langle S(X_5), S(\{5, 6\})\rangle \cap A_6 = \langle (1, 2), (1, 2, 3, 4), (5, 6)\rangle \cap A_6$, and $H_s = \langle S(\{1, 2\}), S(\{3, 4, 5, 6\}) \cap A_6 = \langle (1, 2), (3, 4, 5, 6), (3, 4)\rangle A_6$. We know that subgroups K, H_5 and H_s are pronormal subgroups of A_6. Moreover, $K \cap H_5 = A_4$ and therefore $K \cap H_5 \cap H_s = A_4 \cap H_s = \langle (1, 2)(3, 4)\rangle$, which is not a pronormal subgroup of A_6. Therefore $\mathrm{LPrN}(A_6)$ is not a sublattice of $\mathrm{L}(A_6)$.

Case 3: Let $n \geqslant 7$. For $5 \leqslant i \leqslant n - 1$, let $H_i = \langle S(X_i), S(\{i, n\})\rangle \cap A_n = \langle (1, 2), (1, \ldots, i - 1, i + 1, \ldots, n - 1), (i, n)\rangle \cap A_n$ and let $H_s = \langle S(\{1, 2\}), S(\{3, \ldots, n\})\rangle \cap A_n = \langle (1, 2), (3, 4, \ldots, n), (3, 4)\rangle \cap A_n$. We know, that subgroups $H_{n-1}, H_{n-2}, \ldots, H_5$ and H_s are pronormal subgroups of A_n. It is easy to see that $H_{n-1} \cap H_{n-2} = A_{n-3}$, $H_{n-1} \cap H_{n-2} \cap H_{n-3} = A_{n-4}, \ldots, H_{n-1} \cap H_{n-2} \cap \ldots \cap H_5 = A_4$. Therefore $H_{n-1} \cap H_{n-2} \cap \ldots \cap H_5 \cap H_s = A_4 \cap H_s = \langle (1, 2)(3, 4)\rangle$, which is not a pronormal subgroup of A_n and consequently, $\mathrm{LPrN}(A_n)$ is not a sublattice of $\mathrm{L}(A_n)$ for $n \geqslant 7$.

Note that to use the argument of this case we need to intersect at least 2 subgroups, so that the list $H_{n-1}, \ldots, H_5$ must contain at least 2 groups, which is true for $n \geqslant 7$.

□

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